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SEMILINEAR ELLIPTIC EQUATIONS INVOLVING A GRADIENT TERM IN UNBOUNDED DOMAINS

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ABSTRACT. In this article, we study the existence of a classical solution of semilinear elliptic BVP involving gradient term of the type

$$\begin{gathered} -\Delta u = g(u) + \psi(\nabla u) + f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

where Ω is a (not necessarily bounded) domain in $\mathbb{R}^n$, $n \geq 2$ with smooth boundary $\partial\Omega$. $f \in C^{0,\alpha}_{\rm loc}(\overline{\Omega}), 0 < \alpha < 1$, $\psi \in C^1(\mathbb{R}^n, \mathbb{R})$ and g satisfies certain conditions (well known in the literature as "jumping nonlinearity").

1. INTRODUCTION

Throughout this article, let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a (not necessarily bounded) domain with smooth boundary $\partial\Omega$. Let $\Omega = \cup_{i=1}^{\infty}\Omega_i$, $\overline{\Omega}_i \subseteq \overline{\Omega}_{i+1} \subset \overline{\Omega}$, each $\Omega_i \subset \mathbb{R}^n$ is a bounded domain with smooth boundary and for every $x \in \overline{\Omega}$ there exists a bounded domain $\overline{M}$ with smooth boundary such that $x \in \overline{M} \subset \overline{\Omega}$. Suppose that $\psi \in C^1(\mathbb{R}^n, \mathbb{R})$, $f \in C^{0,\alpha}_{\rm loc}(\overline{\Omega}, \mathbb{R}), 0 < \alpha < 1$, and $g \in C^1(\mathbb{R}, \mathbb{R})$ satisfies certain conditions usually known in the literature as jumping nonlinearity. We make a modest attempt to study the existence of solutions for a class of semilinear elliptic BVP involving gradient term of the form

$$\begin{gathered} -\Delta u = g(u) + \psi(\nabla u) + f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega\,. \end{gathered} \tag{1.1}$$

When $\psi \equiv 0$ and Ω is a bounded domain, problem (1.1) reduces to

$$\begin{gathered} -\Delta u = g(u) + f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.2}$$

Boundary-value problems of the type (1.2) has been studied by many authors assuming the hypothesis

$$\limsup_{s\to-\infty} \frac{g(s)}{s} < \lambda_1 < \liminf_{s\to\infty} \frac{g(s)}{s}, \tag{1.3}$$

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where λ_1 is the first eigenvalue of

$$\begin{aligned} -\Delta u &= \lambda u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega\,. \end{aligned} \tag{1.4}$$

The study of existence and non-existence for BVP (1.2) with hypotheses (1.3) is a well known problem, known as Ambrosetti-Prodi type problem. BVP (1.2) with many variants and formulations has been extensively studied in bounded domains by several authors starting with a pioneering work by Ambrosetti and Prodi [5] in 1973. For a good historical background till 1980, we refer to a lecture notes by De Figueriedo [15]. The existence of solutions for the case of semilinear equations in bounded domains is well documented in [15]. De Figueriedo [16] has studied an Ambrosetti-Prodi problem for the scalar case by using the method of monotone iteration and variational techniques. For more details on Ambrosetti-Prodi type problems, we refer to [1, 2, 3, 6, 7, 8, 9, 10, 11, 14]. The problem of the type (1.1) both for bounded and unbounded domains has not been well studied and hence it needs attention. A few references for quasilinear case with jumping nonlinearities are found in [12, 13, 24]. The problem (1.1) considered in the present study is not a subclass of the problem studied in [12, 13, 24]. For a bounded domain $\Omega \subset \mathbb{R}^n$, we study the existence of solutions for the elliptic BVP (1.1). The idea inspired from the Ambrosetti - Prodi problem given in the lecture note due to De Figueriedo [15] with suitable modifications. Elliptic boundary value problems in unbounded domains present specific difficulties, primarily due to the lack of Rellich-Kondrachov compact embedding. In fact the embedding

$$i : H_0^1(\Omega) \hookrightarrow L^p(\Omega), \quad p \in \Big[2, \frac{2n}{n-2}\Big)$$

(which is compact when Ω is bounded) is not compact when Ω is a general unbounded domain. Due to this lack of compactness the standard variational methods fails. The term $\psi(\nabla u)$ containing the gradient makes the problem slightly harder in addition to the absence of usual compact embedding. Due to the presence of the gradient term the problem is not variational and for instance the critical point theory can not be applied directly. Since the term ∇u is not monotone in u, the standard monotone methods for semilinear elliptic equations may not be applied directly. The idea inspired by papers due to Aman, Crandall[4], De Figueiredo, Girardi, Matzeu[18] used in this paper is associating with problem (1.1) a family of semilinear elliptic problems which has a nontrivial solution via monotone method with no dependence on the gradient of the solution in addition to Leray-Schauder fixed point technique. The main result is stated in terms of the eigenvalues of

$$-\Delta u = \lambda u \quad \text{in } \Omega_i, \quad u = 0 \quad \text{on } \partial\Omega_i. \tag{1.5}$$

Our main purpose is to prove "similar" results in case $\Omega \subset \mathbb{R}^n$ is a (not necessarily bounded) domain. In the present case we have the usual techniques to handle the nonlinearity on one hand and an additional extraction of a solution due to the absence of compactness of Ω. In a way, the problem in the unbounded domain is linked with the existence in its bounded subdomains. The paper is organized as follows. In Section 2, we introduce the necessary notations, hypotheses and known results which are subsequently used. In Section 3, we study BVP (1.1) for the case for a bounded domain Ω. Section 4 deals with (1.1) in the unbounded domain Ω. Essentially the proof consists of finding bounds for the sequence of solutions

$\{u_i, i \geq 1\}$ of (1.1) defined on Ω_i and finally, extraction of a solution of (1.1) in Ω. The proof of the latter part is on the lines of proof given by Swanson and Noussair [25] or [26]. Standard theory of L^p estimates and Schauder estimates for elliptic BVPs, have been used.

