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A SIXTH-ORDER PARABOLIC EQUATION DESCRIBING CONTINUUM EVOLUTION OF FILM FREE SURFACE

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Abstract. In this article, we study the regularity of solutions for a sixth-order parabolic equation. Based on the Schauder type estimates and Campanato spaces, we prove the existence of classical global solutions.

1. Introduction

In the previous fifteen-twenty years, essentially sixth-order nonlinear parabolic partial differential equationa, as models for applications in mechanics and physics, have become more common in the literature on pure and applied PDEs. Evans, Galaktionov and King[1, 2] studied the blow-up behavior and global similarity patterns of solutions for a sixth order thin film equations containing an unstable (backward parabolic) second-order term

$$u_t = \nabla \cdot (|u|^n \nabla \Delta^2 u) - \Delta(|u|^{p-1}u), \quad n > 0, p > 0,$$

with bounded integrable initial data. Jüngel and Milisić[3] proved the global in time existence of weak nonnegative solutions to the following initial value problem in one space dimension with periodic boundary conditions:

$$\begin{gathered} n_t = L[n] := \Big[n\Big(\frac{1}{n}(n(\log n)_{xx})_{xx} + \frac{1}{2}((\log n)_{xx})^2\Big)_x\Big]_x, \quad x \in \mathbb{T}, t > 0, \\ n(x,0) = n_0(x), \quad x \in \mathbb{T}. \end{gathered}$$

In [4], by an extension of the method of matched asymptotic expansions, Korzec, Evans, Münch and Wangner derived the stationary solutions of a 1D driven sixth order Cahn-Hilliard equation which arises as a model for epitaxially growing nano-structures. Li and Liu[5] studied the radial symmetric solutions for the following sixth order thin film equation:

$$u_t = \nabla \cdot [|u|^n \nabla \Delta^2 u], \quad x \text{ in the unit ball of } \mathbb{R}^2,\ n > 0.$$

Recently, based on the Landau-Ginzburg theory, Pawlow and Zajaczkowski [6] proved that a 3D sixth order Cahn-Hilliard equation under consideration is well posed in the sense that it admits a unique global smooth solution which depends

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continuously on the initial datum. We also refer the solvability conditions in $H^6(\mathbb{R}^3)$ for sixth order linearized Cahn-Hilliard problem is also studied in [7].

In the study of a thin, solid film grown on a solid substrate, in order to describe the continuum evolution of the film free surface, there arise a classical surface-diffusion equation (see [8])

$$v_n = \mathcal{D}\Delta_S\mu = \mathcal{D}\Delta_S(\mu_\gamma + \mu_w) = \mathcal{D}\Delta_S(\tilde{\gamma}_{\alpha\beta}C_{\alpha\beta} + \nu\Delta^2 u + \mu_w), \tag{1.1}$$

where v_n is the normal surface velocity, $\mathcal{D} = D_S S_0 \Omega_0 V_0/(RT)^{23}$ (D_s is the surface diffusivity, S_0 is the number of atoms per unit area on the surface, Ω_0 is the atomic volume, V_0 is the molar volume of lattice cites in the film, R is the universal gas constant and T is the absolute temperature), Δ_S is the surface Laplace operator, ν is the regularization coefficient that measures the energy of edges and corners, $C_{\alpha\beta}$ is the surface curvature tensor and μ_w being an exponentially decaying function of u that has a singularity at $u \to 0$ (see [8]).

In the small-slop approximation, in the particular cases of high-symmetry orientations of a crystal with cubic symmetry, then the evolution equation (1.1) for the film thickness can be written in the following form

$$\frac{\partial u}{\partial t} = D\left\{D^5 u + D^3 u - D[|Du|^2 D^2 u] + D[w_0(u) + w_2(u)|Du|^2 + w_3(u)D^2 u]\right\}, \tag{1.2}$$

where $w_{0,2,3}(h)$ are smooth functions, respectively $[w_3(h_0) = 0, 2w_2 = \frac{dw_3}{dh}]$.

We study the sixth-order nonlinear parabolic equation

$$\begin{aligned}\frac{\partial u}{\partial t} = D\big\{m(u)\big[&D^5 u + D^3 u - D(|Du|^2 D^2 u)\\ &+ D(w_0(u) + w_2(u)|Du|^2 + w_3(u)D^2 u)\big]\big\},\end{aligned} \tag{1.3}$$

where $(x,t) \in Q_T$, $Q_T \equiv (0,1)\times(0,T)$. On the basis of physical consideration, Equation (1.3) is supplemented by the following boundary conditions

$$Du(x,t) = D^3u(x,t) = D^5u(x,t) = 0, \quad x = 0, 1, \tag{1.4}$$

and initial condition

$$u(x,0) = u_0(x), \quad x \in [0,1]. \tag{1.5}$$

Our main purpose is to establish the existence of classical global solutions under much general assumptions. The main difficulties for treating the problem (1.3)-(1.5) are caused by the nonlinearity of the principal part and the lack of maximum principle. Due to the nonlinearity of the principal part, there are more difficulties in establishing the global existence of classical solutions. Our method for investigating the regularity of solutions is based on uniform Schauder type estimates for local in time solutions, which are relatively less used for such kind of parabolic equations of sixth order. Our approach lies in the combination of the energy techniques with some methods based on the framework of Campanato spaces. Now, we give the main results in this paper.

