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REMARKS ON REGULARITY CRITERIA FOR THE 3D NAVIER-STOKES EQUATIONS

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Abstract. In this article, we study the regularity criteria for the 3D Navier-Stokes equations involving derivatives of the partial components of the velocity. It is proved that if $\nabla_h \widetilde{u}$ belongs to Triebel-Lizorkin space, ∇u_3 or u_3 belongs to Morrey-Campanato space, then the solution remains smooth on $[0,T]$.

1. Introduction

This article is devoted to the Cauchy problem for the following incompressible 3D Navier-Stokes equation:

$$\begin{gathered} u_t + (u\cdot\nabla)u + \nabla p = \triangle u, \quad x\in\mathbb{R}^3,\ t>0 \\ \operatorname{div} u = 0, \quad x\in\mathbb{R}^3,\ t>0 \end{gathered} \tag{1.1}$$

with initial data

$$u(x,0) = u_0, \quad x\in\mathbb{R}^3, \tag{1.2}$$

where $u = (u_1(x,t), u_2(x,t), u_3(x,t))$ and $p = p(x,t)$ denote the unknown velocity vector and the unknown scalar pressure, respectively. In the last century, Leray [11] and Hopf [8] proved the global existence of a weak solution $u(x,t)\in L^\infty(0,\infty;L^2(\mathbb{R}^3))\cap L^2(0,\infty;H^1(\mathbb{R}^3))$ to (1.1)-(1.3) for any given initial datum $u_0(x)\in L^2(\mathbb{R}^3)$. However, whether or not such a weak solution is regular and unique is still a challenging open problem. From that time on, different criteria for regularity of the weak solutions has been proposed.

The classical Prodi-Serrin conditions (see [16, 18, 19]) say that if

$$u\in L^t(0,T;L^s(\mathbb{R}^3)), \quad \frac{2}{t}+\frac{3}{s}=1, \quad 3\le s\le\infty,$$

then the solution is smooth. Similar results is showed by Beirão da Veiga [1] involving the velocity gradient growth condition:

$$\nabla u\in L^t(0,T;L^s(\mathbb{R}^3)), \quad \frac{2}{t}+\frac{3}{s}=2, \quad \frac{3}{2}<s\le\infty.$$

Actually, whether the weak solution is smooth when a part of the velocity components is involved. As for this direction, later on, criteria just for one velocity

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component appeared. The first result in this direction is due to Neustupa et al [15] (see also Zhou [21]), where the authors showed that if

$$u_3 \in L^t(0,T;L^s(\mathbb{R}^3)), \quad \frac{2}{t}+\frac{3}{s}=\frac{1}{2}, \quad s\in(6,\infty],$$

then the solution is smooth. A similar result, for the gradient of one velocity component, is independently due to Zhou [22] and Pokorný [17]. In [22], Zhou proved that if

$$\nabla u_3 \in L^t(0,T;L^s(\mathbb{R}^3)), \quad \frac{2}{t}+\frac{3}{s}=\frac{3}{2}, \quad 3\le s<\infty.$$

then the solution is smooth on $[0,T]$. This result is extended by Zhou and Pokorný [26]; that is,

$$\nabla u_3 \in L^t(0,T;L^s(\mathbb{R}^3)), \quad \frac{2}{t}+\frac{3}{s}=\frac{23}{12}, \quad 2\le s\le 3.$$

Further criteria, including several components of the velocity gradient, pressure or other quantities, can be found, here we just list some. Zhou and Pokorný [25] proved the regularity condition

$$u_3 \in L^t(0,T;L^s(\mathbb{R}^3)), \quad \frac{2}{t}+\frac{3}{s}=\frac{3}{4}+\frac{1}{2s}, \quad s>\frac{10}{3}.$$

And in [10], Jia and Zhou proved that if a weak solution u satisfies one of the following two conditions:

$$u_3 \in L^\infty(0,T;L^{\frac{10}{3}}(\mathbb{R}^3)); \quad \nabla u_3 \in L^\infty(0,T;L^{30/19}(\mathbb{R}^3)),$$

then u is regular on $[0,T]$. Dong and Zhang [5] proved that if the horizontal derivatives of the two velocity components

$$\int_0^T \|\nabla_h \tilde{u}(s)\|_{\dot{B}^0_{\infty,\infty}}\,ds<\infty,$$

then the solution keeps smoothness up to time T, where $\tilde{u}=(u_1,u_2,0)$, and $\nabla_h\tilde{u}=(\partial_1\tilde{u},\partial_2\tilde{u},0)$. For other kinds of regularity criteria, see [2, 6, 7, 9, 23, 24, 28, 29, 30] and the references cited therein.

Throughout this paper C will denote a generic positive constant which can vary from line to line. For simplicity, we shall use $\int f(x)\,dx$ to denote $\int_{R^3} f(x)\,dx$, use $\|\cdot\|_p$ to denote $\|\cdot\|_{L^p}$.