2. Preliminaries

For $i \geq 1$, let λ_i be the first eigenvalue with corresponding eigenfunction ϕ_i of the Dirichlet BVP

$$-\Delta u = \lambda u \quad \text{in } \Omega_i, \quad u = 0 \quad \text{on } \partial\Omega_i. \tag{2.1}$$

Let N_i denote the span of ϕ_i in $L^2(\Omega_i)$, $\int_{\Omega_i} \phi_i^2 = 1$, and let $f_i = f|_{\overline{\Omega}_i}$ be the restriction of f on $\overline{\Omega}_i$, $i \geq 1$, $f_i = t_i\phi_i + h_i$, where $h_i \in N_i^{\perp}$, for $i \geq 1$. At each step, a generic constant is denoted by c or k_0 to avoid too many suffixes. For convenience, we list the following hypotheses:

(H1) Let $g \in C^1(\mathbb{R}, \mathbb{R})$ satisfy

$$\limsup_{s\to-\infty} \frac{g(s)}{s} < 0 < \lambda_1 < \liminf_{s\to\infty} \frac{g(s)}{s}, \tag{2.2}$$

($-\infty$ and ∞ are also allowed in the above limits).

(H2) Let $\psi \in C^1(\mathbb{R}^n, \mathbb{R})$ with $\psi(0) = 0$, be a bounded and Lipschitz continuous function with Lipschitz constant η, i.e.,

$$|\psi(p) - \psi(q)| \leq \eta|p - q|, \quad \text{for all } p, q \in \mathbb{R}^n.$$

Remark: (i) The least eigenvalue of the Laplacian (with Dirichlet boundary conditions) when Ω is unbounded could be zero and hence the hypotheseis (H1) is natural for unbounded domain.
(ii) We have $\Omega_i \subseteq \Omega_{i+1}$ which implies that $\lambda_1 \geq \lambda_2 \geq \lambda_3, \dots \geq \lambda_i \geq \lambda_{i+1}, \dots$. Then, from (H1), it follows

$$\limsup_{s\to-\infty} \frac{g(s)}{s} < 0 < \dots \leq \lambda_j \dots \leq \lambda_3 \leq \lambda_2 \leq \lambda_1 < \liminf_{s\to\infty} \frac{g(s)}{s}. \tag{2.3}$$

We refer to [21, 22] for properties of λ_i and the corresponding eigenfunctions ϕ_i.

Definition 2.1. A function $u : \Omega \to \mathbb{R}$ is called a solution of (1.1) if
(i) $u \in C^{2,\alpha}(\overline{M})$ for every bound subdomain $M \subset \Omega$,
(ii) u satisfies (1.1) identically.

3. Bounded domains

Let $G \subset \mathbb{R}^n$ be a bounded domain with smooth boundary ∂G. Let $f \in C^{0,\alpha}(\overline{G})$ and ψ satisfies the hypothesis (H2). Let $g \in C^1(\mathbb{R}, \mathbb{R})$ satisfies the hypothesis

(H1')

$$\limsup_{s\to-\infty} \frac{g(s)}{s} < 0 < \hat{\lambda}_1 < \liminf_{s\to\infty} \frac{g(s)}{s}, \tag{3.1}$$

($-\infty$ and ∞ are also allowed in the above limits) where, $\hat{\lambda}_1$ is the first eigenvalue with corresponding eigenfunction ϕ of

$$-\Delta u = \lambda u \quad \text{in } G, \quad u = 0 \quad \text{on } \partial G. \tag{3.2}$$

In this section, we establish the existence of solutions for the BVP

$$\begin{gathered} -\Delta u = g(u) + \psi(\nabla u) + f \quad \text{in } G, \\ u = 0 \quad \text{on } \partial G. \end{gathered} \tag{3.3}$$

Let N denote the span of ϕ in $L^2(G)$, $\int_G \phi^2 = 1$, and $f = t\phi + h$, where $h \in N^\perp$. The proof of the following result is on the same lines of [15, Lemma 1] and hence omitted.

Lemma 3.1. *Assume* (H1'). *Then, there are numbers $\hat{c} > 0$, $\underline{\hat{\mu}}$ and $\hat{\overline{\mu}}$ such that $\underline{\hat{\mu}} < 0 < \hat{\lambda}_1 < \hat{\overline{\mu}}$ and*

$$\begin{gathered} g(s) \geq \underline{\hat{\mu}} s - \hat{c}, \quad \forall s \in \mathbb{R}, \\ g(s) \geq \hat{\overline{\mu}} s - \hat{c}, \quad \forall s \in \mathbb{R}. \end{gathered} \tag{3.4}$$

We begin with the following existence result which is used later.

