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EXISTENCE AND STABILITY OF ALMOST PERIODIC SOLUTIONS FOR SICNNS WITH NEUTRAL TYPE DELAYS

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Abstract. This article concerns the shunting inhibitory cellular neural networks with neutral type delays. Under a weaker condition than the usual Lipschitz condition, we establish the existence and stability of almost periodic solutions for SICNNs with neutral type delays. An example is given to illustrate our main results.

1. Introduction

Since Bouzerdoum and Pinter [2, 3, 4] introduced and analyzed the shunting inhibitory cellular neural networks (SICNNs), they have been extensively applied in psychophysics, speech, perception, robotics, adaptive pattern recognition, vision, and image processing (cf. [10, 15] and references therein).

It is well known that studies on neural networks not only involve a discussion of stability properties, but also involve many dynamic behaviors such as periodic oscillatory behavior and almost periodic oscillatory properties. In applications, if the various constituent components of the temporally nonuniform environment is with incommensurable (nonintegral multiples) periods, then one has to consider the environment to be almost periodic since there is no a priori reason to expect the existence of periodic solutions. Therefore, if we consider the effects of the environmental factors, almost periodicity is sometimes more realistic and more general than periodicity. Also, as pointed out in [14, 11], compared with periodic effects, almost periodic effects are more frequent in many real world applications. In fact, this point of view is partially verified by a recent work [21], where the authors proved that the "amount" of almost periodic functions (not periodic) is far more than the "amount" of continuous periodic functions in the sense of category. Thus, studying the existence of almost periodic solutions for differential equations is natural and necessary.

Recently, many authors have studied the existence and stability of periodic solutions and almost periodic solutions for the following SICNNs:

$$x_{ij}'(t) = -a_{ij}x_{ij}(t) - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} f[x_{kl}(t-\tau(t))]x_{ij}(t) + L_{ij}(t),$$

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and its variants. We refer the reader to [1, 5, 6, 7, 8, 9, 12, 13, 16, 17, 18, 19] and reference therein for some of recent developments on this topic.

Especially, in a very recent work, the authors in [17] investigated the existence and stability of almost periodic solutions for the following SICNNs with neutral type delays:

$$\begin{aligned} x'_{ij}(t) = -a_{ij}(t)x_{ij}(t) - \sum_{C_{kl}\in N_r(i,j)} C^{kl}_{ij}(t)\int_0^{\infty} K_{ij}(u)f(x_{kl}(t-u))du\, x_{ij}(t) \\ - \sum_{D_{kl}\in N_s(i,j)} D^{kl}_{ij}(t)\int_0^{\infty} J_{ij}(u)g(x'_{kl}(t-u))du\, x_{ij}(t) + L_{ij}(t), \end{aligned} \tag{1.1}$$

$i = 1, 2, \dots, m$, $j = 1, 2, \dots n$, where m, n are two fixed positive integers, C_{ij} is the cell at the (i, j) position of the lattice, the r-neighborhood $N_r(i, j)$ of C_{ij} is defined as follows:

$$N_r(i,j) = \{C_{kl} : \max|k-i|, |l-j| \le r, 1 \le k \le m, 1 \le l \le n\},$$

and $N_s(i, j)$ is defined similarly. Here $x_{ij}(t)$ is the activity of cell C_{ij}, $L_{ij}(t)$ is the external input to C_{ij}, the coefficient $a_{ij}(t)$ is the passive decay rate of the cell activity, f, g are continuous activity functions of signal transmission, $C^{kl}_{ij}(t)$ represents the connection or coupling strength of postsynaptic activity of the cell transmitted to the cell C_{ij}, and $D^{kl}_{ij}(t)$ has a similar meaning.

In [17], the activation functions f and g satisfy the global Lipschitz conditions. In this paper, as one will see, we allow for more general activity functions, i.e., we will discuss the existence and stability of almost periodic solutions for the SICNNs (1.1) under a weaker Lipschitz conditions on f and g.

Next, let us recall some basic notation and results about almost periodic functions. For more details, we refer the reader to [11, 14, 20].

Definition 1.1. A continuous function u: $\mathbb{R} \to \mathbb{R}$ is called almost periodic if for each $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that every interval I of length $l(\varepsilon)$ contains a number τ with the property that

$$|u(t+\tau) - u(t)| < \varepsilon.$$

We denote by $AP(\mathbb{R})$ the set of all almost periodic functions from $\mathbb{R}$ to $\mathbb{R}$, and by $AP^1(\mathbb{R})$ the set of all continuously differentiable functions $u : \mathbb{R} \to \mathbb{R}$ satisfying $u, u' \in AP(\mathbb{R})$.

Lemma 1.2. *Let $f, g \in AP(\mathbb{R})$ and $k \in L^1(\mathbb{R}^+)$. Then the following assertions hold:*

(a) $f + g \in AP(\mathbb{R})$ *and* $f \cdot g \in AP(\mathbb{R})$*;*
(b) *the function* $t \mapsto f(t-\tau)$ *belongs to* $AP(\mathbb{R})$ *for every* $\tau \in \mathbb{R}$*;*
(c) $F \in AP(\mathbb{R})$*, where*

$$F(t) = \int_0^{+\infty} k(u)f(t-u)du, \quad t \in \mathbb{R}.$$

(d) $AP(\mathbb{R})$ *is a Banach space under the norm* $\|f\| = \sup_{t\in\mathbb{R}} |f(t)|$.

2. Existence of almost periodic solution

For the rest of this article, we denote

$$J = \{11, \dots, 1n, , \dots, m1, \dots, mn\},$$
$$x(t) = \{x_{ij}(t)\} = (x_{11}(t), \dots, x_{1n}(t), .., x_{m1}(t), \dots, x_{mn}(t)),$$
$$X = \{\varphi : \varphi = \{\varphi_{ij}\},\ \varphi_{ij}, \varphi'_{ij} \in AP(\mathbb{R})\}.$$

For every $\varphi \in X$, we denote

$$\|\varphi\| = \sup_{t\in\mathbb{R}} \max_{ij\in J}\{|\varphi_{ij}(t)|\},$$
$$\|\varphi\|_X = \max\{\|\varphi\|, \|\varphi'\|\} = \max\{\sup_{t\in\mathbb{R}} \max_{ij\in J} |\varphi_{ij}(t)|, \sup_{t\in\mathbb{R}} \max_{ij\in J} |\varphi'_{ij}(t)|\}.$$

It is not difficult to verify that X is a Banach space under the norm $\|\cdot\|_X$. For every $ij \in J$, we denote

$$a^+_{ij} := \sup_{t\in\mathbb{R}} a_{ij}(t), \quad a^-_{ij} := \inf_{t\in\mathbb{R}} a_{ij}(t), \quad L^+_{ij} := \sup_{t\in\mathbb{R}} |L_{ij}(t)|,$$
$$\overline{C^{kl}_{ij}} := \sup_{t\in\mathbb{R}} |C^{kl}_{ij}(t)|, \quad \overline{D^{kl}_{ij}} := \sup_{t\in\mathbb{R}} |D^{kl}_{ij}(t)|.$$

We will use the following assumptions:

(H1) For every $ij \in J$, a_{ij}, C^{kl}_{ij}, D^{kl}_{ij} and L_{ij} are both almost periodic functions, and $a^-_{ij} > 0$.