Theorem 1.1. *Assume that*

- $u_0 \in C^{6+\alpha}[0,1]$, $\alpha \in [0,1)$, $D^i u_0(0) = D^i u_0(1) = 0$ $(i = 0, 2, 4)$;
- $m(s) \in C^{1+\alpha}(\mathbb{R})$, $\inf_{s\in\mathbb{R}} m(s) = m_0 > 0$;
- $w_3(h_0) = 0$, $2w_2(h) = w_3'(h)$, $W_0(s) = \int_0^s w_0(s)ds \geq \frac{3}{4}[w_3(s)]^2$.

Then (1.3)-(1.5) *admits a unique classical solution* $u(x,t) \in C^{6+\alpha, 1+\frac{\alpha}{6}}(\bar{Q}_T)$.

Remark 1.2. During the past few years, many authors studied the properties of solutions (such as blow-up behavior and global similarity patterns of solutions, weak nonnegative solutions, radial symmetric solutions, stationary solutions, solvability conditions and so on) for sixth-order parabolic equation, but only a few papers were devoted to the existence of classical solution for sixth order parabolic equation. In this article, based on the Schauder type estimates, Campanato spaces and a result in [9], we consider the existence of classical solutions for a sixth-order parabolic equation which was introduced in [8].

2. Proof of the main result

Based on the classical approach, it is easy for us to conclude that problem (1.3)-(1.5) admits a unique classical solution local in time. So, it is sufficient to make a priori estimates. First of all, we give the Hölder norm estimate on the local in time solutions.

Lemma 2.1. *Assume that u is a smooth solution of the problem* (1.3)-(1.5). *Then there exists a constant C depending only on the known quantities, such that for any (x_1, t_1), $(x_2, t_2) \in Q_T$ and some $0 < \alpha < 1$,*

$$\begin{gathered}|u(x_1,t_1) - u(x_2,t_2)| \le C(|t_1 - t_2|^{\frac{\alpha}{6}} + |x_1 - x_2|^{\alpha}),\\ |Du(x_1,t_1) - Du(x_2,t_2)| \le C(|t_1 - t_2|^{\frac{1}{12}} + |x_1 - x_2|^{1/2}).\end{gathered}$$

Proof. Now, we set

$$F(t) = \int_0^1 \Big(\frac{1}{2}|D^2u|^2 - \frac{1}{2}|Du|^2 + \frac{1}{12}|Du|^4 + W_0(u) - \frac{1}{2}w_3(u)|Du|^2\Big)dx.$$

Integrating by parts, from the boundary value condition (1.4), we deduce that

$$\begin{aligned}\frac{d}{dt}F(t) &= \int_0^1 [D^2uD^2u_t - DuDu_t + \frac{1}{3}|Du|^2DuDu_t + w_0(u)u_t\\ &\quad - w_3(u)DuDu_t - \frac{1}{2}w_3'(u)|Du|^2u_t]dx\\ &= \int_0^1 [D^2uD^2u_t - DuDu_t + \frac{1}{3}|Du|^2DuDu_t + w_0(u)u_t\\ &\quad + w_3(u)D^2uu_t + \frac{1}{2}w_3'(u)|Du|^2u_t]dx\\ &= \int_0^1 \Big[D^4u + D^2u - \frac{1}{3}D(|Du|^2Du) + w_0(u)\\ &\quad + w_2(u)|Du|^2 + w_3(u)D^2u\Big]u_tdx\\ &= -\int_0^1 m(u)\Big|D\Big[D^4u + D^2u - \frac{1}{3}D(|Du|^2Du) + w_0(u)\\ &\quad + w_2(u)|Du|^2 + w_3(u)D^2u\Big]\Big|^2dx \le 0.\end{aligned}$$

Hence $F(t) \le F(0)$, that is

$$\begin{aligned} &\int_0^1 (\frac{1}{2}|D^2u|^2 + \frac{1}{12}|Du|^4 + W_0(u))dx \\ &\le F(0) + \frac{1}{2}\int_0^1 (|Du|^2 + w_3(u)|Du|^2)dx. \end{aligned} \tag{2.1}$$

It then from Poincaré's inequality and the boundary value condition (1.4) follows that

$$\int_0^1 |Du|^2 dx \le \frac{1}{\pi^2}\int_0^1 |D^2u|^2 dx. \tag{2.2}$$

On the other hand, we have

$$\int_0^1 w_3(u)|Du|^2 dx \le \frac{1}{6}\int_0^1 |Du|^4 dx + \frac{3}{2}\int_0^1 [w_3(u)]^2 dx. \tag{2.3}$$