Lemma 3.2. *Let $G \subset \mathbb{R}^n$ be a bounded domain and β be a positive real number. Let $f \in C^{0,\alpha}(\overline{G}), \alpha \in (0,1)$ and ψ satisfies hypotheses (H2). Then, there exists a solution $u \in C^{2,\alpha}(\overline{G})$ of the BVP*

$$\begin{gathered} -\Delta u + \beta u = \psi(\nabla u) + f \quad \text{in } G, \\ u = 0 \quad \text{on } \partial G. \end{gathered} \tag{3.5}$$

Proof. A part of the proof is similar to the method given in the books by Evans [19, p.505] and Gilbarg and Trudinger [20, p.281]. It uses the Leray-Shauder's fixed point theorem [20, Theorem 11.3]. We divide the proof into three steps for convenience.
Step-1: For a given $u \in H_0^1(G)$, we define

$$h := \psi(\nabla u) + f.$$

Since ψ is bounded and $f \in C^{0,\alpha}(\overline{G})$, we note that $h \in L^2(G)$. Let $w \in H_0^1(G)$ be the unique weak solution of the linear BVP

$$\begin{gathered} -\Delta w + \beta w = h \quad \text{in } G, \\ w = 0 \quad \text{on } \partial G. \end{gathered} \tag{3.6}$$

By the regularity theory, we know that, $w \in H^2(G)$ with an estimate

$$\|w\|_{2,2,G} \leq c|h|_{2,G}, \tag{3.7}$$

for some (generic) constant c. Now, we define a operator $\mathcal{T} : H_0^1(G) \to H_0^1(G)$ by $\mathcal{T}u = w$ for a given $u \in H_0^1(G)$. Also, by a similar argument found in Evans [19, p.506], we note that $\mathcal{T} : H_0^1(G) \to H_0^1(G)$ is continuous and compact.
Step-2: The equation $u = \sigma\mathcal{T}u$, $\sigma \in [0,1]$ in $H^2(G) \cap H_0^1(G)$ is equivalent to the BVP

$$\begin{gathered} -\Delta u + \beta u = \sigma\{\psi(\nabla u) + f\} \quad \text{in } G, \\ u = 0 \quad \text{on } \partial G. \end{gathered} \tag{3.8}$$

Now ψ satisfies hypothesis (H2). Consequently, we have $|\psi(\nabla u)| \le \eta|\nabla u|$. Since $\sigma \le 1$, we have by Hölder's inequality,

$$\begin{aligned}\int_G |\nabla u|^2 + \beta \int_G |u|^2 &= \int_G \psi(\nabla u)u + \int_G fu \le \eta \int_G |\nabla u||u| + \int_G fu \\ &\le \frac{\eta}{2}\Big\{\int_G |\nabla u|^2 + \int_G |u|^2\Big\} + |f|_{2,G}|u|_{2,G} \\ &\le \Big(\frac{\eta}{2} + |f|_{2,G}\Big)\|u\|_{1,2,G}\end{aligned} \tag{3.9}$$

Case I: If $\beta \ge 1$, from (3.9) we have

$$\int_G |\nabla u|^2 + \int_G |u|^2 \le \int_G |\nabla u|^2 + \beta\int_G |u|^2 \le \Big(\frac{\eta}{2} + |f|_{2,G}\Big)\|u\|_{1,2,G}$$

or

$$\|u\|_{1,2,G} \le \frac{\eta}{2} + |f|_{2,G} \tag{3.10}$$

Case II: If $0 < \beta < 1$, from (3.9) we have

$$\beta\int_G |\nabla u|^2 + \beta\int_G |u|^2 \le \int_G |\nabla u|^2 + \beta\int_G |u|^2 \le \Big(\frac{\eta}{2} + |f|_{2,G}\Big)\|u\|_{1,2,G}$$

or

$$\|u\|_{1,2,G} \le \frac{1}{\beta}\Big(\frac{\eta}{2} + |f|_{2,G}\Big). \tag{3.11}$$

So for any β, from (3.10) and (3.11), we have

$$\|u\|_{1,2,G} \le \frac{1}{\beta}\Big(\frac{\eta}{2} + |f|_{2,G}\Big) = c, \tag{3.12}$$

where c is a (generic) constant independent of u and σ. By Leray-Schauder fixed point theorem, $\mathcal{T}$ has a fixed point equivalently, there exists a solution $u \in H^2(G) \cap H_0^1(G)$ of the BVP (3.5).

Step-3: Now, since $f \in C^{0,\alpha}(\overline{G})$, and ψ is bounded, we note $f, \psi(\nabla u) \in L^p(G)$ for any p. We choose $p > n$, so that by L^p regularity $u \in W^{2,p}(G)$. Then, by Sobolev embedding theorem for $p > n$, we have the solution $u \in C^{1,\alpha}(\overline{G})$, $\alpha = 1 - \frac{n}{p}$, and consequently $h \in C^{0,\alpha}(\overline{G})$. Then, by standard Schauder regularity $u \in C^{2,\alpha}(\overline{G})$, which completes the proof. □

We need the following results whose proofs are suitable modifications of corresponding results of [15].

Lemma 3.3 (Existence of a subsolution). *Assume* (H1') *and* (H2) *hold. For any given $f \in C^{0,\alpha}(\overline{G})$, there exists, $w \in C^{2,\alpha}(\overline{G})$ a subsolution of* (3.3) *such that $w < \overline{u}$ in G, where $\overline{u}$ is any super solution of* (3.3).