(H2) There exist four functions $f_1, f_2, g_1, g_2 : \mathbb{R} \to \mathbb{R}$ and four positive constants $L_{f_1}, L_{f_2}, L_{g_1}, L_{g_2}$ such that $f = f_1 f_2$, $g = g_1 g_2$ and for all $u, v \in \mathbb{R}$, there holds

$$|f_i(u) - f_i(v)| \le L_{f_i}|u - v|, \quad |g_i(u) - g_i(v)| \le L_{g_i}|u - v|, \quad i = 1, 2.$$

(H3) There exists a constant $\lambda_0 > 0$ such that

$$\int_0^\infty |K_{ij}(u)|e^{\lambda_0 u} du < +\infty, \quad \int_0^\infty |J_{ij}(u)|e^{\lambda_0 u} du < +\infty, \quad ij \in J.$$

(H4) There exists a constant $d > 0$ such that

$$\max\Big\{\max_{ij\in J}\big\{\frac{A_{ij} + L^+_{ij}}{a^-_{ij}}\big\}, \max_{ij\in J}\big\{\frac{A_{ij} + L^+_{ij}}{a^-_{ij}}(a^+_{ij} + a^-_{ij})\big\}\Big\} \le d,$$
$$\max\Big\{\max_{ij\in J}\big\{\frac{B_{ij}(0)}{a^-_{ij}}\big\}, \max_{ij\in J}\big\{\frac{B_{ij}(0)}{a^-_{ij}}(a^+_{ij} + a^-_{ij})\big\}\Big\} < 1,$$

where $M_{f_i} = \sup_{|x|\le d} |f_i(x)|$, $M_{g_i} = \sup_{|x|\le d} |g_i(x)|$, $i = 1, 2$,

$$A_{ij} = \sum_{C_{kl}\in N_r(i,j)} \overline{C^{kl}_{ij}}(dM_{f_1}M_{f_2}) \int_0^\infty |K_{ij}(u)|du + \sum_{D_{kl}\in N_s(i,j)} \overline{D^{kl}_{ij}}(dM_{g_1}M_{g_2}) \int_0^\infty |J_{ij}(u)|du,$$

and

$$B_{ij}(0) = \sum_{C_{kl}\in N_r(i,j)} \overline{C^{kl}_{ij}}\Big[(M_{f_1}M_{f_2}) \int_0^\infty |K_{ij}(u)|du$$

$$
\begin{aligned}
&+ d(M_{f_1}L_{f_2} + M_{f_2}L_{f_1})\int_0^\infty |K_{ij}(u)|du\Big] \\
&+ \sum_{D_{kl}\in N_s(i,j)} \overline{D_{ij}^{kl}}\Big[(M_{g_1}M_{g_2})\int_0^\infty |J_{ij}(u)|du \\
&+ d(M_{g_1}L_{g_2} + M_{g_2}L_{g_1})\int_0^\infty |J_{ij}(u)|du\Big],
\end{aligned}
$$

Theorem 2.1. *Under assumptions* H1)–(H4)*, there exists a unique continuously differentiable almost periodic solution of* (1.1) *in the region* $\Omega = \{\varphi \in X : \|\varphi\|_X \le d\}$.

Proof. For $\omega \in (0, \lambda_0]$, we denote

$$
\begin{aligned}
B_{ij}(\omega) = &\sum_{C_{kl}\in N_r(i,j)} \overline{C_{ij}^{kl}}\Big[(M_{f_1}M_{f_2})\int_0^\infty |K_{ij}(u)|du \\
&+ d(M_{f_1}L_{f_2} + M_{f_2}L_{f_1})\int_0^\infty |K_{ij}(u)|e^{\omega u}du\Big] \\
&+ \sum_{D_{kl}\in N_s(i,j)} \overline{D_{ij}^{kl}}\Big[(M_{g_1}M_{g_2})\int_0^\infty |J_{ij}(u)|du \\
&+ d(M_{g_1}L_{g_2} + M_{g_2}L_{g_1})\int_0^\infty |J_{ij}(u)|e^{\omega u}du\Big].
\end{aligned}
$$

For each $\varphi \in X$, we consider the almost periodic differential equations

$$
\begin{aligned}
x_{ij}'(t) = &-a_{ij}(t)x_{ij}(t) - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(t)\int_0^\infty K_{ij}(u)f(\varphi_{kl}(t-u))du\varphi_{ij}(t) \\
&- \sum_{D_{kl}\in N_s(i,j)} D_{ij}^{kl}(t)\int_0^\infty J_{ij}(u)g(\varphi_{kl}'(t-u))du\varphi_{ij}(t) + L_{ij}(t), \quad ij \in J.
\end{aligned}
\tag{2.1}
$$

Combining (H1) and Lemma 1.2, we know that the inhomogeneous part of equation (2.1) is an almost periodic function. Noting that $a_{ij}^- > 0$, by [14, Theorem 7.7], we conclude that (2.1) has a unique almost periodic solution x^φ satisfying

$$
\begin{aligned}
x^\varphi(t) = \Big\{\int_{-\infty}^t &e^{-\int_s^t a_{ij}(u)du}\Big[-\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(s)\int_0^\infty K_{ij}(u)f(\varphi_{kl}(s-u))du\varphi_{ij}(s) \\
&- \sum_{D_{kl}\in N_s(i,j)} D_{ij}^{kl}(s)\int_0^\infty J_{ij}(u)g(\varphi_{kl}'(s-u))du\varphi_{ij}(s) + L_{ij}(s)\Big]ds\Big\}_{ij\in J}.
\end{aligned}
$$

Now, define a mapping T on $\Omega = \{\varphi \in X : \|\varphi\|_X \le d\}$ by

$$
(T\varphi)(t) = x^\varphi(t), \quad \forall \varphi \in \Omega.
$$

It is easy to show that $T(\Omega) \subset X$.

