Electronic Journal of Differential Equations, Vol. 2014 (2014), No. 228, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

SOLUTIONS TO QUASILINEAR EQUATIONS OF N-BIHARMONIC TYPE WITH DEGENERATE COERCIVITY

SAMI AOUAOUI

Abstract. In this article we show the existence of multiple solutions for quasilinear equations in divergence form with degenerate coercivity. Our strategy is to combine a variational method and an iterative technique to obtain the solutions.

1. Introduction and statement of main results

In this article, we study the quasilinear equation

$$-\operatorname{div}\big(a(x,u)|\nabla u|^{N-2}\nabla u\big)+V(x)|u|^{N-2}u+\Delta_N^2 u = f(x,u)+h(x), \quad x\in\mathbb{R}^N, \tag{1.1}$$

where $N\geq 2$, $\Delta_N^2 u=\Delta(|\Delta u|^{N-2}\Delta u)$, and $h\in L^{N'}(\mathbb{R}^N)$, $N'=\frac{N}{N-1}$, $h\neq 0$ and $h\geq 0$. Concerning the functions V, f and a, we have the following assumptions:

(V1) $V:\mathbb{R}^N\to\mathbb{R}$ is a continuous function such that

$$V(x)\geq V_0>0, \quad \forall x\in\mathbb{R}^N,$$

where V_0 is a positive constant.

(V2) For every $M>0$, $\operatorname{meas}(\{x\in\mathbb{R}^N,\ V(x)\leq M\})<+\infty$, where "meas" denotes the Lebesgue measure in $\mathbb{R}^N$.

(H1) $f:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function. We assume that for every positive real number $k>0$, there exist two positive constants $\alpha_k>N-1$ and $C_k>0$ such that

$$|f(x,s)|\leq C_k|s|^{\alpha_k}, \quad \text{a.e. } x\in\mathbb{R}^N \text{ and for all } s\in\mathbb{R} \text{ with } |s|\leq k.$$

(H2) There exists $\nu>N$ such that

$$\nu F(x,s)\leq f(x,s)s, \quad \forall (x,s)\in\mathbb{R}^N\times\mathbb{R}, \text{ where } F(x,s)=\int_0^s f(x,t)dt.$$

(H3) There exist two real numbers $A>0$ and $p>N$, such that

$$F(x,s)\geq As^p, \quad \text{a.e. } x\in\mathbb{R}^N,\ \forall s\geq 0.$$

2000 *Mathematics Subject Classification.* 35D30, 35J20, 35J61, 58E05.
Key words and phrases. Variational method; N-biharmonic operator; mountain pass theorem; minimax level; iterative scheme; critical point.

Submitted June 22, 2014. Published October 27, 2014.

(H4) There exist two positive constants β_0 and β_1 such that
$$|f(x,s_1)-f(x,s_2)| \le \beta_0 z^{\beta_1}|s_1-s_2|^{N-1},$$
a.e $x\in\mathbb{R}^N$, $\forall z\in[0,1]$ and $\forall s_1,s_2\in[-z,z]$.

(H5) $a:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ is a continuous function satisfying the following property: for every $k>0$, there exist $0<a_k<a'_k<+\infty$ such that
$$a_k \le a(x,s)\le a'_k, \quad \forall x\in\mathbb{R}^N \text{ and } |s|\le k.$$

(H6) There exists a constant $L>0$ such that
$$|a(x,s_1)-a(x,s_2)|\le L|s_1-s_2|^{N-1}, \quad \text{a.e. } x\in\mathbb{R}^N,\ \forall s_1,s_2\in[-1,1].$$

Examples. When $N=2$, for f, we can choose:

1. $f(x,s)=\lambda|s|^{\alpha-1}s$, $\lambda>0$, $\alpha>1$.
2. $f(x,s)=\lambda|s|^{\alpha-1}s+|s|^{\beta-1}s(e^{p_0s^2}-1)$, $\lambda>0$, $\alpha>1$, $\beta>1$ and $p_0>0$.

For a, we can choose:

1. $a(x,s)=1+|s|^{\sigma-1}s,\ \sigma>1$.
2. $a(x,s)=\frac{1}{1+s^2}$.

Many articles about problems similar to (1.1), having a divergence part of the form $-\operatorname{div}(A(x,u)|\nabla u|^{p-2}\nabla u)$ with degenerate coercivity, have been published. Among them, the following model is of special interest:
$$-\operatorname{div}\Big(\frac{|\nabla u|^{p-2}\nabla u}{(1+|u|)^q}\Big)=f \text{ in } \Omega,$$
where Ω is some open (bounded in the majority of cases) domain of $\mathbb{R}^N$, $N\ge 2$, $q>0$, $p>1$ and f is datum satisfying some summability condition. See for example [4, 7, 8, 9, 10, 11] and references therein. We want to mention also the model
$$-\operatorname{div}(A(x,u)|\nabla u|^{p(x)-2}\nabla u)+|u|^{p(x)-2}u=Z(x,u,\nabla u), \quad \text{in } \mathbb{R}^N,\ N\ge 3,$$
where p is some bounded and Lipschitz continuous function. This model was studied in [6] in the very special framework of the generalized Sobolev space with variable exponents. In the previously cited works, the authors use approximations in order to overcome the lack of coercivity. Then, establish a priori estimates on the sequence of approximative solutions, and then use the passage to the limit to finally obtain a weak solution for the initial equation.