Proof. By Lemma 3.2, let w be a (unique) solution of the BVP

$$\begin{gathered}-\Delta w = \underline{\hat{\mu}} w - \hat{c} + \psi(\nabla w) + f \quad \text{in } G,\\ w = 0 \quad \text{on } \partial G.\end{gathered}$$

where $\underline{\hat{\mu}}$ and $\hat{c}$ are as in Lemma 3.1. We choose $\hat{c}$ in such a way that (3.4) has strict inequality. Notice

$$-\Delta \overline{u} \ge g(\overline{u}) + \psi(\nabla \overline{u}) + f \quad \text{in } G, \quad \overline{u} = 0 \quad \text{on } \partial G.$$

Subtracting and then applying the mean value theorem (along with (3.4))

$$\begin{aligned} -\Delta(\overline{u}-w) &\geq g(\overline{u}) + \psi(\nabla\overline{u}) - \underline{\hat{\mu}}w - \psi(\nabla w) + \hat{c} \\ &> \underline{\hat{\mu}}(\overline{u}-w) + \nabla\psi(\xi(x))(\nabla\overline{u} - \nabla w) \\ &= \underline{\hat{\mu}}(\overline{u}-w) + \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\psi(\xi(x))\frac{\partial}{\partial x_j}(\overline{u}-w) \quad \text{in } G, \end{aligned} \tag{3.13}$$

$$\overline{u} - w = 0 \quad \text{on } \partial G,$$

where $\xi(x)$ lies between $\nabla\overline{u}(x)$ and $\nabla w(x)$. By the maximum principle (refer [20, p.33]) $\overline{u} - w > 0$ in G and so $\overline{u} > w$ in G. We note that

$$\begin{gathered} -\Delta w = \underline{\hat{\mu}}w - \hat{c} + \psi(\nabla w) + f < g(w) + \psi(\nabla w) + f \quad \text{in } G, \\ w = 0 \quad \text{on } \partial G, \end{gathered} \tag{3.14}$$

or that w is a sub solution of (3.3). □

Now, we concentrate on establishing a required supersolution.

Lemma 3.4 (Existence of super solutions)**.** *Let $h \in N^{\perp}$. Then, there exists a $\tau \in \mathbb{R}$ such that*

$$\begin{gathered} -\Delta u = g(u) + \psi(\nabla u) + t\phi + h(x) \quad \textit{in } G, \\ u = 0 \quad \textit{on } \partial G, \end{gathered} \tag{3.15}$$

has a super solution $\overline{v}$, if $t \leq \tau$.

Proof. For a suitable negative t, we prove that, problem (3.15) has a super solution. Fix a positive integer $\mathfrak{K}$ and let

$$m = \max\big\{g(s) + \psi(p) + h(x) : x \in \overline{G}, 0 \leq s \leq \mathfrak{K},\ 0 \leq |p| \leq \mathfrak{K}\big\}. \tag{3.16}$$

Choose sub domains D_1, D_2 such that $D_1 \subseteq \overline{D}_1 \subseteq D_2 \subseteq \overline{D}_2 \subseteq G$ and $\operatorname{vol}(G \setminus D_1) \leq \delta$, where $\delta > 0$ will be chosen shortly. Let $H \in C^{0,\alpha}(\overline{G})$ such that

$$H = m \text{ in } G \setminus D_2, \quad H = 0 \text{ in } D_1, \quad 0 < H < m \text{ in } D_{2,1} (= D_2 \setminus D_1). \tag{3.17}$$

Let $\overline{v}$ be the solution of the BVP

$$-\Delta\overline{v} = H \quad \text{in } G, \quad \overline{v} = 0 \quad \text{on } \partial G.$$

By maximum principle $\overline{v} > 0$ in G, and by the a priori estimates for solutions of elliptic equations in $W^{2,p}(G)$ we obtain

$$\|\overline{v}\|_{2,p,G} \leq c|H|_{p,G} = c\Big\{\int_{G\setminus D_1} |H|^p + \int_{D_1} |H|^p\Big\}^{1/p} \leq cm\delta^{1/p}, \tag{3.18}$$

where c is a generic constant. By the Sobolev imbedding theorem, there exists a constant c' such that

$$\|\overline{v}\|_{C^{1,\alpha}(\overline{G})} \leq c'\|\overline{v}\|_{2,p,G}, \quad \forall \overline{v} \in W^{2,p}(G),\ p > n/2. \tag{3.19}$$

From the inequalities (3.18) and (3.19), we obtain

$$\|\overline{v}\|_{C^{1,\alpha}(\overline{G})} \leq cm\delta^{1/p}, \quad c \text{ a generic constant.}$$

Now, we choose D_1 in such a way that $c\,m\delta^{1/p} \leq \mathfrak{K}$ and under these circumstances we claim that $\overline{v}$ is a supersolution of (3.15) for a large negative t. Let $\phi^* = \min\{\phi(x) : x \in D_2\} > 0$ and we denote $\tau = \frac{-m}{\phi^*}$. We claim that

$$\tau\phi + m \leq H \ \text{ in } G. \tag{3.20}$$

If $x \in D_2$, since $\frac{-m\phi}{\phi^*} \le -m$ then,

$$\tau\phi + m = \frac{-m\phi}{\phi^*} + m \le 0 \le H.$$

Now, for $x \in G \backslash D_2$, from (3.17), we have $\tau\phi + m \le m = H$ and hence, we have the desired claim. Consequently, from (3.16) and (3.20), we have

$$-\Delta\overline{v} = H \ge \ m + \tau\phi \ge g(\overline{v}) + \tau\phi \ + \psi(\nabla\overline{v}) + h \quad \text{in } G.$$

Then, for $t \le \tau$, we obtain

$$-\Delta\overline{v} \ge g(\overline{v}) + \tau\phi \ + \psi(\nabla\overline{v}) + h \ge g(\overline{v}) + t\phi + \psi(\nabla\overline{v}) + h \quad \text{in } G,$$

which completes the proof of the lemma. □

Corollary 3.5. *Assume* (H1') *and* (H2) *hold. Suppose that for a given* $f \in C^{0,\alpha}(\overline{G})$ *problem* (3.3) *has a solution. Then, problem* (3.3) *has a minimal solution* $u_{\min}$*, i.e., given any other solution* u *of* (3.3) *one has* $u_{\min} \le u$ *in* $\overline{G}$*).*

We have the following is a result for the existence of solution of (3.3). The proof follows similar to the arguments of [15] with some difficulties due to the term $\psi(\nabla u)$ (Since ∇u is not monotone in u we can't use exactly the same proof given in [15] which can be overcame by the help of Lemma 3.2). For a similar technique, we refer Amann and Crandall [4].