Next, let us check that $T(\Omega) \subset \Omega$. It suffices to prove that $\|T\varphi\|_X \le d$ for all $\varphi \in \Omega$. By (H2) and (H3), we have

$$
\begin{aligned}
&\|T\varphi\| \\
&= \sup_{t\in\mathbb{R}} \max_{ij\in J}\Big\{\Big|\int_{-\infty}^t e^{-\int_s^t a_{ij}(u)du}
\end{aligned}
$$

$$
\begin{aligned}
&\times \Big[- \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(s) \int_0^\infty K_{ij}(u) f(\varphi_{kl}(s-u)) du \varphi_{ij}(s) \\
&- \sum_{D_{kl}\in N_s(i,j)} D_{ij}^{kl}(s) \int_0^\infty J_{ij}(u) g(\varphi'_{kl}(s-u)) du \varphi_{ij}(s) + L_{ij}(s) \Big] ds \Big| \Big\} \\
\le& \sup_{t\in\mathbb{R}} \max_{ij\in J} \Big\{ \int_{-\infty}^t e^{-\int_s^t a_{ij}(u)du} \\
&\times \Big[\sum_{C_{kl}\in N_r(i,j)} \overline{C_{ij}^{kl}} \int_0^\infty |K_{ij}(u)||f(\varphi_{kl}(s-u))| du |\varphi_{ij}(s)| \\
&+ \sum_{D_{kl}\in N_s(i,j)} \overline{D_{ij}^{kl}} \int_0^\infty |J_{ij}(u)||g(\varphi'_{kl}(s-u))| du |\varphi_{ij}(s)| + |L_{ij}(s)| \Big] ds \Big\} \\
\le& \sup_{t\in\mathbb{R}} \max_{ij\in J} \Big\{ \int_{-\infty}^t e^{a_{ij}^-(s-t)} \Big[\sum_{C_{kl}\in N_r(i,j)} \overline{C_{ij}^{kl}} (dM_{f_1}M_{f_2}) \int_0^\infty |K_{ij}(u)| du \\
&+ \sum_{D_{kl}\in N_s(i,j)} \overline{D_{ij}^{kl}} (dM_{g_1}M_{g_2}) \int_0^\infty |J_{ij}(u)| du + L_{ij}^+ \Big] ds \Big\} \\
\le& \max_{ij\in J} \Big\{ \Big[\sum_{C_{kl}\in N_r(i,j)} \overline{C_{ij}^{kl}} (dM_{f_1}M_{f_2}) \int_0^\infty |K_{ij}(u)| du \\
&+ \sum_{D_{kl}\in N_s(i,j)} \overline{D_{ij}^{kl}} (dM_{g_1}M_{g_2}) \int_0^\infty |J_{ij}(u)| du + L_{ij}^+ \Big] / a_{ij}^- \Big\} \\
=& \max_{ij\in J} \Big\{ \frac{A_{ij} + L_{ij}^+}{a_{ij}^-} \Big\},
\end{aligned}
$$

and

$$
\begin{aligned}
\|(T\varphi)'\| =& \sup_{t\in\mathbb{R}} \max_{ij\in J} \Big\{ \Big| - a_{ij}(t) \int_{-\infty}^t e^{-\int_s^t a_{ij}(u)du} \\
&\times \Big[- \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(t) \int_0^\infty K_{ij}(u) f(\varphi_{kl}(s-u)) du \varphi_{ij}(s) \\
&- \sum_{D_{kl}\in N_s(i,j)} D_{ij}^{kl}(t) \int_0^\infty J_{ij}(u) g(\varphi'_{kl}(s-u)) du \varphi_{ij}(s) + L_{ij}(s) \Big] ds \\
&+ \Big[- \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(t) \int_0^\infty K_{ij}(u) f(\varphi_{kl}(t-u)) du \varphi_{ij}(t) \\
&- \sum_{D_{kl}\in N_s(i,j)} D_{ij}^{kl}(t) \int_0^\infty J_{ij}(u) g(\varphi'_{kl}(t-u)) du \varphi_{ij}(t) + L_{ij}(t) \Big] \Big| \Big\} \\
\le& \sup_{t\in\mathbb{R}} \max_{ij\in J} \Big\{ a_{ij}^+ \cdot \frac{A_{ij} + L_{ij}^+}{a_{ij}^-} + A_{ij} + L_{ij}^+ \Big\} \\
=& \max_{ij\in J} \Big\{ \frac{A_{ij} + L_{ij}^+}{a_{ij}^-} (a_{ij}^+ + a_{ij}^-) \Big\}.
\end{aligned}
$$

Then, from (H4) it follows that

$$\|(T\varphi)\|_X \le \max\Big\{\max_{ij\in J}\Big\{\frac{A_{ij}+L_{ij}^+}{a_{ij}^-}\Big\}, \max_{ij\in J}\Big\{\frac{A_{ij}+L_{ij}^+}{a_{ij}^-}(a_{ij}^+ + a_{ij}^-)\Big\}\Big\} \le d,$$

which implies that $T(\Omega)\subset\Omega$.

Let $\varphi,\psi\in\Omega$, and for $ij\in J$ denote

$$\begin{aligned}\alpha_{ij}(s) = \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(s)\Big(&\int_0^\infty K_{ij}(u)f(\varphi_{kl}(s-u))du\varphi_{ij}(s)\\ &-\int_0^\infty K_{ij}(u)f(\psi_{kl}(s-u))du\psi_{ij}(s)\Big),\end{aligned}$$

and

$$\begin{aligned}\beta_{ij}(s) = \sum_{D_{kl}\in N_s(i,j)} D_{ij}^{kl}(s)\Big(&\int_0^\infty J_{ij}(u)g(\varphi'_{kl}(s-u))du\varphi_{ij}(s)\\ &-\int_0^\infty J_{ij}(u)g(\psi'_{kl}(s-u))du\psi_{ij}(s)\Big).\end{aligned}$$