In this article, we develop a new method to deal with such kind of problems. The main idea in this new method is inspired by the work [14]. In [14], de Figueiredo, Girardi and Matzeu considered the semilinear elliptic equation
$$\begin{gathered} -\Delta u = f(x,u,\nabla u) \quad \text{in } \Omega,\\ u=0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$
where Ω is a bounded smooth domain of $\mathbb{R}^N$, $N\ge 3$. Because the dependence of the nonlinearity on the gradient of the solution, (1.2) is non-variational and a direct attack to it using critical point theory is not possible. The new approach by de Figueiredo, Girardi and Matzeu consists of associating with (1.2) a family of semilinear elliptic problems with no dependence on the gradient. Namely, for each $w\in H_0^1(\Omega)$, they considered the problem
$$\begin{gathered} -\Delta u = f(x,u,\nabla w) \quad \text{in } \Omega,\\ u=0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.3}$$

Thus, the authors have "frozen" the gradient term. Problem (1.3) is of variational nature and could be treated by variational method. We have to mention that this idea was used in some later works dealing also with nonlinear problems involving nonlinearities with dependence on the gradient. We can cite [15, 17, 18, 26, 27, 29].

In this article, we try to use this idea to discuss a completely different kind of problem. In fact, in the problem (1.1) and in contrast with (1.2) (and similar equations), the nonvariational nature is not due to the dependence of the right-hand term on the gradient of the solution but it is in reality due to the presence of the coefficient $a(x, u)$ in the divergence part. Hence, we will try to "freeze" the term $a(x, u)$. The "associated" problem will be variational and consequently could be treated using the critical point theory. An iterative scheme will be performed in order to obtain weak solutions for the initial problem (1.1). This method allows us to obtain a multiplicity result which, knowing that in the majority of cases the classical nonvariational methods give the existence of one solution, could be seen as an interesting result.

The existence of the $N-$biharmonic operator, Δ_N^2, is remarkable. The importance of studying fourth-order equations lies in the fact that they can describe some physical phenomena as the deformations of an elastic beam in equilibrium state (see [24, 36]). Laser and McKenna [23] pointed out that this type of nonlinearity provides a model to study travelling waves in suspension bridges. For this reason, there is a wide literature that deals with existence and multiplicity of solutions for nonlinear fourth-order elliptic problems in bounded and unbounded domains. See for example [19, 20, 25, 28] and references therein. On the other hand, the study of nonlinear equations involving the N-Laplacian operator, $N \geq 2$, which is a borderline case for the Sobolev embedding, could be considered as one of the most interesting topics of research during last decades. A special interest has been given to equation of N-Laplacian type containing nonlinear terms which have a subcritical or critical exponential growth. See [1, 2, 3, 12, 16, 21, 22, 30, 31, 32, 33, 34, 35] and references therein. Here, we highlight the fact that in the present work we deal with a more general type of nonlinearity which includes the case of exponential growth.

The appropriate space in which the problem (1.1) will be studied is the subspace of $W^{2,N}(\mathbb{R}^N)$,

$$E = \Big\{u \in W^{2,N}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^N \,dx < +\infty\Big\},$$

which is a Banach reflexive space equipped with the norm

$$\|u\| = \Big(\int_{\mathbb{R}^N} (|\nabla u|^N + V(x)|u|^N + |\Delta u|^N)\,dx\Big)^{1/N}.$$

In view of (V1), we clearly have

$$E \hookrightarrow W^{2,N}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N), \quad \forall N \leq q \leq +\infty.$$

Also there exists a positive constant $\delta_0 > 0$ such that

$$|u|_{L^\infty(\mathbb{R}^N)} \leq \delta_0 \|u\|, \quad \forall u \in E. \tag{1.4}$$

Furthermore, since (V2) holds, we obtain (see [31]) the compactness of the embedding

$$E \hookrightarrow L^p(\mathbb{R}^N), \quad \text{for all } p \geq N.$$

This compact embedding will be crucial in the proof of our multiplicity result.

Definition 1.1. A function $u \in E$ is said to be a weak solution of the problem (1.1) if it satisfies

$$\int_{\mathbb{R}^N} a(x,u)|\nabla u|^{N-2}\nabla u \nabla v\,dx + \int_{\mathbb{R}^N} V(x)|u|^{N-2}uv\,dx + \int_{\mathbb{R}^N} |\Delta u|^{N-2}\Delta u \Delta v\,dx$$
$$= \int_{\mathbb{R}^N} f(x,u)v\,dx + \int_{\mathbb{R}^N} hv\,dx, \quad \forall v \in E.$$

The main result in the present paper is given by the following theorem.

Theorem 1.2. *Assume that* (V1), (V2), (H1)–(H6) *hold. Then, there exist* $A_0 > 0$ *and* $d_0 > 0$ *with the following property: if* $A > A_0$, *and* $|h|_{L^{N'}(\mathbb{R}^N)} < d_0$, *then problem* (1.1) *admits at least two nontrivial weak solutions.*

2. Proof of main resutls

The proof of Theorem 1.2 will be divided into several steps. First, for $w \in E$, we introduce the functional I_w defined on E by

$$I_w(u) = \int_{\mathbb{R}^N} \frac{a(x,w)|\nabla u|^N + V(x)|u|^N + |\Delta u|^N}{N}\,dx - \int_{\mathbb{R}^N} F(x,u)\,dx - \int_{\mathbb{R}^N} hu\,dx.$$

Lemma 2.1. *Assume that* (V1), (V2), (H1), (H5) *hold. Then, there exist* $0 < \rho < \frac{1}{\delta_0}$, $\mu > 0$, *and* $d > 0$ *independent of* w *such that*

$$I_w(u) \geq \mu, \quad \textit{for } \|u\| = \rho,$$

provided that $\|w\| \leq \frac{1}{\delta_0}$ *and* $|h|_{L^{N'}(\mathbb{R}^N)} < d$.