Adding (2.1), (2.2) and (2.3), noticing that $W_0(u) \ge \frac{3}{4}[w_3(u)]^2$, we obtain

$$\sup_{0<t<T}\int_0^1 |D^2u|^2 dx \le C. \tag{2.4}$$

Combing (2.2) and (2.4) gives

$$\sup_{0<t<T}\int_0^1 |Du|^2 dx \le C, \tag{2.5}$$

The integration of (1.3) over the interval $(0,1)$ yields $\int_0^1 \frac{\partial u}{\partial t}dx = 0$, hence we obtain

$$\int_0^1 u(x,t)dx = \int_0^1 u_0(x)dx.$$

Applying the mean value theorem, we see that for some $x_t^* \in (0,1)$

$$u(x_t^*, t) = \int_0^1 u_0(x)dx = M.$$

Then

$$|u(x,t)| \le |u(x,t) - u(x_t^*,t)| + |u(x_t^*,t)| \le |\int_{x_t^*}^x Du(t,y)dy| + M.$$

Taking this into account, we deduce that

$$\sup_{Q_T}|u(x,t)| \le C, \tag{2.6}$$

On the other hand, a simple calculation shows that

$$\int_0^1 u^2 dx \le \sup_{Q_T}|u(x,t)|^2 \le C. \tag{2.7}$$

Combing (2.7), (2.5) and (2.4) together, using Sobolev's embedding theorem, we derive that

$$\sup_{Q_T}|Du(x,t)| \le C\Big(\int_0^1 u^2 dx + \int_0^1 |Du|^2 dx + \int_0^1 |D^2u|^2 dx\Big)^{1/2} \le C. \tag{2.8}$$

Multiplying both sides of (1.3) by D^4u, integrating the resulting relation with respect to x over $(0,1)$, integrating by parts, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 |D^2u|^2dx + \int_0^1 m(u)|D^5u|^2dx \\
&= -\int_0^1 m(u)D^3uD^5u\,dx + \int_0^1 m(u)D(|Du|^2D^2u)D^5u\,dx \\
&\quad - \int_0^1 m(u)Dw_0(u)D^5u\,dx - \frac{1}{2}\int_0^1 m(u)D(w_3'(u)|Du|^2)D^5du\,dx \\
&\quad - \int_0^1 m(u)D(w_3(u)D^2u)D^5u\,dx \\
&= -\int_0^1 m(u)D^3uD^5u\,dx + \int_0^1 m(u)|Du|^2D^3uD^5u\,dx \\
&\quad + 2\int_0^1 m(u)Du|D^2u|^2D^5u\,dx - \int_0^1 m(u)w_0'(u)DuD^5u\,dx \\
&\quad - \frac{1}{2}\int_0^1 m(u)w_3''(u)|Du|^3D^5u\,dx - 2\int_0^1 m(u)w_3'(u)DuD^2uD^5u\,dx \\
&\quad - \int_0^1 m(u)w_3(u)D^3uD^5u\,dx \\
&=: I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7.
\end{aligned}
\tag{2.9}
$$

By Nirenberg's inequality, we derive that

$$
\begin{aligned}
&\int_0^1 |D^3u|^2dx \\
&\le \Big(C'\Big(\int_0^1 |D^5u|^2dx\Big)^{1/6}\Big(\int_0^1 |D^2u|^2dx\Big)^{1/3} + C''\Big(\int_0^1 |D^2u|^2dx\Big)^{1/2}\Big)^2 \\
&\le \varepsilon\int_0^1 |D^5u|^2dx + C_\varepsilon.
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^1 |D^2u|^4dx \\
&\le \Big(C'\Big(\int_0^1 |D^5u|^2dx\Big)^{1/24}\Big(\int_0^1 |D^2u|^2dx\Big)^{11/24} + C''\Big(\int_0^1 |D^2u|^2dx\Big)^{1/2}\Big)^4 \\
&\le \varepsilon\int_0^1 |D^5u|^2dx + C_\varepsilon.
\end{aligned}
$$

Hence

$$
\begin{aligned}
I_1 &\le \sup_{Q_T}|m(u)|\int_0^1 |D^3uD^5u|dx \le C\int_0^1 |D^3uD^5u|dx \\
&\le \varepsilon\int_0^1 |D^5u|^2dx + C_\varepsilon\int_0^1 |D^3u|^2dx \\
&\le 2\varepsilon\int_0^1 |D^5u|^2dx + C.
\end{aligned}
\tag{2.10}
$$

$$
\begin{aligned}
I_2 &\le \sup_{Q_T} |m(u)(Du)^2| \int_0^1 |D^3uD^5u|dx \\
&\le C\int_0^1 |D^3uD^5u|dx \\
&\le 2\varepsilon\int_0^1 |D^5u|^2dx + C.
\end{aligned}
\tag{2.11}
$$

