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POHOZAEV-TYPE INEQUALITIES AND NONEXISTENCE RESULTS FOR NON C^2 SOLUTIONS OF $p(x)$-LAPLACIAN EQUATIONS

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Abstract. In this article we obtain a Pohozaev-type inequality for Sobolev spaces with variable exponents. This inequality is used for proving the nonexistence of nontrivial weak solutions for the Dirichlet problem

$$\begin{gathered} -\Delta_{p(x)} u = |u|^{q(x)-2}u, \quad x \in \Omega \\ u(x) = 0, \quad x \in \partial\Omega, \end{gathered}$$

with non-standard growth. Our results extend those obtained by Ôtani [16].

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. The domain Ω is said to be *star shaped* (respectively *strictly star shaped*) if $(x\cdot\nu(x)) \geqslant 0$ (respectively if $(x\cdot\nu(x)) \geqslant \rho > 0$) holds for all $x \in \partial\Omega$ with a suitable choice of the origin, where $\nu(x) = (\nu_1(x), \dots, \nu_N(x))$ denotes the outward unit normal at $x \in \partial\Omega$. Consider the problem

$$\begin{gathered} -\Delta_{p(x)} u = f(u), \quad x \in \Omega \\ u(x) = 0, \quad x \in \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Delta_{p(x)} u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$, and f is a non-linear function.

In [4], to obtain nonexistence results for (1.1) for star shaped domains Ω, Pohozaev-type identities are stated and applied to the case in which f does not depend on $p(x)$ and $u \in C^2(\Omega)$. For $f(u) = |u|^{q-2}u$, $1 < q < \infty$, $2 < p < \infty$, and p, q constants, nontrivial solutions of (1.1) do not belong to $C^2(\Omega) \cap C(\overline{\Omega})$, see [11]. The arguments in [11, Proposition 1.1] are easily extended to the case of Sobolev Spaces with variable exponents, so that, in general, results in [4] can not be applied when $\nabla u(x) = 0$, not even for solutions in $W^{2,p(x)}(\Omega) \cap W^{1,p(x)}(\overline{\Omega})$. In this way, solutions to the problem

$$\begin{gathered} -\Delta_{p(x)} u = |u|^{q(x)-2}u, \quad x \in \Omega \\ u(x) = 0, \quad x \in \partial\Omega, \end{gathered} \tag{1.2}$$

in general do not belong to $C^2(\Omega)$.

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The results in the present paper generalize to Sobolev Spaces with Variable Exponents $p(x)$ the work of Ôtani [16], which hold for constant exponents p. The generalization is in the sense that the spaces with p constant are contained in the spaces with variable exponent, more precisely, the classical Lebesgue space $L^p(\Omega)$ coincides with the modular space $(L^0(\Omega))_{\rho_p}$, [3, Example 2.1.8, p. 25]. As a consequence, the Pohozaev-type inequality (3.37), Theorem 3.2, in this paper:

$$\begin{aligned}
&-\int_\Omega \frac{N}{q(x)}|u|^{q(x)}\,dx + \int_\Omega \frac{N-p(x)}{p(x)}|\nabla u|^{p(x)}\,dx \\
&+\int_\Omega x\cdot\nabla p(x)\frac{|\nabla u|^{p(x)}}{p(x)^2}\log\left(e^{-1}|\nabla u|^{p(x)}\right)dx \\
&-\int_\Omega x\cdot\nabla q(x)\frac{|u|^{q(x)}}{q(x)^2}\log\left(e^{-1}|u|^{q(x)}\right)dx + R \le 0,
\end{aligned}$$

holds for p constant in the corresponding Sobolev spaces.

In [16], Ôtani studied the Existence, Regularity and Nonexistence of (1.2). The existence of solutions for (1.2) is proved in [7] and [15]. In [15], the authors studied the existence for the case in which the embbeding from $W_0^{1,p(\cdot)}(\Omega)$ to $L^{q(\cdot)}(\Omega)$ is compact. In the same paper, the authors include the study of the case in which the embbeding from $W_0^{1,p(\cdot)}(\Omega)$ to $L^{q(\cdot)}(\Omega)$ is not compact, provided that certain functional inequality holds. On the other hand, the regularity of solutions of problem (1.2) is studied in [5, Theorem 1.2].