Proof. For $\|w\| \leq \frac{1}{\delta_0}$, by (1.4) it yields $|w|_{L^\infty(\mathbb{R}^N)} \leq 1$ and by (H5) we can assert that there exist $0 < a_1 < a_1' < +\infty$ such that

$$a_1 \leq a(x, w(x)) \leq a_1', \quad \forall x \in \mathbb{R}^N. \tag{2.1}$$

For $\|u\| \leq 1/\delta_0$, then by (1.4) it yields

$$|u(x)| \leq 1, \text{ a.e. } x \in \mathbb{R}^N.$$

By (H1), we get the existence of two constants $\alpha > N-1$ and $c_1 > 0$ such that

$$|f(x,u(x))| \leq c_1|u(x)|^\alpha, \text{ a.e } x \in \mathbb{R}^N. \tag{2.2}$$

This implies

$$\int_{\mathbb{R}^N} F(x,u)\,dx \leq c_2\|u\|^{\alpha+1}.$$

This inequality and (2.1) give

$$I_w(u) \geq \min\{1,\ a_1\}\frac{\|u\|^N}{N} - c_2\|u\|^{\alpha+1} - |h|_{L^{N'}(\mathbb{R}^N)}\|u\|.$$

Since $\alpha + 1 > N$, then one can easily find $0 < \rho < \min\{1,\ \frac{1}{\delta_0}\}$ small enough such that

$$\min\{1, a_1\}\frac{\rho^N}{N} - c_2\rho^{1+\alpha} \geq \min\{1, a_1\}\frac{\rho^N}{2N}.$$

It follows that

$$I_w(u) \geq \min\{1, a_1\}\frac{\rho^N}{2N} - |h|_{L^{N'}(\mathbb{R}^N)}\rho, \quad \text{for } \|u\| = \rho.$$

We complete the proof of Lemma 2.1 by taking $d = \min\{1, a_1\}\frac{\rho^{N-1}}{4N}$ and $\mu = \min\{1, a_1\}\frac{\rho^N}{4N}$. $\square$

Lemma 2.2. *Assume that* (V1), (V2), (H1), (H3), (H5) *hold. Then, there exists $\vartheta \in E$ independent of w such that $\|\vartheta\| > \rho$ and $I_w(\vartheta) < 0$ for all $w \in E$ with $\|w\| \le \frac{1}{\delta_0}$.*

Proof. Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ be such that $\varphi \neq 0$ and $\varphi \ge 0$. For $t > 0$, we have

$$\begin{aligned} I_w(t\varphi) &\le \max\{1, a_1'\}\frac{t^N}{N}\|\varphi\|^N - \int_{\mathbb{R}^N} F(x, t\varphi)\,dx \\ &\le \max\{1, a_1'\}\frac{t^N}{N}\|\varphi\|^N - At^p|\varphi|^p_{L^p(\mathbb{R}^N)}. \end{aligned}$$

Since $p > N$, we have

$$\max\{1, a_1'\}\frac{t^N}{N}\|\varphi\|^N - At^p|\varphi|^p_{L^p(\mathbb{R}^N)} \to -\infty, \quad \text{as } t \to +\infty.$$

Thus, we can choose $\vartheta = t_0\varphi$ where t_0 is large enough such that $\|t_0\varphi\| > 1 > \rho$. This completes the proof. $\square$

Now, by the Mountain Pass Theorem without the Palais-Smale condition (see [5, 37]), there exists a sequence $(u_{n,w}) \subset E$ such that $I_w'(u_{n,w}) \to 0$ and $I_w(u_{n,w}) \to c_w = \inf_{\gamma\in\Gamma}\sup_{0\le t\le 1} I_w(\gamma(t))$, where

$$\Gamma = \{\gamma \in C([0,1],\ E),\ \gamma(0) = 0,\ \gamma(1) = t_0\varphi = \vartheta\}.$$

Lemma 2.3. *Assume that* (V1), (V2), (H1)–(H3), (H5) *hold. Let $w \in E$ with $\|w\| \le \frac{1}{\delta_0}$. Then, for every $0 < \eta < \frac{1}{\delta_0}$, there exist $A_\eta > 0$ and $d_\eta > 0$ such that: if $A > A_\eta$, and $|h|_{L^{N'}(\mathbb{R}^N)} < d_\eta$ then the functional I_w admits a nontrivial critical point $u_w \in E$ such that $0 < \mu \le I_w(u_w) = c_w$, where μ is given by Lemma 2.1. Moreover, $\|u_w\| \le \eta$.*

Proof. We have

$$I_w(u_{n,w}) - \frac{1}{\nu}\langle I_w'(u_{n,w}),\ u_{n,w}\rangle \le c_w + o_n(1)(1 + \|u_{n,w}\|).$$

Using (H2) and (2.1), we have

$$\min\{1, a_1\}(\frac{1}{N} - \frac{1}{\nu})\|u_{n,w}\|^N \le c_w + o_n(1)(1 + \|u_{n,w}\|) + |h|_{L^{N'}(\mathbb{R}^N)}\|u_{n,w}\|. \quad (2.3)$$