The purpose of this article is to improve and extend above known regularity criterion of weak solution for the equations (1.1), (1.2) to the Triebel-Lizorkin space and Morrey-Campanato spaces. The main results of this paper read:

Theorem 1.1. *Assume that* $u_0\in H^1(\mathbb{R}^3)$ *with* $\operatorname{div} u_0=0$. *u(x,t) is the corresponding weak solution to* (1.1) *and* (1.2) *on* $[0,T)$. *If additionally*

$$\int_0^T \|\nabla_h \widetilde{u}(\cdot,t)\|^p_{\dot{F}^0_{q,\frac{2}{3}q}}\,dt<\infty, \quad \text{with } \frac{2}{p}+\frac{3}{q}=2,\ \frac{3}{2}<q\le\infty, \tag{1.3}$$

then the solution remains smooth on $[0,T]$.

Theorem 1.2. *Assume that* $u_0\in H^1(\mathbb{R}^3)$ *with* $\operatorname{div} u_0=0$. $u(x,t)$ *is the corresponding weak solution to* (1.1) *and* (1.2) *on* $[0,T)$. *If additionally*

$$\int_0^T \|\nabla u_3(\cdot,t)\|^{\frac{8}{3(2-r)}}_{\dot{M}^{p,\frac{3}{r}}}\,dt<\infty, \quad \text{with} \quad 0<r\le 1,\ 2\le p\le\frac{3}{r}, \tag{1.4}$$

then the solution remains smooth on $[0,T]$.

Theorem 1.3. *Assume that* $u_0 \in H^1(\mathbb{R}^3)$ *with* $\operatorname{div} u_0 = 0$. *u(x,t) is the corresponding weak solution to* (1.1) *and* (1.2) *on* $[0,T)$. *If additionally*

$$\int_0^T \|u_3(\cdot,t)\|_{\dot{M}^{p,\frac{3}{r}}}^{\frac{8}{3-4r}} dt < \infty, \quad \text{with } 0 < r < \frac{3}{4},\ 2 \le p \le \frac{3}{r}, \tag{1.5}$$

then the solution remains smooth on $[0,T]$.

Remark 1.4. Noticing that the classical Riesz transformation is bounded in $\dot{B}^0_{\infty,\infty}$, if we take $q = \infty$ in Theorem 1.1, then the classical Beal-Kato-Majda criterion for the Navier-Stokes equations is obtained; that is, if

$$\int_0^T \|\nabla_h \widetilde{u}(\cdot,t)\|_{\dot{B}^0_{\infty,\infty}} dt < \infty$$

then the solution remains smooth on $[0,T]$.

Remark 1.5. Since (it is proved in [12, 13])

$$L^q(\mathbb{R}^3) = \dot{M}^{q,q}(\mathbb{R}^3) \subset \dot{M}^{p,q}(\mathbb{R}^3), \quad 1 < p \le q < \infty,$$
$$L^{\frac{3}{r}}(\mathbb{R}^3) \subset \dot{M}^{p,\frac{3}{r}}(\mathbb{R}^3) \subset \dot{X}_r(\mathbb{R}^3) \subset \dot{M}^{2,\frac{3}{r}}(\mathbb{R}^3), \quad 2 < p \le \frac{3}{r},\ 0 < r < \frac{3}{2},$$

the result of Theorem 1.2 is an improved version of [27, Theorem 2]. Also we obtain, if

$$\int_0^T \|u_3(\cdot,t)\|_{\dot{X}_r}^{\frac{8}{3-4r}} dt < \infty, \quad \text{with} \quad 0 < r < \frac{3}{4},$$

then the solution remains smooth on $[0,T]$.

2. Preliminaries

In this section, we shall introduce the Littlewood-Paley decomposition theory, and then give some definitions of the homogeneous Besov space, homogeneous Triebel-Lizorkin space, Morrey-Campanato space and multiplier space as well as some relate spaces used throughout this paper. Before this, let us first recall the weak solutions of (1.1)-(1.3):

Let $u_0 \in L^2(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$, a measurable $\mathbb{R}^3$-valued vector u is said to be a weak solution of (1.1)-(1.3) if the following conditions hold:

(1) $u(x,t) \in L^\infty(0,\infty; L^2(\mathbb{R}^3)) \cap L^2(0,\infty; H^1(\mathbb{R}^3))$;
(2) u solves (1.1)-(1.2) in the sense of distributions;
(3) the energy inequality holds; i.e,

$$\|u\|_2^2 + 2\int_0^t \|\triangle u(\cdot,\tau)\|_2^2 d\tau \le \|u_0\|_2^2, \quad 0 \le t \le T.$$

Let us choose a nonnegative radial function $\varphi \in C^\infty(\mathbb{R}^3)$ be supported in the annulus $\{\xi \in \mathbb{R}^3 : \frac{3}{4} \le |\xi| \le \frac{8}{3}\}$, such that $\sum_{l=-\infty}^{\infty} \varphi(2^{-l}\xi) = 1, \forall \xi \ne 0$. For $f \in S'(\mathbb{R}^3)$, the frequency projection operators $\triangle_l$ is defined as

$$\triangle_l f = \mathscr{F}^{-1}(\varphi(2^{-j}\cdot)) * f,$$

where $\mathscr{F}^{-1}(g)$ is the inverse Fourier transform of g. The formal decomposition

$$f = \sum_{l=-\infty}^{\infty} \triangle_l f. \tag{2.1}$$

is called the homogeneous Littlewood-Paley decomposition. Noticing

$$\triangle_l f=\sum_{j=l-1}^{l+1}\triangle_j(\triangle_l f)$$

and using the Young inequality, we have the following class Bernstein inequality:

Lemma 2.1 ([3]). *Let $\alpha\in N$, then for all $1\le p\le q\le\infty$, $\sup_{|\alpha|=k}\|\partial^\alpha\triangle_l f\|_q\le C2^{lk+3l(\frac{1}{p}-\frac{1}{q})}\|\triangle_l f\|_p$. and C is a constant independent of f,l.*

For $s\in\mathbb{R}$ and $(p,q)\in[1,\infty]\times[1,\infty]$, the homogeneous Besov space $\dot B^s_{p,q}$ is defined by

$$\dot S^s_{p,q}=\{f\in Z'(R^3):\|f\|_{\dot S^s_{p,q}}<\infty\},$$

where

$$\|f\|_{\dot B^s_{p,q}}=\begin{cases}\Big(\sum_{j\in z}2^{jsq}\|\triangle_j f(\cdot)\|_p^q\Big)^{1/q}, & 1\le q<\infty,\\ \sup_{j\in z}2^{js}\|\triangle_j f(\cdot)\|_p, & q=\infty.\end{cases}$$

and $Z'(\mathbb{R}^3)$ denote the dual space of

$$Z'(\mathbb{R}^3)=\{f\in S(\mathbb{R}^3):D^\alpha\hat f(0)=0.\ \forall a\in N^3\}.$$

On the other hand, for $s\in\mathbb{R},(p,q)\in[1,\infty)\times[1,\infty]$, and for $s\in\mathbb{R},p=q=\infty$, the homogenous Triebel-Lizorkin space is defined as

$$\dot F^s_{p,q}=\{f\in Z'(\mathbb{R}^3):\|f\|_{\dot F^s_{p,q}}<\infty\},$$

$$\|f\|_{\dot F^s_{p,q}}=\begin{cases}\|(\sum_{j\in z}2^{jsq}|\triangle_j f(\cdot)|^q)^{1/q}\|_p, & 1\le q<\infty,\\ \|\sup_{j\in z}(2^{js}|\triangle_j f(\cdot)|)\|_p, & q=\infty.\end{cases}$$

Notice that by Minkowski inequality, we have the following two imbedding relations:

$$\dot B^s_{p,q}\subset\dot F^s_{p,q},\quad q\le p;$$
$$\dot F^s_{p,q}\subset\dot B^s_{p,q},\quad p\le q.$$

and the following two inclusions:

$$\dot H^s=\dot B^s_{2,2}=\dot F^s_{2,2},\quad L^\infty\subset\dot F^0_{\infty,\infty}=\dot B^0_{\infty,\infty}.$$

We refer to [20] for more properties.

For $1<q\le p<\infty$, the homogeneous Morrey-Campanato space in $\mathbb{R}^3$ is

$$\dot M^{p,q}=\big\{f\in L^q_{\rm loc}(\mathbb{R}^3);\|f\|_{\dot M^{p,q}}=\sup_{x\in\mathbb{R}^3}\sup_{R>0}R^{\frac{3}{p}-\frac{3}{q}}\|f\|_{q(B(x,R))}<\infty\big\},$$

For $1\le p'\le q'<\infty$, we define the homogeneous space

$$\dot N^{p',q'}=\Big\{f\in L^{q'}|f=\sum_{k\in N}g_k,\ \text{where } g_k\in L^{q'}_{comp}(R^3)\text{ and}$$
$$\sum_{k\in N}d_k^{3(\frac{1}{p'}-\frac{1}{q'})}\|g_k\|_{q'}<\infty,\ \text{where } d_k=\operatorname{diam}(\operatorname{supp}g_k)<\infty\Big\}.$$

For $0<\alpha<3/2$, we say that a function belongs to the multiplier spaces $M(\dot H^\alpha,L^2)$ if it maps, by pointwise multiplication, $\dot H^\alpha$ to L^2:

$$\dot X_\alpha:=M(\dot H^\alpha,L^2):=\big\{f\in S';\|f\cdot g\|_{L^2}\le C\|g\|_{\dot H^\alpha},\forall g\in\dot H^\alpha\big\}.$$

Here, $\dot{H}^\alpha$ is the homogeneous Sobolev space of order α,

$$\dot{H}^\alpha = \big\{f \in L^1_{\rm loc}; \|f\|_{L^2} \equiv \Big(\int_{R^3} |\xi|^{2\alpha}|\hat{u}(\xi)|^2\Big)^{1/2} < \infty\big\}.$$

where $L^p(1 \le p \le \infty)$ is the Lebsgue space endowed with norm $\|\cdot\|_p$.