By (H2), for each $ij\in J$, we obtain

$$\begin{aligned}
&|\alpha_{ij}(s)|\\
&\le \sum_{C_{kl}\in N_r(i,j)} \overline{C_{ij}^{kl}}\Big\{\Big|\int_0^\infty K_{ij}(u)f(\varphi_{kl}(s-u))du\varphi_{ij}(s)\\
&\quad -\int_0^\infty K_{ij}(u)f(\varphi_{kl}(s-u))du\psi_{ij}(s)\Big|\\
&\quad +\Big|\int_0^\infty K_{ij}(u)f(\varphi_{kl}(s-u))du\psi_{ij}(s) - \int_0^\infty K_{ij}(u)f(\psi_{kl}(s-u))du\psi_{ij}(s)\Big|\Big\}\\
&\le \sum_{C_{kl}\in N_r(i,j)} \overline{C_{ij}^{kl}}\Big|\Big[\int_0^\infty K_{ij}(u)f_1(\varphi_{kl}(s-u))f_2(\varphi_{kl}(s-u))du\varphi_{ij}(s)\\
&\quad -\int_0^\infty K_{ij}(u)f_1(\varphi_{kl}(s-u))f_2(\varphi_{kl}(s-u))du\psi_{ij}(s)\Big]\\
&\quad +\Big[\int_0^\infty K_{ij}(u)f_1(\varphi_{kl}(s-u))f_2(\varphi_{kl}(s-u))du\psi_{ij}(s)\\
&\quad -\int_0^\infty K_{ij}(u)f_1(\varphi_{kl}(s-u))f_2(\psi_{kl}(s-u))du\psi_{ij}(s)\Big]\\
&\quad +\Big[\int_0^\infty K_{ij}(u)f_1(\varphi_{kl}(s-u))f_2(\psi_{kl}(s-u))du\psi_{ij}(s)\\
&\quad -\int_0^\infty K_{ij}(u)f_1(\psi_{kl}(s-u))f_2(\psi_{kl}(s-u))du\psi_{ij}(s)\Big]\Big|\\
&\le \sum_{C_{kl}\in N_r(i,j)} \overline{C_{ij}^{kl}}[M_{f_1}M_{f_2} + d(M_{f_1}L_{f_2} + M_{f_2}L_{f_1})]\int_0^\infty |K_{ij}(u)|du\|\varphi-\psi\|_X.
\end{aligned}$$

Similarly, for each $ij\in J$, we have

$$|\beta_{ij}(s)| \le \sum_{D_{kl}\in N_s(i,j)} \overline{D_{ij}^{kl}}[M_{g_1}M_{g_2} + d(M_{g_1}L_{g_2} + M_{g_2}L_{g_1})]\int_0^\infty |J_{ij}(u)|du\|\varphi-\psi\|_X.$$

Thus,

$$|\alpha_{ij}(s)| + |\beta_{ij}(s)| \le B_{ij}(0)||\varphi - \psi||_X.$$

It follows that

$$\begin{aligned}\|T\varphi - T\psi\| &= \sup_{t\in\mathbb{R}} \max_{ij\in J} \Big\{ \Big| \int_{-\infty}^{t} e^{-\int_s^t a_{ij}(u)du} [\alpha_{ij}(s) + \beta_{ij}(s)] ds \Big| \Big\} \\ &\le \sup_{t\in\mathbb{R}} \max_{ij\in J} \int_{-\infty}^{t} e^{-\int_s^t a_{ij}(u)du} (|\alpha_{ij}(s)| + |\beta_{ij}(s)|) ds \\ &\le \sup_{t\in\mathbb{R}} \max_{ij\in J} \Big\{ \int_{-\infty}^{t} e^{-\int_s^t a_{ij}(u)du} ds \Big\} B_{ij}(0) \|\varphi - \psi\|_X \\ &\le \max_{ij\in J} \Big\{ \frac{B_{ij}(0)}{a_{ij}^-} \Big\} \|\varphi - \psi\|_X,\end{aligned}$$

and

$$\begin{aligned}&\|(T\varphi - T\psi)'\| \\ &= \sup_{t\in\mathbb{R}} \max_{ij\in J} \Big\{ \Big| - a_{ij}(t) \int_{-\infty}^{t} e^{-\int_s^t a_{ij}(u)ds} (\alpha_{ij}(s) + \beta_{ij}(s)) ds + (\alpha_{ij}(t) + \beta_{ij}(t)) \Big| \Big\} \\ &\le \sup_{t\in\mathbb{R}} \max_{ij\in J} \Big\{ a_{ij}^+ \Big| \int_{-\infty}^{t} e^{-\int_s^t a_{ij}(u)du} (\alpha_{ij}(s) + \beta_{ij}(s)) ds \Big| + (|\alpha_{ij}(t)| + |\beta_{ij}(t)|) \Big\} \\ &\le \sup_{t\in\mathbb{R}} \max_{ij\in J} \Big\{ a_{ij}^+ \frac{B_{ij}(0)}{a_{ij}^-} \|\varphi - \psi\|_X + B_{ij}(0) \|\varphi - \psi\|_X \Big\} \\ &\le \max_{ij\in J} \Big\{ \frac{B_{ij}(0)}{a_{ij}^-} (a_{ij}^+ + a_{ij}^-) \Big\} \|\varphi - \psi\|_X.\end{aligned}$$

Combining the above two inequalities, we obtain

$$\|T\varphi - T\psi\|_X \le \max \Big\{ \max_{ij\in J} \big\{ \frac{B_{ij}(0)}{a_{ij}^-} \big\}, \max_{ij\in J} \big\{ \frac{B_{ij}(0)}{a_{ij}^-} (a_{ij}^+ + a_{ij}^-) \big\} \Big\} \|\varphi - \psi\|_X.$$

Noticing that

$$max \Big\{ \max_{ij\in J} \big\{ \frac{B_{ij}(0)}{a_{ij}^-} \big\}, \max_{ij\in J} \big\{ \frac{B_{ij}(0)}{a_{ij}^-} (a_{ij}^+ + a_{ij}^-) \big\} \Big\} < 1,$$

By the Banach contraction principle, T has a unique fixed point x in Ω, which is just a continuously differentiable almost periodic solution of Equation (1.1). □

3. Stability of almost periodic solutions

In this section, we will establish some results about the locally exponential stability of the almost periodic solution for Equation (1.1).