To the best of my knowledge, many problems related to Pohozaev-type inequalities and Sobolev Spaces with Variable Exponents remain unstudied, among them, for instance, problem (1.2) in general exterior domains. Hashimoto and Ôtani [11] studied this problem for p constant in an exterior domain $\Omega = \mathbb{R}^N \setminus \bar{\Omega}_0$ where $\bar{\Omega}_0$ is bounded and starshaped.

Many problems related to the $p(x)$-Laplacian remain open, for instance, a characterization of the solutions in dimension one of the eigenvalue problem

$$\begin{gathered}
-\Delta_{p(x)}u = \lambda|u|^{q(x)-2}u, \quad x\in\Omega \\
u = 0, \quad x\in\partial\Omega,
\end{gathered}$$

where λ is an eigenvalue defined by a Rayleigh quotient equation (see for instance [14, equation (2.1), p. 273]). The well known case, p constant [14], has characteristic solutions in terms of $\sin_p(x)$, $\cos_p(x)$ functions, which are generalizations of the ordinary sine and cosine functions, i. e., the solutions of the one dimensional eigenvalue problem for $p = 2$. For $p = p(x)$, the problem seems to be much harder to solve than the constant case.

The reader is referred to [9] for review of applications of $p(x)$-Laplacian equations to ranging from Image Processing to Modeling of Electrorheological fluids.

This paper is organized as follows. In section 2 some necessary background in Sobolev Spaces with Variable Exponents is provided including some required Compact Embedding results. In section 3, Theorem 3.2 we state and prove a Pohozaev-type inequality. In Section 4, we prove some nonexistence results of nontrivial weak solutions of problem (1.2) as a consequence of Pohozaev-type inequality.

2. Variable exponent setting

We recall some definitions and basic properties of the Lebesgue-Sobolev spaces with variable exponent $L^{p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$. For any $p \in C(\overline{\Omega})$, the space of continuous functions in $\overline{\Omega}$, we define

$$p^+ = \sup_{x\in\Omega} p(x) \quad \text{and} \quad p^- = \inf_{x\in\Omega} p(x).$$

The Lebesgue Space with variable exponent for measurable real-valued functions is defined as the set

$$L^{p(\cdot)}(\Omega) = \{u : \int_\Omega |u(x)|^{p(x)}\,dx < \infty\},$$

endowed with the *Luxemburg norm*

$$\|u\|_{p(\cdot)} = \inf\{\mu > 0;\ \int_\Omega |\frac{u(x)}{\mu}|^{p(x)}\,dx \le 1\},$$

which is a separable and reflexive Banach space if $1 < p^- \leqslant p^+ < \infty$. For the basic properties of the Lebesgue Spaces with Variable Exponents we refer to [3] and [12].

Let $L^{p'(\cdot)}(\Omega)$ be the conjugate space of $L^{p(\cdot)}(\Omega)$, obtained by conjugating the exponent pointwise; that is, $1/p(x) + 1/p'(x) = 1$, [12, Corollary 2.7]. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$ the following Hölder type inequality is valid

$$|\int_\Omega uv\,dx| \le \big(\frac{1}{p^-} + \frac{1}{p'^-}\big)\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)}\,. \tag{2.1}$$

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the $p(\cdot)$-*modular* of the $L^{p(\cdot)}(\Omega)$ space, which is the mapping $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(\cdot)}(u) = \int_\Omega |u|^{p(x)}\,dx.$$

If a sequence (u_n), and u are in $L^{p(\cdot)}(\Omega)$ then the following relations hold

$$\|u\|_{p(\cdot)} < 1\ (= 1; > 1) \iff \rho_{p(\cdot)}(u) < 1\ (= 1; > 1) \tag{2.2}$$

$$\|u\|_{p(\cdot)} > 1 \Rightarrow \|u\|_{p(\cdot)}^{p^-} \le \rho_{p(\cdot)}(u) \le \|u\|_{p(\cdot)}^{p^+} \tag{2.3}$$

$$\|u\|_{p(\cdot)} < 1 \Rightarrow \|u\|_{p(\cdot)}^{p^+} \le \rho_{p(\cdot)}(u) \le \|u\|_{p(\cdot)}^{p^-} \tag{2.4}$$

$$\|u_n - u\|_{p(\cdot)} \to 0 \iff \rho_{p(\cdot)}(u_n - u) \to 0, \tag{2.5}$$

since $p^+ < \infty$. For a proof of these facts see [12].