Proposition 3.6 (Monotone method)**.** *Let* $f \in C^{0,\alpha}(\overline{G})$*. Suppose* g *and* ψ *satisfy the hypotheses* (H1') *and* (H2)*, respectively. If there exist functions* $\underline{u}$, $\overline{v} \in C^{2,\alpha}(\overline{G})$ *such that* $\underline{u} \le \overline{v}$ *in* $\overline{G}$ *satisfying*

$$-\Delta\underline{u} \le g(\underline{u}) + \psi(\nabla\underline{u}) + f(x) \quad \text{in } G, \quad \underline{u} \le 0 \quad \text{on } \partial G.$$

and

$$-\Delta\overline{v} \ge g(\overline{v}) + \psi(\nabla\overline{v}) + f(x) \quad \text{in } G, \quad \overline{v} \ge 0 \quad \text{on } \partial G,$$

then, there exist solutions $U, V \in C^{2,\alpha}(\overline{G})$ *of the BVP* (3.3) *such that* $\underline{u}(x) \le U(x) \le V(x) \le \overline{v}(x)$*. Moreover, any solution of* (3.3) *with* $\underline{u}(x) \le u(x) \le \overline{v}(x)$*, is such that* $U(x) \le u(x) \le V(x)$ *(*U *equals to* V *is not ruled out).*

The proof of the following results follows closely the arguments of [15] and hence omitted.

Corollary 3.7. *Let* $h \in N^{\perp}$ *be given. Suppose that problem* (3.15) *has a solution for a given* $\tau \in \mathbb{R}$*. Then, it has a solution for any* $t \le \tau$*.*

Lemma 3.8. *Suppose that problem* (3.15) *has a solution for a given* $f \in C^{0,\alpha}(\overline{G})$*. Then the Dirichlet BVP*

$$\begin{gathered} -\Delta u = g(u) + \psi(\nabla u) + \nu(x) \quad \text{in } G, \\ u = 0 \quad \text{on } \partial G, \end{gathered}$$

where ν *is a given function in* $C^{0,\alpha}(\overline{G})$ *with* $\nu \le f$ *has also a solution.*

Concerning the non-existence of solution, we have the ensuing result.

Lemma 3.9 (Non-existence result)**.** *Assume* (H1') *and* (H2)*. Then, there exists a number* $\zeta \in \mathbb{R}$*, independent of* $h \in N^{\perp}$*, such that the BVP*

$$\begin{gathered} -\Delta u = g(u) + \psi(\nabla u) + t\phi + h(x) \quad \text{in } G, \\ u = 0 \quad \text{on } \partial G, \end{gathered} \tag{3.21}$$

has no solution, for all $t > \zeta$.

The proof follows closely the arguments of [15] with few modifications due to the term $\psi(\nabla u)$.

Now, we establish a main result with above preliminaries:

Theorem 3.10. *Let* $f \in C^{0,\alpha}(\overline{G})$ *and* g, ψ *satisfies the hypotheses* (H1') *and* (H2), *respectively. For any* $h \in N^{\perp}$, *there exists a real number* $\rho = \rho(h)$ *such that, the BVP*

$$\begin{gathered} -\Delta u = g(u) + \psi(\nabla u) + t\phi + h \quad \text{in } G, \\ u = 0 \quad \text{on } \partial G, \end{gathered} \tag{3.22}$$

(i) *has a classical solution* $u = U \in C^{2,\alpha}(\overline{G})$, *if* $t \leq \rho(h)$;
(ii) *has no solution, if* $t > \rho(h)$.

Proof. For a given $h \in N^{\perp}$, the Lemma 3.4 shows that there is a $\tau \in \mathbb{R}$ such that problem (3.22) has a supersolution $\overline{u}$. By Lemma 3.3 we see that, for these given h and τ, BVP (3.22) has a subsolution, $\underline{u} \leq \overline{u}$. So by Proposition 3.6, BVP (3.22) has a solution for the given h and τ determined by Lemma 3.4. Lemma 3.9 shows that the set of t is such that (3.22) has a solution is bounded above and from Corollary 3.7 this set is a half-line. Hence, the proof of Theorem 3.10 is complete by letting $\rho(h)$ to be the supremum of the set of t's for which (3.22) has a solution. □

4. Unbounded domains

Throughout this section, let $f \in C^{0,\alpha}_{\rm loc}(\overline{\Omega})$ and g, ψ satisfies the hypotheses (H1) and (H2), respectively. In this section, we establish the existence of solutions for a class of quasilinear elliptic BVP (1.1). First, we prove a few results that are necessary for establishing the main result of this section. We begin with an attempt to establish the existence of solutions $\{u_j, j \geq 1\}$ to (1.1) in each bounded subdomains $\Omega_j \subset \Omega$ and find a few bounds for them under suitable hypotheses. Finally, extraction of a solution to (1.1) from the sequence $\{u_j\}$ is shown.