Theorem 3.1. *Assume* (H1)–(H4) *hold. Let $x(t) = \{x_{ij}(t)\}$ be the unique continuously differentiable almost periodic solution of* (1.1) *in Ω, and $y(t) = \{y_{ij}(t)\}$ be an arbitrary continuously differentiable solution of Equation* (1.1) *in the region Ω. Then, there exist two constants $\lambda, M > 0$ such that*

$$\|x(t) - y(t)\|_1 \le Me^{-\lambda t}, \quad \forall t \in \mathbb{R},$$

where

$$\|x(t) - y(t)\|_1 := \max\{\max_{ij\in J} |x_{ij}(t) - y_{ij}(t)|, \max_{ij\in J} |x'_{ij}(t) - y'_{ij}(t)|\}.$$

Proof. For $\omega \in [0, \lambda_0]$, we denote

$$T_{ij}(\omega) = a_{ij}^- - \omega - B_{ij}(\omega), \quad S_{ij}(\omega) = a_{ij}^- - \omega - (a_{ij}^+ + a_{ij}^-)B_{ij}(\omega).$$

By (H4), we have $T_{ij}(0) > 0$ and $S_{ij}(0) > 0$ for all $ij \in J$. Then, due to the continuity of $T_{ij}(\omega)$ and $S_{ij}(\omega)$, there exists a sufficiently small positive constant $\lambda < \min\big\{ \min_{ij\in J}\{a_{ij}^-\}, \lambda_0 \big\}$ such that

$$T_{ij}(\lambda) > 0, \quad S_{ij}(\lambda) > 0, \quad ij \in J,$$

which means that

$$\frac{B_{ij}(\lambda)}{a_{ij}^- - \lambda} < 1, \quad \frac{B_{ij}(\lambda)}{a_{ij}^- - \lambda}(a_{ij}^+ + a_{ij}^-) < 1, \quad ij \in J. \tag{3.1}$$

for all $ij \in J$. Setting $M_0 = \max_{ij\in J}\big\{\frac{a_{ij}^-}{B_{ij}(0)}\big\}$, the following three inequalities hold:

$$M_0 > 1, \quad \frac{1}{M_0} - \frac{B_{ij}(\lambda)}{a_{ij}^- - \lambda} \le 0, \quad B_{ij}(\lambda)(\frac{a_{ij}^+}{a_{ij}^- - \lambda} + 1) < 1, \quad ij \in J. \tag{3.2}$$

Now, we denote

$$\begin{aligned}
z(t) &= \big\{ z_{ij}(t) : z_{ij}(t) = x_{ij}(t) - y_{ij}(t) \big\}, \\
R_{ij}(s) &= \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(s)\Big[\int_0^\infty K_{ij}(u) f(y_{kl}(s-u))du y_{ij}(s) \\
&\quad - \int_0^\infty K_{ij}(u) f(x_{kl}(s-u))du x_{ij}(s)\Big], \\
Q_{ij}(s) &= \sum_{D_{kl}\in N_s(i,j)} D_{ij}^{kl}(s)\Big[\int_0^\infty J_{ij}(u) g(y'_{kl}(s-u))du y_{ij}(s) \\
&\quad - \int_0^\infty J_{ij}(u) g(x'_{kl}(s-u))du x_{ij}(s)\Big].
\end{aligned}$$

Since $x(t)$ and $y(t)$ are both solutions to equation (1.1), we have

$$z'_{ij}(s) + a_{ij}(s) z_{ij}(s) = R_{ij}(s) + Q_{ij}(s). \tag{3.3}$$

Multiplying by $e^{\int_0^s a_{ij}(u)du}$ and integrating on $[0, t]$, we obtain

$$z_{ij}(t) = z_{ij}(0) e^{-\int_0^t a_{ij}(u)du} + \int_0^t e^{-\int_s^t a_{ij}(u)du}(R_{ij}(s) + Q_{ij}(s))ds \tag{3.4}$$

Let

$$M := M_0 \cdot \max\big\{ \sup_{t\le 0} \max_{ij\in J} |x_{ij}(t) - y_{ij}(t)|, \sup_{t\le 0}\max_{ij\in J} |x'_{ij}(t) - y'_{ij}(t)| \big\}.$$

Without loss for generality, we can assume that $M > 0$. Then, for all $t \le 0$, noting that $M_0 > 1$, we have

$$\|z(t)\|_1 = \max\big\{ \max_{ij\in J} |x_{ij}(t) - y_{ij}(t)|, \max_{ij\in J} |x'_{ij}(t) - y'_{ij}(t)| \big\} < M e^{-\lambda t}.$$

Next, we prove the inequality

$$\|z(t)\|_1 \le M e^{-\lambda t}, \quad t > 0,$$

by contradiction. If the above inequality is not true, then

$$V := \{t > 0 : \|z(t)\|_1 > M e^{-\lambda t}\} \neq \emptyset.$$

Letting $t_1 = \inf V$, then $t_1 > 0$ and

$$\|z(t)\|_1 \le Me^{-\lambda t}, \quad \forall t \in (-\infty, t_1), \quad \|z(t_1)\|_1 = Me^{-\lambda t_1}. \tag{3.5}$$

For $s \in [0, t_1]$ and $ij \in J$, by the assumptions and (3.5), we have

$$\begin{aligned}
&|R_{ij}(s)| \\
&= \sum_{C_{kl}\in N_r(i,j)} |C_{ij}^{kl}(s)| \Big| \int_0^\infty K_{ij}(u) f(y_{kl}(s-u)) du y_{ij}(s) \\
&\quad - \int_0^\infty K_{ij}(u) f(x_{kl}(s-u)) du x_{ij}(s) \Big| \\
&\le \sum_{C_{kl}\in N_r(i,j)} \overline{C_{ij}^{kl}} \Big\{ \Big| \int_0^\infty K_{ij}(u) f(y_{kl}(s-u)) du y_{ij}(s) \\
&\quad - \int_0^\infty K_{ij}(u) f(y_{kl}(s-u)) du x_{ij}(s) \Big| \\
&\quad + \Big| \int_0^\infty K_{ij}(u) f(y_{kl}(s-u)) du x_{ij}(s) - \int_0^\infty K_{ij}(u) f(x_{kl}(s-u)) du x_{ij}(s) \Big| \Big\} \\
&\le \sum_{C_{kl}\in N_r(i,j)} \overline{C_{ij}^{kl}} \Big\{ \Big| \int_0^\infty K_{ij}(u) f_1(y_{kl}(s-u)) f_2(y_{kl}(s-u)) du y_{ij}(s) \Big| \\
&\quad - \int_0^\infty K_{ij}(u) f_1(y_{kl}(s-u)) f_2(y_{kl}(s-u)) du x_{ij}(s) \Big| \\
&\quad + \Big| \int_0^\infty K_{ij}(u) f_1(y_{kl}(s-u)) f_2(y_{kl}(s-u)) du x_{ij}(s) \\
&\quad - \int_0^\infty K_{ij}(u) f_1(x_{kl}(s-u)) f_2(y_{kl}(s-u)) du x_{ij}(s) \Big| \\
&\quad + \Big| \int_0^\infty K_{ij}(u) f_1(x_{kl}(s-u)) f_2(y_{kl}(s-u)) du x_{ij}(s) \\
&\quad - \int_0^\infty K_{ij}(u) f_1(x_{kl}(s-u)) f_2(x_{kl}(s-u)) du x_{ij}(s) \Big| \Big\} \\
&\le \sum_{C_{kl}\in N_r(i,j)} \overline{C_{ij}^{kl}} \Big\{ (M_{f_1} M_{f_2}) \int_0^\infty |K_{ij}(u)| du |z_{ij}(s)| \\
&\quad + d(M_{f_1} L_{f_2} + M_{f_2} L_{f_1}) \int_0^\infty |K_{ij}(u)| |z_{kl}(s-u)| du \Big\} \\
&\le \sum_{C_{kl}\in N_r(i,j)} \overline{C_{ij}^{kl}} \Big\{ (M_{f_1} M_{f_2}) \int_0^\infty |K_{ij}(u)| du \\
&\quad + d(M_{f_1} L_{f_2} + M_{f_2} L_{f_1}) \int_0^\infty |K_{ij}(u)| e^{\lambda u} du \Big\} Me^{-\lambda s}.
\end{aligned}$$

