*Electronic Journal of Differential Equations*, Vol. 2014 (2014), No. 224, pp. 1–4.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

# A REMARK ON THE RADIAL MINIMIZER OF THE GINZBURG-LANDAU FUNCTIONAL

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Abstract. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with the same area as the unit disk $B_1$ and let

$$E_\varepsilon(u,\Omega) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{4\varepsilon^2}\int_\Omega (|u|^2-1)^2\,dx$$

be the Ginzburg-Landau functional. Denote by $\tilde{u}_\varepsilon$ the radial solution to the Euler equation associated to the problem $\min\{E_\varepsilon(u,B_1) : u\big|_{\partial B_1} = x\}$ and by

$$\mathcal{K} = \Big\{ v = (v_1,v_2) \in H^1(\Omega;\mathbb{R}^2) : \int_\Omega v_1\,dx = \int_\Omega v_2\,dx = 0, \\ \int_\Omega |v|^2\,dx \geq \int_{B_1} |\tilde{u}_\varepsilon|^2\,dx \Big\}.$$

In this note we prove that

$$\min_{v\in\mathcal{K}} E_\varepsilon(v,\Omega) \leq E_\varepsilon(\tilde{u}_\varepsilon, B_1).$$

## 1. Introduction

The Ginzburg-Landau energy has as order parameter a vectorial field $u \in H^1(\Omega;\mathbb{R}^2)$ and it is defined as

$$E_\varepsilon(u,\Omega) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{4\varepsilon^2}\int_\Omega \left(|u|^2-1\right)^2\,dx,$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain and $\varepsilon > 0$. This kind of functionals has been originally introduced as a phenomenological phase-field type free-energy of a superconductor, near the superconducting transition, in absence of an external magnetic field. Moreover these functionals have been used in superfluids such as Helium II. In this context $u$ represents the wave function of the superflluid part of liquid and the parameter $\varepsilon$, which has the dimension of a length, depends on the material and its temperature (see [10, 9, 7]). The Ginzburg-Landau functionals have deserved a great attention by the mathematical community too. Starting from the classical monograph [5] (see also [4]) by Bethuel, Brezis and Hélein, many mathematicians have been interested in studying minimization problems for the Ginzburg-Landau energy with several constraints, also because, besides the physical

2000 *Mathematics Subject Classification.* 35Q56, 35J15.
*Key words and phrases.* Ginzburg-Landau functional, Szegö-Weinberger inequality.

Submitted September 15, 2014. Published October 21, 2014.

motivation, these problems appear as the simplest nontrivial examples of vector field minimization problems.

In [5] the authors consider Dirichlet boundary conditions $g \in C^1(\partial\Omega; \mathbb{S}^1)$ (with $\Omega$ smooth) and study the asymptotic behavior, as $\varepsilon \to 0$, of minimizers $u_\varepsilon$, which satisfy the problem

$$\begin{gathered} -\Delta u_\varepsilon = \frac{1}{\varepsilon^2} u_\varepsilon (1 - |u_\varepsilon|^2) \quad \text{in } \Omega \\ u_\varepsilon = g \quad \text{on } \partial\Omega. \end{gathered} \tag{1.1}$$

It turns out that the value $d = \deg(g, \partial\Omega)$ (i.e., the Brouwer degree or winding number of $g$ considered as a map from $\partial\Omega$ into $\mathbb{S}^1$) plays a crucial role in the asymptotic analysis of $u_\varepsilon$.

In the case $\Omega = B_1$ (the unit ball in $\mathbb{R}^2$ centered at the origin), $g(x) = x$, it is natural to look for radial solutions to (1.1). Indeed, in [12, 5, 15] the authors prove, among other things, that (1.1) has a unique radial solution, that is a solution of the form

$$\tilde{u}_\varepsilon(x) = \tilde{f}_\varepsilon(|x|) \frac{x}{|x|} \tag{1.2}$$

with $\tilde{f}_\varepsilon \geq 0$. Moreover $\tilde{f}'_\varepsilon > 0$; thus, summarizing, $\tilde{f}_\varepsilon$ satisfies

$$\begin{gathered} -\tilde{f}''_\varepsilon - \frac{\tilde{f}'_\varepsilon}{r} + \frac{\tilde{f}_\varepsilon}{r^2} = \frac{1}{\varepsilon^2} \tilde{f}_\varepsilon (1 - \tilde{f}_\varepsilon^2) \quad \text{in } [0,1] \\ \tilde{f}_\varepsilon(0) = 0, \quad \tilde{f}_\varepsilon(1) = 1, \quad \tilde{f}_\varepsilon \geq 0, \quad \tilde{f}'_\varepsilon > 0. \end{gathered} \tag{1.3}$$

It is conjectured that the radial solution (1.2) is the unique minimizer of $E_\varepsilon$ on $B_1$. In [17] (see also [16]) the author gives a partial answer to such a conjecture, proving that $\tilde{u}_\varepsilon$ is stable, in the sense that the quadratic form associated to $E_\varepsilon(\tilde{u}_\varepsilon, B_1)$ is positive definite.

Other types of boundary conditions, for instance prescribed degree boundary conditions, have been considered in [3, 8].

In this article we let $\Omega$ vary among domains with fixed area and prove that the map $\tilde{u}_\varepsilon$ in (1.2) provides an upper bound for the energy $E_\varepsilon$ on the class $\mathcal{K}$ we are going to introduce.