Lemma 2.2 ([4, 12]). *Let $1 \le p' \le q' < \infty$, and p, q such that $\frac{1}{p}+\frac{1}{p'} = 1$, $\frac{1}{q}+\frac{1}{q'} = 1$. Then $\dot{M}^{p,q}$ is the dual space of $\dot{N}^{p',q'}$.*

Lemma 2.3 ([4, 7, 12]). *Let $1 < p' \le q' < 2, m \ge 2$, and $\frac{1}{p} + \frac{1}{p'} = 1$. Denote $\alpha = -\frac{n}{2} + \frac{n}{p} + \frac{n}{m} \in (0, 1]$ Then there exists a constant $C > 0$, such that for any $u \in L^m(\mathbb{R}^n), v \in \dot{H}^\alpha(\mathbb{R}^n)$,*

$$\|u \cdot v\|_{\dot{N}^{p',q'}} \le C\|u\|_{L^m}\|v\|_{\dot{H}^\alpha}.$$

Lemma 2.4 ([13]). *For $0 \le r \le \frac{3}{2}$, let the space $\mathscr{M}(\dot{B}^{r,1}_2 \to L^2)$ be the space of functions which are locally square integrable on $\mathbb{R}^3$ and such that pointwise multiplication with these functions maps boundedly the Besov space $\dot{B}^{r,1}_2(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$. The norm in $\mathscr{M}(\dot{B}^{r,1}_2 \to L^2)$ is given by the operator norm of pointwise multiplication:*

$$\|f\|_{\mathscr{M}(\dot{B}^{r,1}_2 \to L^2)} = \sup\{\|fg\|_2 : \|g\|_{\dot{B}^{r,1}_2} \le 1\}.$$

Then, f belongs to $\mathscr{M}(\dot{B}^{r,1}_2 \to L^2)$ if and only if f belongs to $\dot{M}^{2,\frac{3}{r}}$ (with equivalence of norms).

3. The proof of main results

Proof of Theorem 1.1. Multiplying $(1.1)_1$ by $-\triangle u$, integrating by parts, noting that $\nabla \cdot u = 0$, we have

$$\frac{1}{2}\frac{d}{dt}\int |\nabla u|^2\,dx + \int |\triangle u|^2\,dx = \int [(u\cdot\bigtriangledown)u]\cdot\triangle u\,dx =: I. \tag{3.1}$$

Next we estimate the right-hand side of (3.1), with the help of integration by parts and $-\partial_3 u_3 = \partial_1 u_1 + \partial_2 u_2$, one shows that

$$\begin{aligned}
I =& -\sum_{i,j,k=1}^{3}\int \partial_k u_i \partial_i u_j \partial_k u_j\,dx \\
=& -\sum_{i,j=1}^{2}\sum_{k=1}^{3}\int \partial_k u_i \partial_i u_j \partial_k u_j\,dx - \sum_{i,k=1}^{2}\int \partial_k u_i \partial_i u_3 \partial_k u_3\,dx \\
& - \sum_{j,k=1}^{2}\int \partial_k u_3 \partial_3 u_j \partial_k u_j\,dx - \sum_{k=1}^{3}\int \partial_k u_3 \partial_3 u_3 \partial_k u_3\,dx \\
& - \sum_{i=1}^{2}\int \partial_3 u_i \partial_i u_3 \partial_3 u_3\,dx - \sum_{j=1}^{2}\int \partial_3 u_3 \partial_3 u_j \partial_3 u_j\,dx \\
\le& C\int |\nabla_h \widetilde{u}||\nabla u|^2\,dx.
\end{aligned}$$

Thus, the above inequality implies

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2 + \|\triangle u\|_2^2 \le C\int |\nabla_h \widetilde{u}||\nabla u|^2\,dx. \tag{3.2}$$

Using the Littlewood-Paley decomposition (2.1), $\nabla_h\widetilde{u}$ can be written as

$$\nabla_h\widetilde{u} = \sum_{j<-N}\triangle_j(\nabla_h\widetilde{u}) + \sum_{j=-N}^{N}\triangle_j(\nabla_h\widetilde{u}) + \sum_{j>N}\triangle_j(\nabla_h\widetilde{u}).$$

where N is a positive integer to be chosen later. Substituting this into (3.2), one obtains

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2 + \|\triangle u\|_2^2 \\
&\le C\sum_{j<-N}\int |\triangle_j(\nabla_h\widetilde{u})||\nabla u|^2\,dx + C\sum_{j=-N}^{N}\int |\triangle_j(\nabla_h\widetilde{u})||\nabla u|^2\,dx \\
&\quad + C\sum_{j>N}\int |\triangle_j(\nabla_h\widetilde{u})||\nabla u|^2\,dx \\
&=: K_1 + K_2 + K_3.
\end{aligned} \tag{3.3}$$

For K_i $(i=1,2,3)$, we now give the estimates one by one. For K_1, using the Hölder inequality, the Young inequality and Lemma 2.1, it follows that

$$\begin{aligned}
K_1 &\le C\sum_{j<-N}\|\triangle_j(\nabla_h\widetilde{u})\|_\infty\|\nabla u\|_2^2 \\
&\le C\sum_{j<-N}2^{3j/2}\|\triangle_j(\nabla_h\widetilde{u})\|_2\|\nabla u\|_2^2 \\
&\le C\Big(\sum_{j<-N}2^{3j}\Big)^{1/2}\Big(\sum_{j<-N}\|\triangle_j(\nabla_h\widetilde{u})\|_2^2\Big)^{1/2}\|\nabla u\|_2^2 \\
&\le C2^{-3N/2}\|\nabla u\|_2^3.
\end{aligned} \tag{3.4}$$

Where in the last inequality, we use the fact that for all $s\in\mathbb{R}$, $\dot{H}^s = \dot{B}^s_{2,2}$.