$$
\begin{aligned}
I_3 &\le 2\sup_{Q_T} |m(u)Du| \int_0^1 |(D^2u)^2D^5u|dx \\
&\le \varepsilon\int_0^1 |D^5u|^2dx + C_\varepsilon\int_0^1 |D^2u|^4dx \le 2\varepsilon\int_0^1 |D^5u|^2dx + C.
\end{aligned}
\tag{2.12}
$$

$$
\begin{aligned}
I_4 &\le \sup_{Q_T} |m(u)w_0'(u)| \int_0^1 |DuD^5u|dx \le C\int_0^1 |DuD^5u|dx \\
&\le \varepsilon\int_0^1 |D^5u|^2dx + C_\varepsilon\int_0^1 |Du|^2dx \le \varepsilon\int_0^1 |D^5u|^2dx + C.
\end{aligned}
\tag{2.13}
$$

$$
\begin{aligned}
I_5 &\le \sup_{Q_T} |m(u)w_3''(u)|Du|^2| \int_0^1 |DuD^5u|dx \le C\int_0^1 |DuD^5u|dx \\
&\le \varepsilon\int_0^1 |D^5u|^2dx + C_\varepsilon\int_0^1 |Du|^2dx \le \varepsilon\int_0^1 |D^5u|^2dx + C.
\end{aligned}
\tag{2.14}
$$

$$
\begin{aligned}
I_6 &\le 2\sup_{Q_T} |m(u){w_3}'(u)Du| \int_0^1 |D^2uD^5u|dx \le C\int_0^1 |D^2uD^5u|dx \\
&\le \varepsilon\int_0^1 |D^5u|^2dx + C_\varepsilon\int_0^1 |D^2u|^2dx \le \varepsilon\int_0^1 |D^5u|^2dx + C.
\end{aligned}
\tag{2.15}
$$

$$
\begin{aligned}
I_7 &\le \sup_{Q_T} |m(u)w_3(u)| \int_0^1 |D^3uD^5u|dx \le C\int_0^1 |D^3uD^5u|dx \\
&\le \varepsilon\int_0^1 |D^5u|^2dx + C_\varepsilon\int_0^1 |D^3u|^2dx \le 2\varepsilon\int_0^1 |D^5u|^2dx + C.
\end{aligned}
\tag{2.16}
$$

Summing up, noticing that $m(s) \ge m_0 > 0$, we obtain

$$
\frac{d}{dt}\int_0^1 |D^2u|^2dx + (2m_0 - 22\varepsilon)\int_0^1 |D^5u|^2dx \le C,
$$

where ε is small enough, it satisfies $2m_0 - 10\varepsilon > 0$. Therefore,

$$
\int\int_{Q_T} |D^5u|^2dxdt \le C. \tag{2.17}
$$

Multiplying both sides of the equation (1.3) by D^6u, integrating the resulting relation with respect to x over $(0,1)$, after integrating by parts, and using the boundary value conditions, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 |D^3u|^2dx + \int_0^1 D(m(u)D^5u)D^6u\,dx + \int_0^1 D(m(u)D^3u)D^6u\,dx \\
&= \int_0^1 D[m(u)D(|Du|^2D^2u)]D^6u\,dx
\end{aligned}
$$

$$-\int_0^1 D[m(u)D(w_0(u)+w_2(u)|Du|^2+w_3(u)D^2u)]D^6u\,dx.$$

Simple calculations show that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1|D^3u|^2dx+\int_0^1 m(u)|D^6u|^2dx\\
&=-\int_0^1 m'(u)DuD^5uD^6u\,dx-\int_0^1 m(u)D^4uD^6u\,dx-\int_0^1 m'(u)DuD^3uD^6u\,dx\\
&\quad+\int_0^1 m(u)(|Du|^2D^4u\,dx+6DuD^2uD^3u+2|D^2u|^2D^2u)D^6u\,dx\\
&\quad+\int_0^1 m'(u)Du(|Du|^2D^3u+2Du|D^2u|^2)D^6u\,dx\\
&\quad-\int_0^1 m(u)(w_0'(u)D^2u+w_0''(u)|Du|^2)D^6u\,dx-\int_0^1 m'(u)w_0'(u)|Du|^2D^6u\,dx\\
&\quad-\int_0^1 m'(u)Du(w_2'(u)|Du|^2Du+2w_2(u)DuD^2u)D^6u\,dx\\
&\quad-\int_0^1 m(u)[w_2''(u)|Du|^4+5w_2'(u)|Du|^2D^2u+2w_2(u)|D^2u|^2\\
&\quad+2w_2(u)DuD^3u]D^6u\,dx\\
&\quad-\int_0^1 m(u)[w_3''(u)DuD^2u+2w_3'(u)D^3u+w_3'(u)D^4u]D^6u\,dx\\
&\quad-\int_0^1 m'(u)Du(w_3'(u)D^2u+w_3(u)D^3u)D^6u\,dx\\
&=: I_8+I_9+I_{10}+I_{11}+I_{12}+I_{13}+I_{14}+I_{15}+I_{16}+I_{17}+I_{18}.
\end{aligned}$$