Similarly, for $s \in [0, t_1]$ and $ij \in J$, we have

$$\begin{aligned}
|Q_{ij}(s)| \le \sum_{D_{kl}\in N_s(i,j)} \overline{D_{ij}^{kl}} \Big\{ &(M_{g_1} M_{g_2}) \int_0^\infty |J_{ij}(u)| du |z_{ij}(s)| \\
&+ d(M_{g_1} L_{g_2} + M_{g_2} L_{g_1}) \int_0^\infty |J_{ij}(u)| |z'_{kl}(s-u)| du \Big\}
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{D_{kl}\in N_s(i,j)} \overline{D_{ij}^{kl}}\Big\{(M_{g_1}M_{g_2})\int_0^\infty |J_{ij}(u)|du \\
&\quad + d(M_{g_1}L_{g_2}+M_{g_2}L_{g_1})\int_0^\infty |J_{ij}(u)|e^{\lambda u}du\Big\}Me^{-\lambda s}.
\end{aligned}
$$

Then we have

$$
|R_{ij}(s)|+|Q_{ij}(s)|\le Me^{-\lambda s}B_{ij}(\lambda),\quad s\in[0,t_1],\ ij\in J. \tag{3.6}
$$

Combining (3.4) and (3.6), we have

$$
\begin{aligned}
|z_{ij}(t_1)| &= \big|z_{ij}(0)e^{-\int_0^{t_1} a_{ij}(u)du}+\int_0^{t_1} e^{-\int_s^{t_1} a_{ij}(u)du}(R_{ij}(s)+Q_{ij}(s))ds\big| \\
&\le \frac{M}{M_0}e^{-a_{ij}^- t_1}+\int_0^{t_1} e^{(a_{ij}^- -\lambda)s-a_{ij}^- t_1}ds\cdot MB_{ij}(\lambda) \\
&= \frac{M}{M_0}e^{-a_{ij}^- t_1}+\frac{(e^{-\lambda t_1}-e^{-a_{ij}^- t_1})B_{ij}(\lambda)}{a_{ij}^- -\lambda}\cdot M \\
&\le Me^{-\lambda t_1}\Big\{\frac{e^{(\lambda-a_{ij}^-)t_1}}{M_0}+\frac{[1-e^{(\lambda-a_{ij}^-)t_1}]B_{ij}(\lambda))}{a_{ij}^- -\lambda}\Big\} \\
&= Me^{-\lambda t_1}\Big\{\Big(\frac{1}{M_0}-\frac{B_{ij}(\lambda)}{a_{ij}^- -\lambda}\Big)e^{(\lambda-a_{ij}^-)t_1}+\frac{B_{ij}(\lambda)}{a_{ij}^- -\lambda}\Big\}.
\end{aligned} \tag{3.7}
$$

Then, by (3.1) and (3.2), we deduce that

$$
|z_{ij}(t_1)|<Me^{-\lambda t_1},\quad ij\in J. \tag{3.8}
$$

Recalling that

$$
z_{ij}'(t)=-a_{ij}(t)z_{ij}(t)+R_{ij}(t)+Q_{ij}(t),
$$

by (3.6) and (3.7), we have

$$
\begin{aligned}
|z_{ij}'(t_1)| &= |-a_{ij}(t_1)z_{ij}(t_1)+R_{ij}(t_1)+Q_{ij}(t_1)| \\
&\le a_{ij}^+|z_{ij}(t_1)|+|R_{ij}(t_1)|+|Q_{ij}(t_1)| \\
&< a_{ij}^+Me^{-\lambda t_1}\Big\{\Big(\frac{1}{M_0}-\frac{B_{ij}(\lambda)}{a_{ij}^- -\lambda}\Big)e^{(\lambda-a_{ij}^-)t_1}+\frac{B_{ij}(\lambda)}{a_{ij}^- -\lambda}\Big\}+Me^{-\lambda t_1}B_{ij}(\lambda) \\
&= Me^{-\lambda t_1}\Big\{a_{ij}^+\Big(\frac{1}{M_0}-\frac{B_{ij}(\lambda)}{a_{ij}^- -\lambda}\Big)e^{(\lambda-a_{ij}^-)t_1}+B_{ij}(\lambda)(\frac{a_{ij}^+}{a_{ij}^- -\lambda}+1)\Big\}
\end{aligned}
$$

Then, from (3.2) it follows that

$$
|z_{ij}'(t_1)|<Me^{-\lambda t_1},\quad ij\in J. \tag{3.9}
$$

Combining (3.8) and (3.9), we obtain

$$
\|z(t_1)\|_1<Me^{-\lambda t_1},
$$

which contradicts with (3.5). Thus, we obtain

$$
\|z(t)\|_1\le Me^{-\lambda t},
$$

for all $t\in\mathbb{R}$. This completes the proof. □

Remark 3.2. Compared with the results in [17], our Lipschitz conditions are weaker, and thus our results may have a wider range of applications.

Next, we give an example to illustrate our main results.