For K_2, by the Hölder inequality and the Young inequality, one has

$$\begin{aligned}
K_2 &= C\int\sum_{j=-N}^{N}|\triangle_j(\nabla_h\widetilde{u})||\nabla u|^2\,dx \\
&\le CN^{\frac{2q-3}{2q}}\int\Big(\sum_{j=-N}^{N}|\triangle_j(\nabla_h\widetilde{u})|^{2q/3}\Big)^{3/(2q)}|\nabla u|^2\,dx \\
&\le CN^{\frac{2q-3}{2q}}\|\nabla_h\widetilde{u}\|_{\dot{F}^0_{q,\frac{2q}{3}}}\|\nabla u\|^2_{\frac{2q}{q-1}} \\
&\le CN^{\frac{2q-3}{2q}}\|\nabla_h\widetilde{u}\|_{\dot{F}^0_{q,\frac{2q}{3}}}\|\nabla u\|_2^{\frac{2q-3}{q}}\|\triangle u\|_2^{3/q} \\
&\le \frac{1}{2}\|\triangle u\|_2^2 + CN\|\nabla_h\widetilde{u}\|_{\dot{F}^0_{q,\frac{2q}{3}}}^{\frac{2q}{2q-3}}\|\nabla u\|_2^2.
\end{aligned} \tag{3.5}$$

where we used the interpolation inequality

$$\|u\|_s \le C\|u\|_2^{\frac{3}{s}-\frac{1}{2}}\|u\|_{\dot{H}^1}^{\frac{3}{2}-\frac{3}{s}},$$

for $2\le s\le 6$.

Finally, using Hölder inequality and Lemma 2.1, K_3 can be estimated as

$$\begin{aligned} K_3 &= C\sum_{j>N}\int |\triangle_j(\nabla_h\widetilde{u})||\nabla u|^2\,dx \\ &\le C\sum_{j>N}\|\triangle_j(\nabla_h\widetilde{u})\|_3\|\nabla u\|_3^2 \\ &\le C\sum_{j>N}2^{\frac{j}{2}}\|\triangle_j(\nabla_h\widetilde{u})\|_2\|\nabla u\|_3^2 \\ &\le C(\sum_{j>N}2^{-j})^{1/2}(\sum_{j>N}2^{2j}\|\triangle_j(\nabla_h\widetilde{u})\|_2^2)^{1/2}\|\nabla u\|_2\|\triangle u\|_2 \\ &\le C2^{-N/2}\|\nabla u\|_2\|\triangle u\|_2^2. \end{aligned} \tag{3.6}$$

Substituting (3.4), (3.5) and (3.6) in (3.3), we obtain

$$\begin{aligned} &\frac{d}{dt}\|\nabla u\|_2^2+\|\triangle u\|_2^2 \\ &\le C2^{-\frac{3}{2}N}\|\nabla u\|_2^3+CN\|\nabla_h\widetilde{u}\|_{\dot F^0_{q,\frac{2q}{3}}}^{\frac{2q}{2q-3}}\|\nabla u\|_2^2+C2^{-N/2}\|\nabla u\|_2\|\triangle u\|_2^2. \end{aligned} \tag{3.7}$$

Now we choose N such that $C2^{-N/2}\|\nabla u\|_2\le\frac{1}{2}$; that is

$$N\ge\frac{\ln(\|\nabla u\|_2^2+e)+\ln C}{\ln 2}+2.$$

Thus (3.7) implies

$$\frac{d}{dt}\|\nabla u\|_2^2\le C+C\|\nabla_h\widetilde{u}\|_{\dot F^0_{q,\frac{2q}{3}}}^p\ln(\|\nabla u\|_2^2+e)\|\nabla u\|_2^2.$$

Taking the Gronwall inequality into consideration, we obtain

$$\ln(\|\nabla u\|_2^2+e)\le C\Big[1+\int_0^T\|\nabla_h\widetilde{u}\|_{\dot F^0_{q,\frac{2q}{3}}}^p(\tau)d\tau\cdot e^{\int_0^T\|\nabla_h\widetilde{u}\|_{\dot F^0_{q,\frac{2q}{3}}}^p(\tau)d\tau}\Big].$$

The proof of Theorem 1.1 is complete under the condition (1.3). □

Proof of Theorem 1.2. Multiplying $(1.1)_1$ by $-\triangle_h u$, integrating by parts, noting that $\nabla\cdot u=0$, we have

$$\frac{1}{2}\frac{d}{dt}\int|\nabla_h u|^2\,dx+\int|\nabla\nabla_h u|^2\,dx=\int[(u\cdot\bigtriangledown)u]\cdot\triangle_h u\,dx=:J. \tag{3.8}$$