Then, $(u_{n,w})$ is a bounded sequence in E. Now, by Young's inequality, there exists $c_3 > 0$ such that

$$|h|_{L^{N'}(\mathbb{R}^N)}\|u_{n,w}\| \le \frac{\min\{1, a_1\}}{2}(\frac{1}{N} - \frac{1}{\nu})\|u_{n,w}\|^N + c_3|h|^{N'}_{L^{N'}(\mathbb{R}^N)}.$$

Putting this inequality in (2.3), we obtain

$$\frac{\min\{1, a_1\}}{2}(\frac{1}{N} - \frac{1}{\nu})\|u_{n,w}\|^N \le c_w + o_n(1)(1 + \|u_{n,w}\|) + c_3|h|^{N'}_{L^{N'}(\mathbb{R}^N)}.$$

By passing to the upper limit, we obtain

$$\limsup_{n\to+\infty}\|u_{n,w}\|^N \le \frac{2c_w}{\min\{1, a_1\}(\frac{1}{N} - \frac{1}{\nu})} + c_4|h|^{N'}_{L^{N'}(\mathbb{R}^N)}. \quad (2.4)$$

Now, observe that by the even definition of c_w, we have

$$c_w \leq \max_{t\geq 0} I_w(t\varphi) \leq \max_{t\geq 0}\Big(\frac{\max\{1,a_1'\}t^N\|\varphi\|^N}{N} - At^p|\varphi|^p_{L^p(\mathbb{R}^N)}\Big).$$

It is clear that the function

$$K(t) = \frac{\max\{1,a_1'\}t^N\|\varphi\|^N}{N} - At^p|\varphi|^p_{L^p(\mathbb{R}^N)}$$

defined on $[0,+\infty[$ attains its maximum at

$$t_{\max} = \Big(\frac{\max\{1,a_1'\}\|\varphi\|^N}{Ap|\varphi|^p_{L^p(\mathbb{R}^N)}}\Big)^{\frac{1}{p-N}}.$$

Thus,

$$\max_{t\geq 0} K(t) = \max\{1,a_1'\}\|\varphi\|^N\Big(\frac{1}{N}-\frac{1}{p}\Big)\Big(\frac{\max\{1,a_1'\}\|\varphi\|^N}{pA|\varphi|^p_{L^p(\mathbb{R}^N)}}\Big)^{\frac{N}{p-N}}.$$

Hence,

$$c_w \leq \max\{1,a_1'\}\|\varphi\|^N\Big(\frac{1}{N}-\frac{1}{p}\Big)\Big(\frac{\max\{1,a_1'\}\|\varphi\|^N}{pA|\varphi|^p_{L^p(\mathbb{R}^N)}}\Big)^{\frac{N}{p-N}}. \tag{2.5}$$

Denote

$$\Sigma(A) = \max\{1,a_1'\}\|\varphi\|^N\Big(\frac{1}{N}-\frac{1}{p}\Big)\Big(\frac{\max\{1,a_1'\}\|\varphi\|^N}{pA|\varphi|^p_{L^p(\mathbb{R}^N)}}\Big)^{\frac{N}{p-N}}.$$

Fix $0<\eta<\frac{1}{\delta_0}$. It is clear that there exists $A_\eta>0$ large enough such that

$$\Sigma(A) \leq \frac{\min\{1,a_1\}}{4}\Big(\frac{1}{N}-\frac{1}{\nu}\Big)\eta^N,$$

provided that $A>A_\eta$. On the other hand, we can choose $|h|_{L^{N'}(\mathbb{R}^N)}$ small enough such that

$$c_4|h|^{N'}_{L^{N'}(\mathbb{R}^N)} \leq \frac{\eta^N}{2}.$$

Hence, by (2.4) and (2.5) we deduce that

$$\limsup_{n\to+\infty}\|u_{n,w}\|^N \leq \eta^N.$$

It follows, that there exists $n_0>1$ large enough such that

$$\|u_{n,w}\| \leq \Big(\frac{2\Sigma(A)}{\min\{1,a_1\}(\frac{1}{N}-\frac{1}{\nu})} + c_4|h|^{N'}_{L^{N'}(\mathbb{R}^N)}\Big)^{1/N} \leq \eta < \frac{1}{\delta_0}, \quad \forall n\geq n_0.$$

Up to a subsequence, $(u_{n,w})$ is weakly convergent to some point u_w in E. We claim that, up to a subsequence, $(u_{n,w})$ is strongly convergent to u_w in E. First, observe that by (2.2) we have

$$\int_{\mathbb{R}^N}|f(x,u_{n,w})|^{N'}\,dx \leq c_5\int_{\mathbb{R}^N}|u_{n,w}|^{\alpha N'}\,dx.$$

Thus, we get the boundedness of the sequence $(f(\cdot,u_{n,w}))$ in $L^{N'}(\mathbb{R}^N)$. This fact together with the compact embedding $E\hookrightarrow\hookrightarrow L^N(\mathbb{R}^N)$ imply

$$\int_{\mathbb{R}^N}|f(x,u_{n,w})(u_{n,w}-u_w)|\,dx \to 0, \quad n\to+\infty. \tag{2.6}$$