The following two results are about the existence of subsolutions and supersolutions in Ω_j, and they are generalization of Lemmas 3.3 and 3.4, respectively. The proof is an application of these lemmas with G replaced by Ω_j and $f = f_j$, $j \geq 1$ and hence omitted.

Lemma 4.1 (Existence of subsolutions)**.** *For any given* $f_j \in C^{0,\alpha}(\overline{\Omega}_j)$, $j \geq 1$, *there exists,* $w_j \in C^{2,\alpha}(\overline{\Omega}_j)$ *a subsolution of*

$$\begin{gathered} -\Delta u = g(u) + \psi(\nabla u) + f_j \quad \text{in } \Omega_j, \\ u = 0 \quad \text{on } \partial\Omega_j. \end{gathered} \tag{4.1}$$

such that $w_j < \overline{v}_j$ *in* Ω_j, *where* $\overline{v}_j$ *is any super solution of* (4.1).

Now we turn our attention towards establishing the required supersolution.

Lemma 4.2 (Existence of super solutions)**.** *Let* $h_j \in N_j^{\perp}$, $j \geq 1$. *Then, there exists* $\tau_j \in \mathbb{R}$ *such that*

$$\begin{gathered} -\Delta u = g(u) + \psi(\nabla u) + t_j\phi_j + h_j(x) \quad \text{in } \Omega_j, \\ u = 0 \quad \text{on } \partial\Omega_j, \end{gathered} \tag{4.2}$$

has a super solution $\overline{v}_j$, *if* $t_j \leq \tau_j$.

Lemma 4.3. *For any $h_j \in N_j^{\perp}$, $j \geq 1$, and for $t_j \leq \tau_j$, the BVP* (4.2) *has a solution $u = U_j \in C^{2,\alpha}(\overline{\Omega}_j)$, $j \geq 1$.*

The proof of the above lemma is an application of Theorem 3.10 with G replaced by Ω_j and hence omitted. A useful consequence is as follows.

Corollary 4.4. *Let $h_j \in N_j^{\perp}$ be given. Suppose that problem* (4.2) *has a solution for a given $\tau_j \in \mathbb{R}$. Then, it has a solution for any $s_j < \tau_j$.*

To summarize, we have the following result.

Corollary 4.5. *Let the hypotheses of Lemma 4.3 be satisfied. Then, for $j \geq 1$, and for $t_j \leq \tau_j$, the sequence denoted as*

$$u_j(x) = \begin{cases} U_j(x) & \text{if } x \in \overline{\Omega}_j, \\ 0 & \text{if } x \in \Omega \setminus \overline{\Omega}_j. \end{cases} \tag{4.3}$$

has the following properties:

(i) $u_j \in C^{2,\alpha}(\overline{\Omega}_j)$;
(ii) $u_j = 0$ *on* $\partial\Omega_j$;
(iii) $-\Delta u_j = g(u_j) + \psi(\nabla u_j) + f_j$ *in* Ω_j;
(iv) $\underline{u}_j(x) \leq u_j(x) \leq \overline{v}_j(x)$ *for all* $x \in \Omega_j$

The following is a vital result is about the boundedness of $\{u_j\}$ in $C^{2,\alpha}_{\rm loc}(\overline{\Omega})$ given that $\|u_j\|_{\infty,\Omega_j}$ are uniformly bounded.

Lemma 4.6. *Let $\{u_j, j \geq 1\}$ be the sequence defined by* (4.3) *in Corollary 4.3. Let $M \subset \Omega$ be an arbitrary bounded domain with smooth boundary and let i be a positive integer such that $\overline{M} \subseteq \Omega_i$. Suppose that $\|u_j\|_{\infty,\Omega_i} \leq c'$, $\forall\, j \geq i$, where c' is a constant independent of j. Then, there exists a constant $k > 0$ (independent of j) such that*

$$\|u_j\|_{C^{2,\alpha}(\overline{M})} \leq k, \quad \text{for all } j \geq i.$$

Proof. Let Q and R be bounded domains such that $\overline{M} \subseteq Q$, $\overline{Q} \subseteq R$, $\overline{R} \subseteq \Omega_i$ with boundaries ∂Q and ∂R in $C^{2,\alpha}$. By the hypotheses, for $j \geq i$, we have

$$-\Delta u_j = g(u_j) + \psi(\nabla u_j) + f_j \quad \text{in } \Omega_j,$$

(since $\Omega_i \subseteq \Omega_j$ for $j \geq i$). Let v_j be the unique solution of the BVP

$$\begin{gathered} -\Delta v = g(u_j) + \psi(\nabla u_j) + f_j \quad \text{in } R, \\ v = 0 \quad \text{on } \partial R. \end{gathered} \tag{4.4}$$

By the hypotheses of Lemma 4.6 and (H2), the sequences $\{u_j\}$ and $\{\psi(\nabla u_j)\}$ are uniformly bounded in $L^\infty(R)$ for $j \geq i$, respectively. As a consequence, the function g_j, defined by, $g_j(x) = g(u_j) + \psi(\nabla u_j) + f_j$ is uniformly bounded on R and so $g_j \in L^p(R)$ for any $p \geq 1$ and

$$|g_j|_{p,R} \leq c, \quad \text{for all } j \geq i.$$

where, c is (a generic constant) independent of j. By the L^p theory for elliptic equations, we have $\|v_j\|_{2,p,R} \leq c$ for all $j \geq i$, where c is (a generic constant) independent of j. Also, by Sobolev embedding theorem (with the choice $p > n$ and $\alpha = 1 - \frac{n}{p}$), we have

$$\|v_j\|_{C^{1,\alpha}(\overline{R})} \leq \|v_j\|_{2,p,R} \leq c, \quad \forall j \geq i, \tag{4.5}$$

where, c is a constant independent of j. For $j \geq i$, let w_j be the unique solution of the BVP

$$-\Delta w = 0 \quad \text{in } R, \quad w = u_j \quad \text{on } \partial R. \tag{4.6}$$