By Nirenberg's inequality, we deduce that

$$\begin{aligned}
&\int_0^1|D^5u|^2dx\\
&\le\Big(C'\Big(\int_0^1|D^6u|^2dx\Big)^{\frac{3}{8}}\Big(\int_0^1|D^2u|^2dx\Big)^{\frac{1}{8}}+C''\Big(\int_0^1|D^2u|^2dx\Big)^{1/2}\Big)^2\\
&\le\varepsilon\int_0^1|D^6u|^2dx+C_\varepsilon.
\end{aligned}$$

$$\begin{aligned}
&\int_0^1|D^4u|^2dx\\
&\le\Big(C'\Big(\int_0^1|D^6u|^2dx\Big)^{\frac{1}{4}}\Big(\int_0^1|D^2u|^2dx\Big)^{\frac{1}{4}}+C''\Big(\int_0^1|D^2u|^2dx\Big)^{1/2}\Big)^2\\
&\le\varepsilon\int_0^1|D^6u|^2dx+C_\varepsilon.
\end{aligned}$$

$$\begin{aligned}
&\int_0^1|D^3u|^2dx\\
&\le\Big(C'\Big(\int_0^1|D^6u|^2dx\Big)^{1/8}\Big(\int_0^1|D^2u|^2dx\Big)^{3/8}+C''\Big(\int_0^1|D^2u|^2dx\Big)^{1/2}\Big)^2
\end{aligned}$$

$$
\le \varepsilon \int_0^1 |D^6u|^2dx + C_\varepsilon.
$$

$$
\begin{aligned}
&\int_0^1 |D^2u|^4dx \\
&\le \Big(C'\Big(\int_0^1 |D^6u|^2dx\Big)^{\frac{1}{32}}\Big(\int_0^1 |D^2u|^2dx\Big)^{\frac{15}{32}} + C''\Big(\int_0^1 |D^2u|^2dx\Big)^{1/2}\Big)^4 \\
&\le \varepsilon \int_0^1 |D^6u|^2dx + C_\varepsilon.
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^1 |D^2u|^6dx \\
&\le \Big(C'\Big(\int_0^1 |D^6u|^2dx\Big)^{\frac{1}{24}}\Big(\int_0^1 |D^2u|^2dx\Big)^{\frac{11}{24}} + C''\Big(\int_0^1 |D^2u|^2dx\Big)^{1/2}\Big)^4 \\
&\le \varepsilon \int_0^1 |D^6u|^2dx + C_\varepsilon.
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^1 |D^3u|^4dx \\
&\le \Big(C'\Big(\int_0^1 |D^6u|^2dx\Big)^{\frac{5}{32}}\Big(\int_0^1 |D^2u|^2dx\Big)^{\frac{11}{32}} + C''\Big(\int_0^1 |D^2u|^2dx\Big)^{1/2}\Big)^4 \\
&\le \varepsilon \int_0^1 |D^6u|^2dx + C_\varepsilon.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
I_8 &\le \sup_{Q_T} |m'(u)Du| \int_0^1 |D^5uD^6u|dx \le C\int_0^1 |D^5uD^6u|dx \\
&\le \varepsilon \int_0^1 |D^6u|^2dx + C_\varepsilon \int_0^1 |D^5u|^2dx \le 2\varepsilon \int_0^1 |D^6u|^2dx + C'_\varepsilon.
\end{aligned}
$$

$$
\begin{aligned}
I_9 &\le \sup_{Q_T} |m(u)| \int_0^1 |D^4uD^6u|dx \le C\int_0^1 |D^4uD^6u|dx \\
&\le \varepsilon \int_0^1 |D^6u|^2dx + C_\varepsilon \int_0^1 |D^4u|^2dx \le 2\varepsilon \int_0^1 |D^6u|^2dx + C'_\varepsilon.
\end{aligned}
$$

$$
\begin{aligned}
I_{10} &\le \sup_{Q_T} |m'(u)Du| \int_0^1 |D^3uD^6u|dx \le C\int_0^1 |D^3uD^6u|dx \\
&\le \varepsilon \int_0^1 |D^6u|^2dx + C_\varepsilon \int_0^1 |D^3u|^2dx \le 2\varepsilon \int_0^1 |D^6u|^2dx + C'_\varepsilon.
\end{aligned}
$$

$$
\begin{aligned}
I_{11} &\le \sup_{Q_T} |m(u)(Du)^2| \int_0^1 |D^4uD^6u|dx + 6\sup_{Q_T} |m(u)Du| \int_0^1 |D^2uD^3uD^6u|dx \\
&\quad + 2\sup_{Q_T} |m(u)| \int_0^1 |D^2u|^3 D^6u\, dx
\end{aligned}
$$