Using (2.6) and the weak convergence of $(u_{n,w})$ to u_w in E, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^N} a(x,w)(|\nabla u_{n,w}|^{N-2}\nabla u_{n,w} - |\nabla u_w|^{N-2}\nabla u_w)\nabla(u_{n,w}-u_w)\,dx\\
&+\int_{\mathbb{R}^N} V(x)(|u_{n,w}|^{N-2}u_{n,w} - |u_w|^{N-2}u_w)(u_{n,w}-u_w)\,dx\\
&+\int_{\mathbb{R}^N} (|\Delta u_{n,w}|^{N-2}\Delta u_{n,w} - |\Delta u_w|^{N-2}\Delta u_w)\Delta(u_{n,w}-u_w)\,dx\\
&\to 0, \quad \text{as } n\to+\infty.
\end{aligned}
$$

Recalling the standard inequality

$$
(|x|^{N-2}x - |y|^{N-2}y)(x-y) \geq 2^{-N}|x-y|^N, \quad \forall x,y\in\mathbb{R}^r,\ \forall r\geq 1, \tag{2.7}
$$

we can deduce that, up to a subsequence, $(u_{n,w})$ is strongly convergent to u_w in E. Consequently, u_w is a critical point of I_w and $I_w(u_w) = c_w \geq \mu > 0$. Moreover, taking into account that

$$
\|u_{n,w}\| \leq \eta, \quad \forall n\geq n_0,
$$

and passing to the limit as $n\to+\infty$, we obtain $\|u_w\|\leq\eta$. □

Lemma 2.4. *Assume that* (V1)–(V2), (H1), (H5) *hold. Let $w\in E$ be such that $\|w\|\leq\frac{1}{\delta_0}$. Then, the functional I_w admits a nontrivial weak solution $U_w\in E$ such that $I_w(U_w)\leq-\sigma<0$ and $\|U_w\|\leq\rho$, where ρ is given by Lemma 2.1 and σ is some positive constant independent of w.*

Proof. Let φ be the function introduced and used in Lemma 2.2. Clearly, we can choose $0\leq\varphi(x)\leq 1$ for all $x\in\mathbb{R}^N$. For $0<t<1$, we have

$$
I_w(t\varphi) \leq \max\{1,a_1'\}\frac{t^N}{N}\|\varphi\|^N - \int_{\mathbb{R}^N} F(x,t\varphi)\,dx - t\int_{\mathbb{R}^N} h\varphi\,dx. \tag{2.8}
$$

By (H1), we can easily obtain

$$
\lim_{t\to 0^+}\int_{\mathbb{R}^N}\frac{F(x,t\varphi)}{t}\,dx = 0.
$$

Moreover, since

$$
\int_{\mathbb{R}^N} h\varphi\,dx > 0,
$$

by (2.8) one can easily find $0<t_1<\inf(1,\frac{\rho}{\|\varphi\|})$ small enough and independent of w, and $\sigma>0$ also independent of w such that

$$
I_w(t_1\varphi)\leq-\sigma<0.
$$

Now, denote

$$
\theta_w = \inf\{I_w(u),\ \|u\|\leq\rho\}.
$$

In view of Lemma 2.1 and by the Ekeland's variational principle (see [13]), there exists a sequence $(U_{n,w})\subset E$ such that

$$
\|U_{n,w}\|\leq\rho,\ I_w(U_{n,w})\to\theta_w, \quad \text{and} \quad I_w'(U_{n,w})\to 0.
$$

Up to a subsequence, $(U_{n,w})$ is weakly convergent to some point U_w in E. Observe that $\rho<\frac{1}{\delta_0}$ and arguing as for (2.6), we can prove that

$$
\int_{\mathbb{R}^N} f(x,U_{n,w})(U_{n,w}-U_w)\,dx \to 0, \quad \text{as } n\to+\infty.
$$

Proceeding exactly as for the sequence $(u_{n,w})$, we can easily show that, up to a subsequence, $(U_{n,w})$ is strongly convergent to U_w in E. Therefore, the point U_w is a critical point of I_w satisfying

$$I_w(U_w) = \theta_w \le -\sigma < 0, \quad \text{and} \quad \|U_w\| \le \rho.$$

This completes the proof. $\square$

Proof of Theorem 1.2 completed. To conclude the proof, an iterative scheme will be performed. Let $0 < \eta < 1/\delta_0$ and fix $u_0 \in E$ such that $\|u_0\| \le \eta$. By Lemma 2.3, under the condition $A > A_\eta$, and $|h|_{L^{N'}(\mathbb{R}^N)} < d_\eta$, the functional I_{u_0} admits a nontrivial critical point $u_1 \in E$ such that

$$I_{u_0}(u_1) \ge \mu > 0, \quad \|u_1\| \le \eta.$$

Similarly, the functional I_{u_1} admits a critical point u_2 such that

$$I_{u_1}(u_2) \ge \mu > 0, \quad \|u_2\| \le \eta.$$

This way, we construct a sequence $(u_n) \subset E$ such that

$$\|u_n\| \le \eta, \quad I_{u_{n-1}}(u_n) \ge \mu > 0,$$

and u_n is a critical point of the functional $I_{u_{n-1}}$. Thus, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} a(x,u_{n-1})|\nabla u_n|^{N-2}\nabla u_n \nabla v\,dx \\
&+ \int_{\mathbb{R}^N} V(x)|u_n|^{N-2}u_n v\,dx + \int_{\mathbb{R}^N} |\Delta u_n|^{N-2}\Delta u_n \Delta v\,dx \\
&= \int_{\mathbb{R}^N} f(x,u_n)v\,dx + \int_{\mathbb{R}^N} hv\,dx, \quad \forall v \in E.
\end{aligned} \tag{2.9}$$