Since $\{u_j\}$ is uniformly bounded in $L^\infty(R)$ for $j \geq i$, by the maximum principle, we have $\|w_j\|_{C^0(\overline{R})} \leq c$, $j \geq i$, where c is (a generic constant) independent of j. Now the classical interior Schauder estimates yields

$$\|w_j\|_{C^{2,\alpha}(\overline{Q})} \leq c \quad \|w_j\|_{C^0(\overline{R})} \leq c, \quad \forall j \geq i, \tag{4.7}$$

where, c is a constant independent of j. Coupling the equations (4.4) and (4.6), we note $u = v_j + w_j$ satisfies

$$\begin{gathered} -\Delta u = g(u_j) + \psi(\nabla u_j) + f_j \quad \text{in } R, \\ u = u_j \quad \text{on } \partial R. \end{gathered} \tag{4.8}$$

Since $u = u_j$ is also a solution of (4.8), consequently, by uniqueness for elliptic BVP, we have $u_j = v_j + w_j$ and it follows from (4.5) and (4.7)

$$\|u_j\|_{C^{1,\alpha}(\overline{Q})} \leq c, \quad \forall j \geq i. \tag{4.9}$$

where, c is a constant independent of j. By interior Schauder estimate with aid of the inequality (4.9) and regularity assumptions on g, ψ, we have

$$\|u_j\|_{C^{2,\alpha}(\overline{M})} \leq c\big[\|u_j\|_{C^0(\overline{Q})} + \|f_j\|_{C^{0,\alpha}(\overline{Q})}\big] = k, \quad \forall j \geq i,$$

where k is a constant independent of j. This completes the proof. □

Now, we establish the main result in Theorem 4.8 of this section with above preliminaries.

Remark 4.7. For any $h_j \in N_j^\perp$, $j \geq 1$ by Lemma 4.3 we know that there exists a $\tau_j \in \mathbb{R}$ such that, the BVP

$$-\Delta u = g(u) + \psi(\nabla u) + t_j\phi_j + h_j \quad \text{in } \Omega_j, \quad u = 0 \quad \text{on } \partial\Omega_j, \tag{4.10}$$

has a solution $U_j \in C^{2,\alpha}(\overline{\Omega}_j)$ for $t_j \leq \tau_j$. The condition $\tau = \inf \tau_j > -\infty$ in the following theorem is a link between the solution in the subdomains $\Omega_j, j \geq 1$ and a solution in Ω.

Theorem 4.8. *For $t_j \leq \tau_j, j \geq 1$, let $u = U_j \in C^{2,\alpha}(\overline{\Omega}_j)$ be the solution of* (4.10). *We assume the condition*

$$\tau = \inf \tau_j > -\infty. \tag{4.11}$$

Then, under the hypotheses of Lemma 4.6, BVP (1.1) *has a classical solution $u \in C^{2,\alpha}_{\rm loc}(\overline{\Omega})$, if $t_j \leq \tau$.*

Proof. If $t_j \leq \tau$, from condition (4.11) we have $t_j \leq \tau_j$, for each $j = 1, 2 \dots$. For each $j \geq 1$, by Lemma 4.3, we get a sequence of solutions $\{U_j, j \geq 1\}$ to the BVP (4.10), for $t_j \leq \tau_j$. The idea of the proof is to extract a solution of (1.1) in unbounded domain from a sequence of solutions $\{U_j, j \geq 1\}$ of (4.10) in bounded subdomains. So, to get a sequence of solutions $\{U_j\}$ of (4.10) for all j, we need $t = t_j$ should be less than or equal to each τ_j for $j \geq 1$. By Lemma 4.2, we know each τ_j are negative numbers. So $\inf \tau_j$ should not converges to $-\infty$ and hence we need the condition (4.11).

Since we have a sequence of solutions $\{U_j, j \geq 1\}$ of (4.10), we define the sequence $\{u_j\}$ by (4.3) in Corollary 4.5. For each integer $i = 1, 2 \dots$, by Lemma 4.6, there exists a positive constant k, independent of j, such that

$$\|u_j\|_{C^{2,\alpha}(\overline{\Omega}_i)} \leq k, \quad \forall\, j \geq i.$$

The inclusion $C^{2,\alpha}(\overline{\Omega}_1) \hookrightarrow\hookrightarrow C^2(\overline{\Omega}_1)$ is compact (cf. [20, p.282]) and hence $\{u_j, j \geq i\}$ has a convergent sequence $\{u_j^1, j \geq i\}$ in $C^2(\overline{\Omega}_1)$ and let $\{u_j^1, j \geq i\}$ converges uniformly to u^1 in $C^2(\overline{\Omega}_1)$. Now, inductively, let $\{u_j^i\}$ be a subsequence of $\{u_j^{i-1}, j \geq i, i \geq 2\}$ such that the $\{u_j^i\}$ converge uniformly to u^i in $C^2(\overline{\Omega}_i)$.