$$
\begin{aligned}
&\le C\int_0^1 |D^3uD^6u|dx + C\int_0^1 |D^2uD^3uD^6u|dx + C\int_0^1 |D^2u|^3 D^6u\,dx \\
&\le \varepsilon\int_0^1 |D^6u|^2dx + C_\varepsilon(\int_0^1 |D^3u|^2dx + \int_0^1 |D^2u|^4dx + \int_0^1 |D^3u|^4dx \\
&\quad + \int_0^1 |D^2u|^6dx \\
&\le 2\varepsilon\int_0^1 |D^6u|^2dx + C.
\end{aligned}
$$

$$
\begin{aligned}
I_{12} &\le \sup_{Q_T}|m'(u)(Du)^3|\int_0^1 |D^3uD^6u|dx + 2\sup_{Q_T}|m'(u)(Du)^2|\int_0^1 |(D^2u)^2D^6u|dx \\
&\le C\int_0^1 |D^3uD^6u|dx + C\int_0^1 |(D^2u)^2D^6u|dx \\
&\le \varepsilon\int_0^1 |D^6u|^2dx + C_\varepsilon\Big(\int_0^1 |D^3u|^2dx + \int_0^1 |D^2u|^4dx\Big) \\
&\le 2\varepsilon\int_0^1 |D^6u|^2dx + C.
\end{aligned}
$$

$$
\begin{aligned}
I_{13} &\le \sup_{Q_T}|m(u)w_0'(u)|\int_0^1 |D^2uD^6u|dx + \sup_{Q_T}|m(u)w_0''(u)Du|\int_0^1 |DuD^6u|dx \\
&\le C\int_0^1 |D^2uD^6u|dx + C\int_0^1 |DuD^6u|dx \\
&\le \varepsilon\int_0^1 |D^6u|^2dx + C_\varepsilon\Big(\int_0^1 |D^2u|^2dx + \int_0^1 |Du|^2dx\Big) \\
&\le \varepsilon\int_0^1 |D^6u|^2dx + C.
\end{aligned}
$$

$$
I_{14} \le \sup_{Q_T}|m'(u)w_0'(u)Du|\int_0^1 |DuD^6u|dx \le C\int_0^1 DuD^6u\,dx \le \varepsilon\int_0^1 |D^6u|^2dx + C.
$$

$$
\begin{aligned}
I_{15} &\le \sup_{Q_T}|m'(u)w_2'(u)(Du)^3|\int_0^1 |DuD^6u|dx \\
&\quad + 2\sup_{Q_T}|m'(u)w_2(u)(Du)^2|\int_0^1 |D^2uD^6u|dx \\
&\le C\int_0^1 |DuD^6u|dx + C\int_0^1 |D^2uD^6u|dx \\
&\le \varepsilon\int_0^1 |D^6u|^2dx + C_\varepsilon(\int_0^1 |Du|^2dx + \int_0^1 |D^2u|^2dx) \\
&\le \varepsilon\int_0^1 |D^6u|^2dx + C.
\end{aligned}
$$

$$
I_{16} \le \sup_{Q_T}|m(u)w_2''(u)(Du)^3|\int_0^1 |DuD^6u|dx
$$

$$
\begin{aligned}
&+5\sup_{Q_T}|m(u)w_2'(u)(Du)^2|\int_0^1|D^2uD^6u|dx\\
&+2\sup_{Q_T}|m(u)w_2(u)|\int_0^1|(D^2u)^2D^6u|dx\\
&+2\sup_{Q_T}|m(u)w_2(u)Du|\int_0^1|D^3uD^6u|dx\\
&\le C\int_0^1|DuD^6u|dx+C\int_0^1|D^2uD^6u|dx+C\int_0^1|(D^2u)^2D^6u|dx\\
&+C\int_0^1|D^3uD^6u|dx\\
&\le \varepsilon\int_0^1|D^6u|^2dx+C_\varepsilon\Big(\int_0^1|Du|^2dx+\int_0^1|D^2u|^2dx+\int_0^1|D^2u|^4dx\\
&+\int_0^1|D^3u|^2dx\Big)\\
&\le 2\varepsilon\int_0^1|D^6u|^2dx+C.
\end{aligned}
$$

$$
\begin{aligned}
I_{17}&\le \sup_{Q_T}|m(u)w_3''(u)Du|\int_0^1|D^2uD^6u|dx+2\sup_{Q_T}|m(u)w_3'(u)|\int_0^1|D^3uD^6u|dx\\
&+\sup_{Q_T}|m(u)w_3(u)|\int_0^1|D^4uD^6u|dx\\
&\le C\int_0^1|D^2uD^6u|dx+C\int_0^1|D^3uD^6u|dx+C\int_0^1|D^4uD^6u|dx\\
&\le \varepsilon\int_0^1|D^6u|^2dx+C_\varepsilon(\int_0^1|D^2u|^2dx+\int_0^1|D^3u|^2dx+\int_0^1|D^4u|^2dx)\\
&\le 2\varepsilon\int_0^1|D^6u|^2dx+C.
\end{aligned}
$$

$$
\begin{aligned}
I_{18}&\le \sup_{Q_T}|m'(u)w_3'(u)Du|\int_0^1|D^2uD^6u|dx+\sup_{Q_T}|m'(u)w_3(u)Du|\int_0^1|D^3uD^6u|dx\\
&\le C\int_0^1|D^2uD^6u|dx+C\int_0^1|D^3uD^6u|dx\\
&\le \varepsilon\int_0^1|D^6u|^2dx+C_\varepsilon\Big(\int_0^1|D^2u|^2dx+\int_0^1|D^3u|^2dx\Big)\\
&\le 2\varepsilon\int_0^1|D^6u|^2dx+C.
\end{aligned}
$$