**Theorem 1.1.** *Let $\varepsilon > 0$ and $\Omega \subset \mathbb{R}^2$ be a bounded domain such that $|\Omega| = |B_1|$. Denoted by*

$$\begin{aligned} \mathcal{K} = \Big\{ v = (v_1, v_2) \in H^1(\Omega; \mathbb{R}^2) : \int_\Omega v_1 \, dx = \int_\Omega v_2 \, dx = 0, \\ \int_\Omega |v|^2 \, dx \geq \int_{B_1} |\tilde{u}_\varepsilon|^2 \, dx \Big\}, \end{aligned}$$

*it holds*

$$\min_{v \in \mathcal{K}} E_\varepsilon(v, \Omega) \leq E_\varepsilon(\tilde{u}_\varepsilon, B_1). \tag{1.4}$$

## 2. Proof of Theorem 1.1

Define the following continuous extension of $\tilde{f}_\varepsilon$,

$$f_\varepsilon(r) = \begin{cases} \tilde{f}_\varepsilon(r) & \text{if } 0 \leq r \leq 1 \\ 1 & \text{if } r > 1 \end{cases}$$

and the correspondent vector field extending $\tilde{u}_\varepsilon$ to the whole $\mathbb{R}^2$

$$\phi_\varepsilon(x) = \big(\phi_{\varepsilon,1}(x), \phi_{\varepsilon,2}(x)\big) = f_\varepsilon(|x|)\frac{x}{|x|}.$$

It is possible (see [19], see also [1]) to choose the origin in such a way that

$$\int_\Omega \phi_{\varepsilon,1}\,dx = \int_\Omega \phi_{\varepsilon,2}\,dx = 0. \tag{2.1}$$

Note that $\phi_\varepsilon \in \mathcal{K}$. Indeed, besides (2.1), it holds

$$\int_\Omega |\phi_\varepsilon|^2\,dx = \int_{\Omega\cap B_1} |\tilde{u}_\varepsilon|^2\,dx + |\Omega \setminus B_1| \geq \int_{B_1} |\tilde{u}_\varepsilon|^2\,dx,$$

since $|\tilde{u}_\varepsilon| \leq 1$ in $B_1$. A direct computation yields

$$\begin{aligned} E_\varepsilon(\phi_\varepsilon, \Omega) &= \frac{1}{2}\int_\Omega \Big(f_\varepsilon'(|x|)^2 + \frac{f_\varepsilon(|x|)^2}{|x|^2}\Big)\,dx + \frac{1}{4\varepsilon^2}\int_\Omega \big(f_\varepsilon(|x|)^2 - 1\big)^2\,dx \\ &= \int_\Omega B_\varepsilon(|x|)\,dx, \end{aligned}$$

where

$$B_\varepsilon(r) = \frac{1}{2}\Big(f_\varepsilon'(r)^2 + \frac{f_\varepsilon(r)^2}{r^2}\Big) + \frac{1}{4\varepsilon^2}\big(f_\varepsilon(r)^2 - 1\big)^2.$$

Using (1.3) it is straightforward to verify that

$$B_\varepsilon'(r) = -\frac{2}{\varepsilon^2} f_\varepsilon(r) f_\varepsilon'(r)\left(1 - f_\varepsilon(r)^2\right) - \frac{1}{r}\big(f_\varepsilon'(r) - \frac{f_\varepsilon(r)}{r}\big)^2, \quad 0 < r < 1,$$

while, when $r > 1$, it holds $B_\varepsilon(r) = \frac{1}{2r^2}$. Thus $B_\varepsilon(r)$ is a decreasing function in $(0, +\infty)$. By Hardy-Littlewood inequality (see for instance [13]) we finally obtain

$$E_\varepsilon(\phi_\varepsilon, \Omega) = \int_\Omega B_\varepsilon(|x|)\,dx \leq \int_{B_1} B_\varepsilon(|x|)\,dx = E_\varepsilon(\tilde{u}_\varepsilon, B_1)$$

and hence (1.4).

**Remark 2.1.** The appearance of the function $\tilde{u}_\varepsilon$ (i.e., the candidate to be the unique minimizer of $E_\varepsilon$ in $B_1$ under the Dirichlet boundary condition $g(x) = x$) in (1.4) as an upper bound of the energy in the class $\mathcal{K}$ is somehow unexpected. On the other hand such a phenomenon becomes more transparent if one realizes the analogy between the problem under consideration and the maximization problem of the first nontrivial eigenvalue $\mu_1(\Omega)$ of the Neumann Laplacian among sets with prescribed area. As well-known, if $\Omega$ is a smooth, bounded domain of $\mathbb{R}^2$, $\mu_1(\Omega)$ can be variationally characterized as

$$\mu_1(\Omega) = \Big\{\int_\Omega |\nabla z|^2 : z \in H^1(\Omega;\mathbb{R}),\ \int_\Omega |z|^2\,dx = 1,\ \int_\Omega z\,dx = 0\Big\}.$$

If $|\Omega| = |B_1|$ the celebrated Szegö-Weinberger inequality in the plane (see [19], see also [18, 2, 1, 14, 11, 6]) states

$$\mu_1(\Omega) \leq \mu_1(B_1). \tag{2.2}$$

Moreover, $\mu_1(B_1)$ is achieved by the functions $J_1(j_{1,1}'|x|)\frac{x_1}{|x|}$ or $J_1(j_{1,1}'|x|)\frac{x_2}{|x|}$, where $J_1$ is the Bessel function of the first kind and $j_{1,1}'$ is the first zero of its derivative. The role played by $J_1$ in (2.2) is now played by the function $\tilde{f}_\varepsilon$.

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