Example 3.3. Consider the following SICNNs with neutral delays:

$$\begin{aligned} x'_{ij}(t) &= -a_{ij}(t)x_{ij}(t) - \sum_{C_{kl}\in N_1(i,j)} C_{ij}^{kl}(t)\int_0^\infty K_{ij}(u)f(x_{kl}(t-u))du\,x_{ij}(t) \\ &\quad - \sum_{D_{kl}\in N_1(i,j)} D_{ij}^{kl}(t)\int_0^\infty J_{ij}(u)g(x'_{kl}(t-u))du\,x_{ij}(t) + L_{ij}(t), \end{aligned} \tag{3.10}$$

where $i=1,2,3$, $j=1,2,3$. For $i,j=1,2,3$ and $t\in\mathbb{R}$, let

$$f(t)=\frac{1}{4}|t|\cos t,\quad g(t)=\frac{1}{16}(t^2-1),\quad L_{ij}(t)=\frac{1}{8}|\sin\sqrt{2}t+\sin t|,$$
$$K_{ij}(t)=e^{-6t},\quad J_{ij}(t)=e^{-4t}.$$

Moreover,

$$\begin{pmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t)\\ a_{21}(t) & a_{22}(t) & a_{23}(t)\\ a_{31}(t) & a_{32}(t) & a_{33}(t)\end{pmatrix} = \begin{pmatrix} 5+|\sin t| & 5+|\sin\sqrt{2}t| & 9+|\sin\sqrt{3}t|\\ 6+|\cos t| & 6+\frac{|\cos t+\cos\sqrt{2}t|}{2} & 7+|\cos\sqrt{2}t|\\ 8+|\sin t| & 8+|\sin\sqrt{2}t| & 5+|\sin\sqrt{3}t|\end{pmatrix},$$

$$\begin{pmatrix} C_{11}(t) & C_{12}(t) & C_{13}(t)\\ C_{21}(t) & C_{22}(t) & C_{23}(t)\\ C_{31}(t) & C_{32}(t) & C_{33}(t)\end{pmatrix} = \begin{pmatrix} 0.1|\sin t| & 0.3|\sin\sqrt{2}t| & 0.5|\sin\sqrt{3}t|\\ 0.2|\cos t| & 0.1|\cos\sqrt{2}t| & 0.2|\cos\sqrt{3}t|\\ 0.1|\sin t| & 0.2|\sin\sqrt{2}t| & 0.1|\cos\sqrt{3}t|\end{pmatrix}$$

$$\begin{pmatrix} D_{11}(t) & D_{12}(t) & D_{13}(t)\\ D_{21}(t) & D_{22}(t) & D_{23}(t)\\ D_{31}(t) & D_{32}(t) & D_{33}(t)\end{pmatrix} = \begin{pmatrix} 0.1|\cos t| & 0.3|\cos\sqrt{2}t| & 0.5|\cos\sqrt{3}t|\\ 0.2|\sin t| & 0.1|\sin\sqrt{2}t| & 0.2|\sin\sqrt{3}t|\\ 0.1|\cos t| & 0.2|\cos\sqrt{2}t| & 0.1|\cos\sqrt{3}t|\end{pmatrix}.$$

Let $d=1$, $\lambda_0=1$, $f_1(t)=\frac{1}{2}|t|$, $f_2(x)=\frac{1}{2}\cos t$, $g_1(t)=\frac{1}{4}(t-1)$, $g_2(t)=\frac{1}{4}(t+1)$. By a direct calculation, we obtain $L_{f_1}=L_{f_2}=\frac{1}{2}$, $L_{g_1}=L_{g_2}=\frac{1}{4}$, $M_{f_1}=M_{f_2}=\frac{1}{2}$, $M_{g_1}=M_{g_2}=\frac{1}{2}$, $L_{ij}^+=\frac{1}{4}$. Then, it is easy to see that (H1)–(H3) hold.

Next, let us verify (H4). We have

$$\begin{pmatrix} a_{11}^- & a_{12}^- & a_{13}^-\\ a_{21}^- & a_{22}^- & a_{23}^-\\ a_{31}^- & a_{32}^- & a_{33}^-\end{pmatrix} = \begin{pmatrix} 5 & 5 & 9\\ 6 & 6 & 7\\ 8 & 8 & 5\end{pmatrix},\quad \begin{pmatrix} a_{11}^+ & a_{12}^+ & a_{13}^+\\ a_{21}^+ & a_{22}^+ & a_{23}^+\\ a_{31}^+ & a_{32}^+ & a_{33}^+\end{pmatrix} = \begin{pmatrix} 6 & 6 & 10\\ 7 & 7 & 8\\ 9 & 9 & 6\end{pmatrix},$$

$$\begin{pmatrix} \sum_{C_{kl}\in N_1(1,1)}\overline{C_{11}^{kl}} & \sum_{C_{kl}\in N_1(1,1)}\overline{C_{12}^{kl}} & \sum_{C_{kl}\in N_1(1,1)}\overline{C_{13}^{kl}}\\ \sum_{C_{kl}\in N_1(1,1)}\overline{C_{21}^{kl}} & \sum_{C_{kl}\in N_1(1,1)}\overline{C_{22}^{kl}} & \sum_{C_{kl}\in N_1(1,1)}\overline{C_{23}^{kl}}\\ \sum_{C_{kl}\in N_1(1,1)}\overline{C_{31}^{kl}} & \sum_{C_{kl}\in N_1(1,1)}\overline{C_{32}^{kl}} & \sum_{C_{kl}\in N_1(1,1)}\overline{C_{33}^{kl}}\end{pmatrix} = \begin{pmatrix} 0.7 & 1.4 & 1.1\\ 1 & 1.8 & 1.4\\ 0.6 & 0.9 & 0.6\end{pmatrix},$$

$$\begin{pmatrix} \sum_{D_{kl}\in N_1(1,1)}\overline{D_{11}^{kl}} & \sum_{D_{kl}\in N_1(1,1)}\overline{D_{12}^{kl}} & \sum_{D_{kl}\in N_1(1,1)}\overline{D_{13}^{kl}}\\ \sum_{D_{kl}\in N_1(1,1)}\overline{D_{21}^{kl}} & \sum_{D_{kl}\in N_1(1,1)}\overline{D_{22}^{kl}} & \sum_{D_{kl}\in N_1(1,1)}\overline{D_{23}^{kl}}\\ \sum_{D_{kl}\in N_1(1,1)}\overline{D_{31}^{kl}} & \sum_{D_{kl}\in N_1(1,1)}\overline{D_{32}^{kl}} & \sum_{D_{kl}\in N_1(1,1)}\overline{D_{33}^{kl}}\end{pmatrix} = \begin{pmatrix} 0.7 & 1.4 & 1.1\\ 1 & 1.8 & 1.4\\ 0.6 & 0.9 & 0.6\end{pmatrix},$$