Next we estimate the right-hand side of (3.8), with the help of integration by parts and $-\partial_3u_3=\partial_1u_1+\partial_2u_2$, one shows that

$$\begin{aligned} J &= -\sum_{i,j=1}^{3}\sum_{k=1}^{2}\int\partial_ku_i\partial_iu_j\partial_ku_j\,dx \\ &= -\sum_{i,j,k=1}^{2}\int\partial_ku_i\partial_iu_j\partial_ku_j\,dx-\sum_{i,k=1}^{2}\int\partial_ku_i\partial_iu_3\partial_ku_3\,dx \\ &\quad -\sum_{j=1}^{3}\sum_{k=1}^{2}\int\partial_ku_3\partial_3u_j\partial_ku_j\,dx \\ &=: J_1+J_2+J_3. \end{aligned} \tag{3.9}$$

For J_2 and J_3, we obtain

$$|J_2 + J_3| \le C \int |\nabla u_3||\nabla_h u||\nabla u|\,dx. \tag{3.10}$$

J_1 is a sum of eight terms, using the fact $-\partial_3 u_3 = \partial_1 u_1 + \partial_2 u_2$, we can estimate it as

$$\begin{aligned} J_1 &= -\int (\partial_1 u_1 + \partial_2 u_2)[(\partial_1 u_1)^2 - \partial_1 u_1 \partial_2 u_2 + (\partial_2 u_2)^2]\,dx \\ &\quad - \int (\partial_1 u_1 + \partial_2 u_2)[(\partial_2 u_1)^2 + \partial_1 u_2 \partial_2 u_1 + (\partial_1 u_2)^2]\,dx \\ &= \int \partial_3 u_3[(\partial_1 u_1)^2 - \partial_1 u_1 \partial_2 u_2 + (\partial_2 u_2)^2 + (\partial_2 u_1)^2 + \partial_1 u_2 \partial_2 u_1 + (\partial_1 u_2)^2]\,dx \\ &\le C \int |\nabla u_3||\nabla_h u||\nabla u|\,dx. \end{aligned} \tag{3.11}$$

Substituting the estimates (3.9)-(3.11) in (3.8), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla_h u\|_2^2 + \|\nabla\nabla_h u\|_2^2 \le C \int |\nabla u_3||\nabla_h u||\nabla u|\,dx =: L. \tag{3.12}$$

when $2 < p \le \frac{3}{r}$, using Lemmas 2.2 and 2.3, and the Young inequality, we obtain

$$\begin{aligned} L &\le C\|\nabla u_3\|_{\dot{M}^{p,\frac{3}{r}}} \||\nabla u| \cdot |\nabla_h u|\|_{\dot{N}^{p',\frac{3}{3-r}}} \\ &\le C\|\nabla u_3\|_{\dot{M}^{p,\frac{3}{r}}} \|\nabla_h u\|_{\dot{H}^r} \|\nabla u\|_2 \\ &\le C\|\nabla u_3\|_{\dot{M}^{p,\frac{3}{r}}} \|\nabla u\|_{L^2} \|\nabla_h u\|_2^{1-r} \|\nabla\nabla_h u\|_2^r \\ &\le \frac{1}{2}\|\nabla\nabla_h u\|_2^2 + C\|\nabla u_3\|_{\dot{M}^{p,\frac{3}{r}}}^{\frac{2}{2-r}} \|\nabla u\|_2^2. \end{aligned} \tag{3.13}$$

where we used the inequality

$$\|f\|_{\dot{H}^r} = \||\xi|^r \hat{f}\|_2 = (\int |\xi|^{2r}|\hat{f}|^{2r}|\hat{f}|^{2-2r} d\xi)^{1/2} \le \|f\|_2^{1-r}\|\nabla f\|_2^r,$$

with $0 < r \le 1$.

In the case $p = 2$, using Hölders inequality, Lemma 2.4, and the Young inequality, we can estimate L as

$$\begin{aligned} L &\le C\||\nabla u_3| \cdot |\nabla_h u|\|_2 \|\nabla u\|_2 \\ &\le C\|\nabla u_3\|_{\dot{M}^{2,\frac{3}{r}}} \|\nabla_h u\|_{\dot{B}_2^{r,1}} \|\nabla u\|_2 \\ &\le C\|\nabla u_3\|_{\dot{M}^{2,\frac{3}{r}}} \|\nabla_h u\|_2^{1-r} \|\nabla\nabla_h u\|_2^r \|\nabla u\|_2 \\ &\le \frac{1}{2}\|\nabla\nabla_h u\|_2^2 + C\|\nabla u_3\|_{\dot{M}^{2,\frac{3}{r}}}^{\frac{2}{2-r}} \|\nabla u\|_2^2. \end{aligned} \tag{3.14}$$

where we used the following interpolation inequality [14]: for $0 \le r \le 1$, $\|f\|_{\dot{B}_2^{r,1}} \le \|f\|_2^{1-r}\|\nabla f\|_2^r$.