We define, $u : \Omega \to \mathbb{R}$ by

$$u(x) = u^i(x) \quad \text{if } x \in \overline{\Omega}_i,\ i = 1, 2 \dots.$$

We note that this definition is consistent since $\Omega_i \subseteq \Omega_{i+1}$ and $u^{i+1} = u^i$ on $\overline{\Omega}_i$. We note that the diagonal sequence $\{u_j^j(x)\}$ converges to $u(x)$ for all $x \in \Omega$. Let $M \subseteq \Omega$ be any bounded domain, then, $\overline{M} \subseteq \overline{\Omega}_i$ for some i. Now we have $\{u_j^i, j \geq i\}$ converges uniformly in $C^2(\overline{M})$ norm to $u^i = u$ on $\overline{M}$. Since $\{u_j^j, j \geq i\}$ be a subsequence of $\{u_j^i, j \geq i\}$ consequently, $\{u_j^j, j \geq i\}$ converges uniformly in $C^2(\overline{M})$ norm to $u^i = u$ on $\overline{M}$, which shows that $u_j^j \to u$, and $-\Delta u_j^j \to -\Delta u$ in $\overline{M}$. By (iii) of Corollary 4.5, we know

$$-\Delta u_j^j(x) = g(u_j^j(x)) + \psi(\nabla u_j^j(x)) + f_j(x), \quad x \in \overline{M}$$

and letting $j \to \infty$ in the above, we have

$$-\Delta u(x) = g(u(x)) + \psi(\nabla u(x)) + f(x), \quad x \in \overline{M}.$$

Since M is arbitrary, and by the standard regularity argument based on Schauder estimates, $u \in C^{2,\alpha}(\overline{M})$. We have for every $x \in \overline{\Omega}$ there exists a bounded domain $\overline{M}$ with smooth boundary such that $x \in \overline{M} \subset \overline{\Omega}$. Also $\{u_j^j, j \geq i\}$ converges uniformly to u on $\overline{M}$. It follows from (4.3) that $u_j^j = 0$ on $\partial\Omega$ for all $j \geq i$ and since $u_j^j \to u$ on $\overline{M}$, $u = 0$ on $\partial\Omega$. Thus, u is indeed a solution of (1.1), if $t_j \leq \tau$. $\square$

Presently we do not have a clear idea regarding when the condition (4.11) necessarily holds. But in the following example, we see that the condition (4.11) holds and in fact $\tau = \inf \tau_j = -\frac{2\sqrt{\pi}}{3}$. Hence, the condition (4.11) in Theorem 4.8 at least holds for some class of BVPs.

Example 4.9. Let $\Omega := \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : |x| > 1\}$ and

$$\Omega_i := \{x \in \mathbb{R}^3 : 1 < |x| < i + 1\}, \quad i \geq 1$$

Suppose that $f(x) = -|x|^{-6}$, $x \in \Omega$. Then, $f \in C^{0,\alpha}_{\rm loc}(\overline{\Omega})$, $0 < \alpha < 1$. We consider the quasilinear elliptic BVP

$$\begin{gathered} -\Delta u(x) = u^2(x) + \sin(|\nabla u(x)|^2) - |x|^{-6} \quad x \in \Omega, \\ u(x) = 0 \quad x \in \partial\Omega. \end{gathered} \tag{4.12}$$

The functions $g(u) = u^2$ and $\psi(\nabla u) = \sin(|\nabla u|^2)$ satisfy (H1) and (H2), respectively. Let $\phi_i > 0$ be such that $\phi_i \in H_0^1(\Omega_i) \cap C^\infty(\Omega_i)$,

$$-\Delta \phi_i = \lambda_i \phi_i, \quad \text{and} \quad \int_{\Omega_i} {\phi_i}^2\, dx = 1.$$

Now,

$$f|_{\overline{\Omega}_i} = f_i = t\phi_i + h_i, \quad \text{or} \quad t = \int_{\Omega_i} f_i\phi_i\,dx,$$

where $\int_{\Omega_i} h_i\phi_i\,dx = 0$. So in Ω_1,

$$\begin{aligned} -t &= \int_{\Omega_1} |x|^{-6}\phi_1\,dx \le \Big(\int_{\Omega_1} |x|^{-12}\,dx\Big)^{1/2}\Big(\int_{\Omega_1} \phi_1^2\,dx\Big)^{1/2} \\ &= \Big(\int_{\Omega_1} |x|^{-12}\,dx\Big)^{1/2} \\ &= \Big(\int_0^{2\pi}\int_0^{\pi}\int_1^2 r^{-12}r^2\sin\phi\,drd\theta d\phi\Big)^{1/2}, \quad (\text{where } r = |x|) \\ &= \big[\frac{4\pi}{9}(1-\frac{1}{2^9})\big]^{1/2}, \end{aligned}$$

which implies

$$t \ge -\frac{2\sqrt{\pi}}{3}.$$

Similarly, in each Ω_i, $t \ge -\frac{2\sqrt{\pi}}{3}$. By Theorem 3.10, we know there exists a solution of the BVP

$$\begin{gathered} -\Delta u(x) = u^2(x) + \sin(|\nabla u(x)|^2) - |x|^{-6} \\ = u^2(x) + \sin(|\nabla u(x)|^2) - (t\phi_i(x) + h_i(x)), \quad x \in \Omega_i, \\ u(x) = 0, \quad x \in \partial\Omega_i, \end{gathered} \tag{4.13}$$

for all $t \le \tau_i$ (say). Then, each $\tau_i \ge -\frac{2\sqrt{\pi}}{3}$ and so $\inf \tau_i > -\infty$.

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