Summing up, noticing that $m(s)\ge m_0>0$, we obtain

$$
\frac{d}{dt}\int_0^1|D^3u|^2dx+(2m_0-38\varepsilon)\int_0^1|D^6u|^2dx\le C,
$$

where ε is small enough, it satisfies $2m_0 - 38\varepsilon > 0$. Hence

$$\sup_{0<t<T} \int_0^1 |D^3 u|^2 dx \le C. \tag{2.18}$$

Combing (2.7), (2.5), (2.4) and (2.18) together, using Sobolev's embedding theorem, we derive that

$$\sup_{Q_T} |D^2 u(x,t)| \le C\Big(\int_0^1 [u^2 dx + |Du|^2 |D^2 u|^2 |D^3 u|^2] dx\Big)^{1/2} \le C. \tag{2.19}$$

By (2.5) and (2.6), we deduce that

$$|u(x_1,t) - u(x_2,t)| \le C|x_1 - x_2|^{\alpha}, \quad 0 \le \alpha < 1. \tag{2.20}$$

Integrating the equation (1.3) with respect to x over $(y, y+(\Delta t)^{1/6}) \times (t_1,t_2)$, where $0 < t_1 < t_2 < T$, $\Delta t = t_2 - t_1$, we deduce that

$$\begin{aligned}
&\int_y^{y+(\Delta t)^{1/6}} [u(z,t_2) - u(z,t_1)] dz \\
&= \int_{t_1}^{t_2} \Big[m(u(y',s)) \Big(D^5 u(y',s) + D^3 u(y',s) - D(|Du(y',s)|^2 D^2 u(y',s)) \\
&\quad + D(w_0(u(y',s)) + w_2(u(y',s))|Du(y',s)|^2 + w_3(u(y',s)) D^2 u(y',s)) \Big) \\
&\quad - m(u(y,s)) \Big(D^5 u(y,s) + D^3 u(y,s) - D(|Du(y,s)|^2 D^2 u(y,s)) \\
&\quad + D(w_0(u(y,s)) + w_2(u(y,s))|Du(y,s)|^2 + w_3(u(y,s)) D^2 u(y,s)) \Big) \Big] ds.
\end{aligned}$$

Set

$$\begin{aligned}
N(s,y) &= m(u(y',s)) \Big(D^5 u(y',s) + D^3 u(y',s) - D(|Du(y',s)|^2 D^2 u(y',s)) \\
&\quad + D(w_0(u(y',s)) + w_2(u(y',s))|Du(y',s)|^2 + w_3(u(y',s)) D^2 u(y',s)) \Big) \\
&\quad - m(u(y,s)) \Big(D^5 u(y,s) + D^3 u(y,s) - D(|Du(y,s)|^2 D^2 u(y,s)) \\
&\quad + D(w_0(u(y,s)) + w_2(u(y,s))|Du(y,s)|^2 + w_3(u(y,s)) D^2 u(y,s)) \Big),
\end{aligned}$$

where $y' = y + (\Delta t)^{1/6}$. Then, the above equality is converted into

$$(\Delta t)^{1/6} \int_0^1 [u(y + \theta(\Delta t)^{1/6}, t_2) - u(y + \theta(\Delta t)^{1/6}, t_1)] d\theta = \int_{t_1}^{t_2} N(s,y) ds.$$

Integrating above equality with respect to y over $(x, x + (\Delta t)^{1/6})$, we immediately obtain

$$(\Delta t)^{1/3} (u(x^*, t_2) - u(x^*, t_1)) = \int_{t_1}^{t_2} \int_x^{x+(\Delta t)^{1/6}} N(s,y) dy ds.$$

Here, we have used the mean value theorem, where $x^* = y^* + \theta^* (\Delta t)^{1/6}$, $y^* \in (x, x + (\Delta t)^{1/6})$, $\theta \in (0,1)$. Then, by Hölder's inequality and (2.4), (2.6), (2.17), we obtain

$$|u(x^*, t_2) - u(x^*, t_1)| \le C(\Delta t)^{\frac{\alpha}{6}}, \ 0 < \alpha < 1.$$

Similar to the above discussion, we have

$$|Du(x_1,t_1) - Du(x_2,t_2)| \le C(|x_1 - x_2|^{1/2} + |t_1 - t_2|^{\frac{1}{12}}). \tag{2.21}$$

The proof is complete. □

To prove Theorem 1.1, the key estimate is the Hölder estimate for D^2u. Now, we give the following lemma which can be seen in [9].