Similarly, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} a(x,u_n)|\nabla u_{n+1}|^{N-2}\nabla u_{n+1} \nabla v\,dx \\
&+ \int_{\mathbb{R}^N} V(x)|u_{n+1}|^{N-2}u_{n+1} v\,dx + \int_{\mathbb{R}^N} |\Delta u_{n+1}|^{N-2}\Delta u_{n+1} \Delta v\,dx \\
&= \int_{\mathbb{R}^N} f(x,u_{n+1})v\,dx + \int_{\mathbb{R}^N} hv\,dx, \quad \forall v \in E.
\end{aligned} \tag{2.10}$$

Taking $v = u_{n+1} - u_n$ as test function in (2.9) and (2.10), and subtracting one equation from the other, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} a(x,u_n)\Big(|\nabla u_{n+1}|^{N-2}\nabla u_{n+1} - |\nabla u_n|^{N-2}\nabla u_n\Big)\nabla(u_{n+1}-u_n)\,dx \\
&+ \int_{\mathbb{R}^N} (a(x,u_n) - a(x,u_{n-1}))|\nabla u_n|^{N-2}\nabla u_n \nabla(u_{n+1}-u_n)\,dx \\
&+ \int_{\mathbb{R}^N} V(x)\Big(|u_{n+1}|^{N-2}u_{n+1} - |u_n|^{N-2}u_n\Big)(u_{n+1}-u_n)\,dx \\
&+ \int_{\mathbb{R}^N} \Big(|\Delta u_{n+1}|^{N-2}\Delta u_{n+1} - |\Delta u_n|^{N-2}\Delta u_n\Big)\Delta(u_{n+1}-u_n)\,dx \\
&= \int_{\mathbb{R}^N} (f(x,u_{n+1}) - f(x,u_n))(u_{n+1}-u_n)\,dx.
\end{aligned} \tag{2.11}$$

Since $\|u_n\| \leq \eta$, for all $n \geq 1$, it follows by (1.4) that

$$|u_n|_{L^\infty(\mathbb{R}^N)},\ |u_{n+1}|_{L^\infty(\mathbb{R}^N)} \leq \delta_0\eta < 1, \quad \forall n \geq 0.$$

By (H_4), it yields

$$|f(x,u_{n+1}(x)) - f(x,u_n(x))| \leq \beta_0(\delta_0\eta)^{\beta_1}|u_{n+1}(x) - u_n(x)|^{N-1},$$

a.e. $x \in \mathbb{R}^N$, for all $n \geq 0$. Consequently

$$\int_{\mathbb{R}^N} (f(x,u_{n+1}) - f(x,u_n))(u_{n+1} - u_n)\,dx \leq \beta_0(\delta_0\eta)^{\beta_1} \int_{\mathbb{R}^N} |u_{n+1} - u_n|^N\,dx. \tag{2.12}$$

If we take η small enough such that

$$\beta_0(\delta_0\eta)^{\beta_1} \leq \frac{V_0 \min\{1,a_1\}2^{-N}}{4},$$

then from (2.12) we infer

$$\begin{aligned} &\int_{\mathbb{R}^N} (f(x,u_{n+1}) - f(x,u_n))(u_{n+1} - u_n)\,dx \\ &\leq \frac{\min\{1,a_1\}2^{-N}}{4} \int_{\mathbb{R}^N} V(x)|u_{n+1} - u_n|^N\,dx \\ &\leq \frac{\min\{1,a_1\}2^{-N}}{4} \|u_{n+1} - u_n\|^N. \end{aligned} \tag{2.13}$$

On the other hand, by Young's inequality we have

$$\begin{aligned} &\int_{\mathbb{R}^N} |a(x,u_n) - a(x,u_{n-1})|\ |\nabla u_n|^{N-1}|\nabla(u_{n+1} - u_n)|\,dx \\ &\leq \frac{\min\{1,a_1\}2^{-N}}{4} \int_{\mathbb{R}^N} |\nabla(u_{n+1} - u_n)|^N\,dx \\ &\quad + c_6 \int_{\mathbb{R}^N} |a(x,u_n) - a(x,u_{n-1})|^{N'}|\nabla u_n|^N\,dx, \end{aligned}$$

and by (H6) it follows that

$$\begin{aligned} &\int_{\mathbb{R}^N} |a(x,u_n) - a(x,u_{n-1})|\ |\nabla u_n|^{N-1}|\nabla(u_{n+1} - u_n)|\,dx \\ &\leq \frac{\min\{1,a_1\}2^{-N}}{4} \|u_{n+1} - u_n\|^N + c_6 L^{N'} |u_n - u_{n-1}|^N_{L^\infty(\mathbb{R}^N)} \|u_n\|^N \\ &\leq \frac{\min\{1,a_1\}2^{-N}}{4} \|u_{n+1} - u_n\|^N + (c_6 L^{N'} \delta_0^N \eta^N) \|u_n - u_{n-1}\|^N. \end{aligned} \tag{2.14}$$

Using (2.7), (2.11), (2.13) and (2.14), we obtain

$$\frac{\min\{1,a_1\}}{2^{N+1}} \|u_{n+1} - u_n\|^N \leq (c_6 L^{N'} \delta_0^N \eta^N) \|u_n - u_{n-1}\|^N. \tag{2.15}$$

Set

$$\Gamma(\eta) = \Big(\frac{c_6 L^{N'} \delta_0^N \eta^N 2^{N+1}}{\min\{1,a_1\}}\Big)^{1/N}.$$