$$\begin{aligned}
A_{ij} &= \sum_{C_{kl}\in N_1(i,j)} \overline{C_{ij}^{kl}}(dM_{f_1}M_{f_2})\int_0^\infty |K_{ij}(u)|du \\
&\quad + \sum_{D_{kl}\in N_1(i,j)} \overline{D_{ij}^{kl}}(dM_{g_1}M_{g_2})\int_0^\infty |J_{ij}(u)|du \\
&= \frac{5}{48}\sum_{C_{kl}\in N_1(i,j)} \overline{C_{ij}^{kl}},
\end{aligned}$$

$$\begin{aligned}
B_{ij}(0) &= \sum_{C_{kl}\in N_1(i,j)} \overline{C_{ij}^{kl}}\Big[(M_{f_1}M_{f_2})\int_0^\infty |K_{ij}(u)|du \\
&\quad + d(M_{f_1}L_{f_2}+M_{f_2}L_{f_1})\int_0^\infty |K_{ij}(u)|du\Big] \\
&\quad + \sum_{D_{kl}\in N_1(i,j)} \overline{D_{ij}^{kl}}\Big[(M_{g_1}M_{g_2})\int_0^\infty |J_{ij}(u)|du \\
&\quad + d(M_{g_1}L_{g_2}+M_{g_2}L_{g_1})\int_0^\infty |J_{ij}(u)|du\Big] \\
&= \frac{1}{4}\sum_{C_{kl}\in N_1(i,j)} \overline{C_{ij}^{kl}}.
\end{aligned}$$

Then, we obtain

$$\begin{pmatrix} A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{pmatrix} = \begin{pmatrix} \frac{3.5}{48} & \frac{7}{48} & \frac{5.5}{48}\\ \frac{5}{48} & \frac{9}{48} & \frac{7}{48}\\ \frac{3}{48} & \frac{4.5}{48} & \frac{3}{48}\end{pmatrix},$$

$$\begin{pmatrix} B_{11}(0) & B_{12}(0) & B_{13}(0)\\ B_{21}(0) & B_{22}(0) & B_{23}(0)\\ B_{31}(0) & B_{32}(0) & B_{33}(0)\end{pmatrix} = \begin{pmatrix} \frac{0.7}{4} & \frac{1.4}{4} & \frac{1.1}{4}\\ \frac{1}{4} & \frac{1.8}{4} & \frac{1.4}{4}\\ \frac{0.6}{4} & \frac{0.9}{4} & \frac{0.6}{4}\end{pmatrix}$$

It follows that

$$\begin{aligned}
&\max\Big\{\max_{ij\in J}\big\{\frac{A_{ij}+L_{ij}^+}{a_{ij}^-}\big\}, \max_{ij\in J}\big\{\frac{A_{ij}+L_{ij}^+}{a_{ij}^-}(a_{ij}^+ + a_{ij}^-)\big\}\Big\} \\
&= \max_{ij\in J}\Big\{\frac{A_{ij}+L_{ij}^+}{a_{ij}^-}(a_{ij}^+ + a_{ij}^-)\Big\} \\
&\le \big(\max_{ij\in J} A_{ij}^+ + \max_{ij\in J} L_{ij}^+\big)\max_{ij\in J}\big\{\frac{a_{ij}^+ + a_{ij}^-}{a_{ij}^-}\big\} \\
&= \big(A_{22}+\frac{1}{4}\big)\frac{a_{11}^+ + a_{11}^-}{a_{11}^-} \\
&= \big(\frac{9}{48}+\frac{1}{4}\big)\cdot\frac{11}{5} < 1 = d,
\end{aligned}$$

and

$$\begin{aligned}
&\max\Big\{\max_{ij\in J}\big\{\frac{B_{ij}(0)}{a_{ij}^-}\big\}, \max_{ij\in J}\big\{\frac{B_{ij}(0)}{a_{ij}^-}(a_{ij}^+ + a_{ij}^-)\big\}\Big\} \\
&= \max_{ij\in J}\big\{\frac{B_{ij}(0)}{a_{ij}-}(a_{ij}^+ + a_{ij}^-)\big\}
\end{aligned}$$

$$
\begin{aligned}
&\le \max_{ij\in J}\{B_{ij}(0)\}\max_{ij\in J}\Big\{\frac{a_{ij}^{+}+a_{ij}^{-}}{a_{ij}^{-}}\Big\}\\
&= B_{22}(0)\frac{a_{11}^{+}+a_{11}^{-}}{a_{11}^{-}}\\
&= \frac{1.8}{4}\cdot\frac{11}{5} < 1.
\end{aligned}
$$

Thus, condition (H4) holds. By Theorem 2.1 and Theorem 3.1, Equation (3.10) admits a unique differentiable almost periodic solution $x^*(t)$ in the region $\Omega = \{\varphi \in X : \|\varphi\|_X \le 1\}$, and $x^*(t)$ is locally exponentially stable in Ω.

Remark 3.4. In Example 3.3, let the iterative sequences $x_n(t) = \{x_{n_{ij}}(t)\}$ be

$$
x_{0_{ij}}(t) = 0, \quad \forall t \in \mathbb{R},\ i = 1,2,3,\ j = 1,2,3,
$$

and

$$
\begin{aligned}
x_{n_{ij}}(t) = \int_{-\infty}^{t} & e^{-\int_s^t a_{ij}(u)du}\\
&\times\Big[-\sum_{C_{kl}\in N_1(i,j)} C_{ij}^{kl}(s)\int_0^{\infty} K_{ij}(u)f(x_{n-1_{kl}}(s-u))du\cdot x_{n-1_{ij}}(s)\\
&- \sum_{D_{kl}\in N_1(i,j)} D_{ij}^{kl}(s)\int_0^{\infty} J_{ij}(u)g(x'_{n-1_{kl}}(s-u))du\cdot x_{n-1_{ij}}(s) + L_{ij}(s)\Big]ds,
\end{aligned}
$$

for all $t \in \mathbb{R}$, $i = 1,2,3$, $j = 1,2,3$, and $n = 1,2,3,\dots$. From the proof of Theorem 2.1, it follows

$$
\|x_n - x^*\|_X \to 0, \quad n \to \infty,
$$

where x^* is the unique almost periodic solution of (3.10) in the region $\Omega = \{\varphi \in X : \|\varphi\|_X \le 1\}$. So one can use this method to compute numerically the almost periodic solution x^*.

Remark 3.5. In the above example, f and g do not satisfy the global Lipschitz condition. So the results in [17] can not be applied to this example.

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