Lemma 2.2. *Assume that* $\sup|f| < +\infty$, $a(x,t) \in C^{\alpha,\frac{\alpha}{6}}(\bar{Q}_T)$, $0 < \alpha < 1$, *and there exist two constants* a_0, b_0, A_0, B_0 *such that* $0 < a_0 \le a(x,t) \le A_0$, $0 < b_0 \le b(x,t) \le B_0$ *for all* $(x,t) \in Q_T$. *If* u *is a smooth solution for the linear problem*

$$\begin{gathered}\frac{\partial u}{\partial t} - D^3(a(x,t)D^3u) + D^3(b(x,t)Du) = D^3f, \quad (x,t) \in Q_T,\\ Du(x,t)|_{x=0,1} = D^3u(x,t)|_{x=0,1} = D^5u(x,t)|_{x=0,1} = 0, \quad t \in [0,T],\\ u(x,0) = u_0(x), \quad x \in [0,1],\end{gathered}$$

then, for any $\delta \in (0,\frac{1}{2})$, *there is a constant* K *depending on* a_0, b_0, A_0, B_0, δ, T, $\iint_{Q_T} u^2dxdt$ *and* $\iint_{Q_T} |D^3u|^2dxdt$, *such that*

$$|u(x_1,t_1) - u(x_2,t_2)| \le K(1+\sup|f|)(|x_1-x_2|^\delta + |t_1-t_2|^{\frac{\delta}{6}}).$$

Now, we prove the main result.

Proof of Theorem 1.1. Suppose that $w = D^2u - D^2u_0$. Then w satisfies the problem

$$\begin{aligned}\frac{\partial w}{\partial t} - D^3(a(x,t)D^3w) + D^3(b(x,t)Dw) &= D^3f,\\ w(x,t) = D^2w(x,t) = D^4w(x,t) &= 0, \quad x = 0,1,\\ w(x,0) &= 0, \quad x \in [0,1],\end{aligned}$$

where $a(x,t) = m(u)$, $b(x,t) = m(u)$ and

$$f(x,t) = m(u)[-D^5u_0 - D^3u_0 + D(|Du|^2D^2u) - D(w_0(u) + w_2(u)|Du|^2 + w_3(u)D^2u)].$$

It then follows from (2.4)-(2.19) and Lemma 2.2 that

$$|D^2u(x_1,t_1) - D^2u(x_2,t_2)| \le C(|x_1-x_2|^{\alpha/2} + |t_1-t_2|^{\alpha/12}).$$

The conclusion follows immediately from the classical theory, since we can transform the equation (1.3) into the form

$$\begin{aligned}&\frac{\partial u}{\partial t} + a_1(x,t)D^6u + a_2(x,t)D^5u + a_3(x,t)D^4u(x,t)\\ &+ a_4(x,t)D^3u(x,t) + a_5(x,t)D^2u(x,t) + a_6(x,t)Du(x,t) = 0.\end{aligned}$$

with the Hölder norms on

$$\begin{aligned}a_1(x,t) &= -m(u(x,t)), \quad a_2(x,t) = -m'(u(x,t))Du(x,t),\\ a_3(x,t) &= m(u(x,t))(|Du(x,t)|^2 + w_3(u(x,t)) - 1),\\ a_4(x,t) &= m'(u(x,t))[|Du(x,t)|^2Du(x,t) - Du(x,t)]\\ &\quad + m(u(x,t))[6Du(x,t)D^2u(x,t) + 2w_2(u(x,t))Du(x,t)\\ &\quad + 2w_3'(u)Du(x,t)],\\ a_5(x,t) &= m(u(x,t))[2|D^2u(x,t)|^2 + w_0'(u(x,t)) + 5w_2'(u(x,t))|Du(x,t)|^2\\ &\quad + 2w_2(u(x,t))D^2u(x,t) + w_3''(u(x,t))|Du(x,t)|^2 + w_3'(u(x,t))D^2u(x,t)]\\ &\quad + m'(u(x,t))[2|Du(x,t)|^2D^2u(x,t) + 2w_2(u(x,t))|Du(x,t)|^2\\ &\quad + w_3'(u(x,t))|Du(x,t)|^2 + w_3(u(x,t))Du(x,t)]\end{aligned}$$

$$
\begin{aligned}
a_6(x,t) = m'(u(x,t))[w_0'(u(x,t))Du(x,t) + w_2'(u(x,t))|Du(x,t)|^2 Du(x,t)] \\
+ m(u(x,t))[w_0'(u(x,t))Du(x,t) + w_2''(u(x,t))|Du(x,t)|^2 Du(x,t)].
\end{aligned}
$$

have been estimated in the above discussion. Then, the proof is complete □

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