Now, gathering (3.13) and (3.14) together and substituting into (3.12), we obtain

$$\frac{d}{dt}\|\nabla_h u\|_2^2 + \|\nabla\nabla_h u\|_2^2 \le C\|\nabla u_3\|_{\dot{M}^{p,\frac{3}{r}}}^{\frac{2}{2-r}} \|\nabla u\|_2^2. \tag{3.15}$$

Multiplying $(1.1)_1$ by $-\triangle u$, integrating by parts, noting that $\nabla\cdot u=0$, we have (see [25])

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2+\|\triangle u\|_2^2&=\int[(u\cdot\bigtriangledown)u]\cdot\triangle u\,dx\\&\le C\int|\nabla_h u||\nabla u|^2\,dx\\&\le C\|\nabla_h u\|_2\|\nabla u\|_4^2\\&\le C\|\nabla_h u\|_2\|\nabla u\|_2^{1/2}\|\nabla u\|_6^{\frac{3}{2}}\\&\le C\|\nabla_h u\|_2\|\nabla u\|_2^{1/2}\|\nabla\nabla_h u\|_2\|\triangle u\|_2^{1/2}.\end{aligned}$$

Integrating, with respect to t, yields

$$\begin{aligned}&\frac{1}{2}\|\nabla u(t)\|_2^2+\int_0^t\|\triangle u(\tau)\|_2^2d\tau\\&\le\frac{1}{2}\|\nabla u_0\|_2^2+C\sup_{0\le\tau\le t}\|\nabla_h u(\tau)\|_2(\int_0^t\|\nabla u(\tau)\|_2^2d\tau)^{1/4}\\&\quad\times\Big(\int_0^t\|\nabla\nabla_h u(\tau)\|_2^2d\tau\Big)^{1/2}\Big(\int_0^t\|\triangle u(\tau)\|_2^2d\tau\Big)^{1/4}.\end{aligned}\tag{3.16}$$

Substituting (3.15) in (3.16), using Hölders inequality and the Young inequality, we obtain

$$\begin{aligned}&\frac{1}{2}\|\nabla u(t)\|_2^2+\int_0^t\|\triangle u(\tau)\|_2^2d\tau\\&\le\frac{1}{2}\|\nabla u_0\|_2^2+(C+C\int_0^t\|\nabla u_3\|_{\dot M^{p,\frac{3}{r}}}^{\frac{2}{2-r}}\|\nabla u(\tau)\|_2^2d\tau)\Big(\int_0^t\|\triangle u(\tau)\|_2^2d\tau\Big)^{1/4}\\&\le C+C\Big(\int_0^t\|\nabla u_3\|_{\dot M^{p,\frac{3}{r}}}^{\frac{2}{2-r}}\|\nabla u(\tau)\|_2^{\frac{3}{2}}\|\nabla u(\tau)\|_2^{1/2}d\tau\Big)^{4/3}\\&\quad+\frac{1}{2}\int_0^t\|\triangle u(\tau)\|_2^2d\tau\\&\le C+C\Big(\int_0^t\|\nabla u_3\|_{\dot M^{p,\frac{3}{r}}}^{\frac{8}{3(2-r)}}\|\nabla u(\tau)\|_2^2d\tau\Big)\Big(\int_0^t\|\nabla u(\tau)\|_2^2d\tau\Big)^{1/3}\\&\quad+\frac{1}{2}\int_0^t\|\triangle u(\tau)\|_2^2d\tau\\&\le C+C\int_0^t\|\nabla u_3\|_{\dot M^{p,\frac{3}{r}}}^{\frac{8}{3(2-r)}}\|\nabla u(\tau)\|_2^2d\tau+\frac{1}{2}\int_0^t\|\triangle u(\tau)\|_2^2d\tau.\end{aligned}\tag{3.17}$$

Absorbing the last term into the left hand side, applying the Gronwall inequality and combining with the standard continuation argument, we conclude that the solutions u can be extended beyond $t=T$ provided that $\nabla u_3\in L^{\frac{8}{3(2-r)}}(0,T;\dot M^{p,\frac{3}{r}}(\mathbb{R}^3))$. This completes the proof of Theorem 1.2. □

Proof of Theorem 1.3. We start from (3.9), we can estimate J_2 and J_3 as

$$|J_2+J_3|\le C\int|u_3||\nabla u||\nabla\nabla_h u|\,dx.\tag{3.18}$$

From (3.11), we find that

$$J_1 \le C \int |u_3||\nabla u||\nabla\nabla_h u|\,dx. \tag{3.19}$$

Combining (3.9),(3.17), (3.18) with (3.8), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla_h u\|_2^2 + \|\nabla\nabla_h u\|_2^2 \le C\int |u_3||\nabla u||\nabla\nabla_h u|\,dx =: V. \tag{3.20}$$