By (2.15), it yields

$$\|u_{n+1} - u_n\| \leq \Gamma(\eta)\|u_n - u_{n-1}\|, \quad \forall n \geq 1. \tag{2.16}$$

Clearly, we can choose η small enough such that $\Gamma(\eta) < 1$. Therefore, by (2.16) (u_n) is a Cauchy sequence and by consequence it is strongly convergent to some point $u \in E$. Passing to the limit as $n \to +\infty$ in (2.9), we conclude that u satisfies

$$\begin{aligned}&\int_{\mathbb{R}^N} a(x,u)|\nabla u|^{N-2}\nabla u \nabla v\,dx + \int_{\mathbb{R}^N} V(x)|u|^{N-2}uv\,dx + \int_{\mathbb{R}^N} |\Delta u|^{N-2}\Delta u \Delta v\,dx\\ &= \int_{\mathbb{R}^N} f(x,u)v\,dx + \int_{\mathbb{R}^N} hv\,dx, \quad \forall v \in E.\end{aligned}$$

According to Definition 1.1, this means that u is a weak solution of problem (1.1). On the other hand, we have $I_{u_{n-1}}(u_n) \geq \mu > 0$, for all $n \geq 2$. Hence,

$$\begin{aligned}&\int_{\mathbb{R}^N} \frac{a(x,u_{n-1})|\nabla u_n|^N + V(x)|u_n|^N + |\Delta u_n|^N}{N}\,dx\\ &- \int_{\mathbb{R}^N} F(x,u_n)\,dx - \int_{\mathbb{R}^N} hu_n\,dx \geq \mu > 0.\end{aligned}$$

Passing to the limit as $n \to +\infty$, it follows

$$\begin{aligned}\Psi(u) = &\int_{\mathbb{R}^N} \frac{a(x,u)|\nabla u|^N + V(x)|u|^N + |\Delta u|^N}{N}\,dx\\ &- \int_{\mathbb{R}^N} F(x,u)\,dx - \int_{\mathbb{R}^N} hu\,dx \geq \mu > 0.\end{aligned}$$

Now, using Lemma 2.4 it is immediate that an iterative scheme could be performed to construct a sequence $(U_n) \subset E$ such that, for all $n \geq 1$,

$$\|U_n\| \leq \rho < \frac{1}{\delta_0}, \quad I_{U_{n-1}}(U_n) \leq -\sigma < 0,$$

and U_n is a critical point of the functional $I_{U_{n-1}}$. Moreover, using the same arguments as for the sequence (u_n), we can easily prove that the sequence (U_n) is strongly convergent to some point $U \in E$ which is a weak solution of problem (1.1). Furthermore, we have

$$\begin{aligned}\Psi(U) = &\int_{\mathbb{R}^N} \frac{a(x,U)|\nabla U|^N + V(x)|U|^N + |\Delta U|^N}{N}\,dx\\ &- \int_{\mathbb{R}^N} F(x,U)\,dx - \int_{\mathbb{R}^N} hU\,dx \leq -\sigma < 0.\end{aligned}$$

Hence, $u \neq U$. This completes the proof of Theorem 1.2.

References

[1] Adimurthi; *Existence of positive solutions of the semilinear Dirichlet problem with critical growth for the N-Laplacian*, Ann. Sc. Norm. Super. Pisa, Cl. Sci., 17(3) (1990), 393-413.

[2] C. O. Alves, G. M. Figueiredo; *On multiplicity and concentration of positive solutions for a class of quasilinear problems with critical exponential growth in $\mathbb{R}^N$*, J. Differential Equations, 246 (2009), 1288-1311.

[3] C. O. Alves, M. A. S. Souto; *Multiplicity of positive solutions for a class of problems with exponential critical growth in $\mathbb{R}^2$*, J. Differential Equations, 244 (2008), 1502-1520.

[4] A. Alvino, L. Boccardo, V. Ferone, L. Orsina, G. Trombetti; *Existence results for nonlinear elliptic equations with degenerate coercivity*, Ann. Mat. Pura Appl., 182 (2003), 53-79.

[5] A. Ambrosetti, P. H. Rabinowitz; *Dual variational methods in critical point theory and applications*, J. Funct. Anal., 14 (1973), 349-381.

[6] S. Aouaoui; *On some nonhomogeneous degenerate quasilinear equations arising in the continuum mechanics*, Comm. Appl. Nonlinear Anal., Vol. 20 No. 4 (2013), 87-108.

[7] L. Boccardo; *Some elliptic problems with degenerate coercivity*, Vol. 6, No. 1 (2006), 1-12.

[8] L. Boccardo, A. Dall'Aglio, L. Orsina; *Existence and regularity results for some elliptic equations with degenerate coercivity*, Atti Semin. Mat. Fis. Univ. Modena Reggio Emilia, 46 (1998), 51-81.

[9] L. Boccardo, H. Brezis; *Some remarks on a class of elliptic equations with degenerate coercivity*, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat., (8) 6:3 (2003), 521-530.

[10] L. Boccardo, G. Corce, L. Orsina; *Existence of solutions for some noncoercive elliptic problems involving derivatives of nonlinear terms*, Differ. Equ. Appl. Vol. 4, No.1 (2012), 3-9.

[11] L. Boccardo, G. Corce, L. Orsina; *Nonlinear degenerate elliptic problems with $W_0^{1,1}(\Omega)$ solutions*, Manuscripta Math., Vol. 137: 3-4 (2012), 419-439.