When $2 < p \le \frac{3}{r}$, similarly as in the proof of L in (3.13), we obtain

$$\begin{aligned} V &\le C\|u_3\|_{\dot M^{p,\frac{3}{r}}} \||\nabla u|\cdot|\nabla\nabla_h u|\|_{\dot N^{p',\frac{3}{3-r}}} \\ &\le C\|u_3\|_{\dot M^{p,\frac{3}{r}}}\|\nabla u\|_{\dot H^r}\|\nabla\nabla_h u\|_2 \\ &\le C\|u_3\|_{\dot M^{p,\frac{3}{r}}}\|\nabla\nabla_h u\|_{L^2}\|\nabla u\|_2^{1-r}\|\triangle u\|_2^r \\ &\le \frac{1}{2}\|\nabla\nabla_h u\|_2^2 + C\|u_3\|^2_{\dot M^{p,\frac{3}{r}}}\|\nabla u\|_2^{2(1-r)}\|\triangle u\|_2^{2r}. \end{aligned} \tag{3.21}$$

When $p = 2$, similarly as in the proof of L in (3.14), we have

$$\begin{aligned} V &\le C\||u_3|\cdot|\nabla u|\|_2\|\nabla\nabla_h u\|_2 \\ &\le C\|u_3\|_{\dot M^{2,\frac{3}{r}}}\|\nabla u\|_{\dot B_2^{r,1}}\|\nabla\nabla_h u\|_2 \\ &\le C\|u_3\|_{\dot M^{2,\frac{3}{r}}}\|\nabla u\|_2^{1-r}\|\triangle u\|_2^r\|\nabla\nabla_h u\|_2 \\ &\le \frac{1}{2}\|\nabla\nabla_h u\|_2^2 + C\|u_3\|^2_{\dot M^{2,\frac{3}{r}}}\|\nabla u\|_2^{2(1-r)}\|\triangle u\|_2^{2r}. \end{aligned} \tag{3.22}$$

Substituting (3.21) and (3.22) in (3.20), we find that

$$\frac{d}{dt}\|\nabla_h u\|_2^2 + \|\nabla\nabla_h u\|_2^2 \le C\|u_3\|^2_{\dot M^{2,\frac{3}{r}}}\|\nabla u\|_2^{2(1-r)}\|\triangle u\|_2^{2r}. \tag{3.23}$$

Substituting (3.23) in (3.16), we obtain

$$\begin{aligned} &\frac{1}{2}\|\nabla u(t)\|_2^2 + \int_0^t \|\triangle u(\tau)\|_2^2 d\tau \\ &\le \frac{1}{2}\|\nabla u_0\|_2^2 + \Big(C + C\int_0^t \|u_3\|^2_{\dot M^{p,\frac{3}{r}}}\|\nabla u(\tau)\|_2^{2(1-r)}\|\triangle u(\tau)\|_2^{2r}d\tau\Big) \\ &\quad\times\Big(\int_0^t \|\triangle u(\tau)\|_2^2 d\tau\Big)^{1/4} \\ &\le C + \frac{1}{4}\int_0^t \|\triangle u(\tau)\|_2^2 d\tau + C\Big(\int_0^t \|u_3\|^2_{\dot M^{p,\frac{3}{r}}}\|\nabla u(\tau)\|_2^{2(1-r)}\|\triangle u(\tau)\|_2^{2r}d\tau\Big)^{4/3} \\ &\le C + \frac{1}{4}\int_0^t \|\triangle u(\tau)\|_2^2 d\tau + C\Big(\int_0^t \|\triangle u(\tau)\|_2^2 d\tau\Big)^{4r/3} \\ &\quad\times\Big(\int_0^t \|u_3\|^{\frac{2}{1-r}}_{\dot M^{p,\frac{3}{r}}}\|\nabla u(\tau)\|_2^2 d\tau\Big)^{\frac{4(1-r)}{3}} \\ &\le C + \frac{1}{2}\int_0^t \|\triangle u(\tau)\|_2^2 d\tau + C\Big(\int_0^t \|u_3\|^{\frac{2}{1-r}}_{\dot M^{p,\frac{3}{r}}}\|\nabla u(\tau)\|_2^{\frac{3-4r}{2(1-r)}}\|\nabla u(\tau)\|_2^{\frac{1}{2(1-r)}}d\tau\Big)^{\frac{4(1-r)}{3-4r}} \\ &\le C + \frac{1}{2}\int_0^t \|\triangle u(\tau)\|_2^2 d\tau + C\Big(\int_0^t \|u_3\|^{\frac{8}{3-4r}}_{\dot M^{p,\frac{3}{r}}}\|\nabla u(\tau)\|_2^2 d\tau\Big)\Big(\int_0^t \|\nabla u(\tau)\|_2^2 d\tau\Big)^{\frac{1}{3-4r}} \\ &\le C + \frac{1}{2}\int_0^t \|\triangle u(\tau)\|_2^2 d\tau + C\int_0^t \|u_3\|^{\frac{8}{3-4r}}_{\dot M^{p,\frac{3}{r}}}\|\nabla u(\tau)\|_2^2 d\tau. \end{aligned}$$

By a similar argument as in the proof of Theorem 1.2, provided that $u_3 \in L^{\frac{8}{3-4r}}(0,T;\dot{M}^{p,\frac{3}{r}}(\mathbb{R}^3))$, we complete the proof of Theorem 1.3. □

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