[12] D. Cao; *Nontrivial solution of semilinear elliptic equation with critical exponent in $\mathbb{R}^2$*, Commun. Partial Differ. Equ., 17(34) (1992), 407-435.

[13] I. Ekeland; *On the variational principle*, J. Math. Anal. App. 47 (1974), 324-353.

[14] D. G. de Figueiredo, M. Girardi, M. Matzeu; *Semilinear elliptic equations with dependence on the gradient via mountain-pass techniques*, Differ. Integral Equ., 17 (12) (2004), 119-126.

[15] G. M. Figueiredo; *Quasilinear equations with dependence on the gradient via Mountain Pass techniques in $\mathbb{R}^N$*, Appl. Math. Comput., 203 (2008), 14-18.

[16] L. R. de Freitas; *Multiplicity of solutions for a class of quasilinear equations with exponential critical growth*, Nonlinear Anal., 95 (2014), 607-624.

[17] M. Girardi, M. Matzeu, Positive and negative solutions of a quasi-linear elliptic equation by a mountain pass method and truncature techniques, Nonlinear Anal. 59 (2004) 199-210.

[18] M. Girardi, M. Matzeu; *A compactness result for quasilinear elliptic equations by mountain pass techniques*, Rend. Mat. Appl., 29 (1) (2009), 83-95.

[19] Y. Huang, X. Liu; *Sign-changing solutions for p-biharmonic equations with Hardy potential*, J. Math. Anal. Appl., 412 (2014), 142-154.

[20] A. Khalil; *On a class of PDE involving p-biharmonic operator*, ISRN Math. Anal., Vol. 2011 (2011), Article ID 630754, pp. 1-11.

[21] N. Lam, G. Lu; *Existence and multiplicity of solutions to equations of N-Laplacian type with critical exponential growth in $\mathbb{R}^N$*, J. Funct. Anal., 262 (2012), 1132-1165.

[22] N. Lam, G. Lu; *N-Laplacian equations in $\mathbb{R}^N$ with subcritical and critical growth without the Ambrosetti-Rabinowitz condition*, Adv. Nonlinear Stud., Vol. 13, Number 2 (2013), 289-308.

[23] A. C. Lazer, P. J. Mckenna; *Large amplitude periodic oscillations in suspension bridges: Some new connections with nonlinear analysis*, SIAM Rev., 32 (1990), 537578.

[24] CH. Li, C-L. Tang; *Three solutions for a Navier boundary value problem involving the p-biharmonic*, Nonlinear Anal., 72 (2010), 1339-1347.

[25] L. Li, L. Ding, W.-W. Pan; *Existence of multiple solutions for a $p(x)$-biharmonic equation*, Electron. J. Differential Equations, Vol. 2013 (2013) No. 139, pp. 1-10.

[26] G. Liu, S. Shi, Y. Wei; *Semilinear elliptic equations with dependence on the gradient*, Electronic J. Differential Equations, Vol. 2012 (2012), No. 139, 1-9.

[27] M. Matzeu, R. Servadei; *A Variational Approach to a class of quasilinear elliptic equations not in divergence form*, Discrete Contin. Dyn. Syst. Ser. S, Vol. 5, No. 4 (2012), 819-831.

[28] G. Molica Bisci, D. Repovš; *Multiple solutions of p-biharmonic equations with Navier boundary conditions*, Complex Var. Elliptic Equ., 59 (2014), 271-284.

[29] R. Servadei; *A semilinear elliptic PDE not in divergence form via variational methods*, J. Math. Anal. Appl., 383 (2011), 190-199.

[30] J. M. do Ó; *Semilinear Dirichlet problems for the N-Laplacian in $\mathbb{R}^N$ with nonlinearities in the critical growth range*, Differ. Integral Equ., 9(5) (1996), 967-979.

[31] J. M. do Ó, E. Medeiros, U. Severo; *On a quasilinear nonhomogeneous elliptic equation with critical growth in $\mathbb{R}^N$*, J. Differential Equations, 246(4) (2009), 1363-1386.

[32] J. M. do Ó; *N-Laplacian equations in $\mathbb{R}^N$ with critical growth*, Abstr. Appl. Anal. Vol. 2, Issue 3-4, (1997), 301-315.

[33] J. M. do Ó, E. Medeiros, U. Severo; *A nonhomogeneous elliptic problem involving critical growth in dimension two*, J. Math. Anal. Appl. vol. 345, no. 1 (2008), 286-304.

[34] E. Tonkes; *Solutions to a perturbed critical semilinear equation concerning the N-Laplacian in $\mathbb{R}^N$*, Comment. Math. Univ. Carolin. 40 (1999) 679-699.

[35] Y. Wang, J. Yang, Y. Zhang; *Quasilinear elliptic equations involving the N-Laplacian with critical exponential growth in $\mathbb{R}^N$*, Nonlinear Anal. 71 (2009), 6157-6169.

[36] W. Wang, P. Zhao; *Nonuniformly nonlinear elliptic equations of p-biharmonic type*, J. Math. Anal. Appl., 348 (2008), 730-738.

[37] M. Willem; *Minimax Theorem*, Birkhäuser, Boston, 1996.

Sami Aouaoui
Institut Supérieur des Mathématiques Appliquées et de l'Informatique de Kairouan, Avenue Assad Iben Fourat, 3100 Kairouan, Tunisie
Phone +216 77 226 575; Fax +216 77 226 575
E-mail address: sami_aouaoui@yahoo.fr