The set $W_0^{1,p(x)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\| = \|\nabla u\|_{p(x)}.$$

The space $(W_0^{1,p(x)}(\Omega), \|\cdot\|_{p(x)})$ is a separable and reflexive Banach space if $1 < p^- \leqslant p^+ < \infty$. We note that if $q \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$, then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is continuous, where $p^*(x) = Np(x)/(N - p(x))$ if $p(x) < N$ or $p^*(x) = +\infty$ if $p(x) \ge N$ [12, Theorem 3.9 and 3.3] (see also [6, Theorem 1.3 and 1.1]).

The bounded variable exponent p is said to be Log-Hölder continuous if there is a constant $C > 0$ such that

$$|p(x) - p(y)| \leqslant \frac{C}{-\log(|x - y|)} \tag{2.6}$$

for all $x, y \in \mathbb{R}^N$, such that $|x - y| \leq 1/2$. A bounded exponent p is Log-Hölder continuous in Ω if and only if there exists a constant $C > 0$ such that

$$|B|^{p_B^- - p_B^+} \leq C \tag{2.7}$$

for every ball $B \subset \Omega$ [3, Lemma 4.1.6, page 101], where $|B|$ is the Lebesgue measure of B. Under the Log-Hölder condition smooth functions are dense in Sobolev Spaces with Variable Exponents [3, Proposition 11.2.3, page 346].

Finally, the compact embedding results, as many other facts, are a very delicate and interesting matters in spaces with variable exponents. For instance, in [15, prop 3.1] is shown that, for certain exponents with $p^*(x) > q(x) > p^*(x) - \epsilon$ (in our notation) with x in some subset of Ω, the embedding from $W_0^{1,p(\cdot)}(\Omega)$ to $L^{q(\cdot)}(\Omega)$ is not compact. On the other hand, if $q(x) = p^*(x)$ at some point $x \in \Omega$, it is known that the embedding is compact in $\mathbb{R}^N$ (see [3, Theorem 8.4.6] and references therein). In this paper, we will use [15, Proposition 3.3] which, in our notation, can be stated as the following proposition.

Proposition 2.1 (Mizuta et al [15])**.** *Let $p(\cdot)$ satisfying the log-Hölder condition on the open and bounded set $\Omega \subset \mathbb{R}^N$. Suppose that $\partial\Omega \in C^1$ or Ω satisfies the cone condition, and $p^+ < N$. Let $q(\cdot)$ be a variable exponent on Ω such that $1 \leqslant q^-$ and*

$$\operatorname{ess\,inf}_{x\in\Omega} \big(p^*(x) - q(x)\big) > 0. \tag{2.8}$$

Then $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{q(\cdot)}(\Omega)$, i. e. $W_0^{1,p(\cdot)}(\Omega)$ is compactly embedded in $L^{q(\cdot)}(\Omega)$.

For a definition of the cone condition used in the above theorem, see [19, p. 159]. In the next section we also require the following Lemma.

Lemma 2.2. *Let $1 < p(x) < q^- < q(x) < q^+ < \infty$ a.e. in Ω. Assume that $\|u_n\|_r < C$ for $1 \leqslant r < \infty$ and $u_n \to u$ as $n \to \infty$ in $L^{p(\cdot)}(\Omega)$. Then $u_n \to u$ as $n \to \infty$ in $L^{q(\cdot)}(\Omega)$, up to a subsequence.*

Proof. Given (2.2) to (2.5), it is enough to show that $\rho_{q(\cdot)}(u_n - u) \to 0$ as $n \to \infty$. We have

$$\rho_{q(\cdot)}(u_n - u) = \int_\Omega |u_n - u|^{q(x)}\,dx \leqslant \int_\Omega |u_n - u|^{q^-}\,dx, \tag{2.9}$$

for n big enough. In deed, the inequality holds since convergence in $L^p(x)(\Omega)$ implies convergence in $L^{p^-}(\Omega)$; i.e., $\|u_n - u\|_{L^{p^-}} \to 0$. So that, up to a subsequence, $|u_n - u| \to 0$ *a.e.* in Ω by [2, Théorème IV.9]. In this way, there exist N_o such that if $n > N_o$, $|u_n - u| < 1$, a.e. in Ω. Therefore, up to a subsequence, $|u_n - u|^{q(x)} < |u_n - u|^{q^-}$, a.e. in Ω, so that the inequality (2.9) holds. Hence, for some $\theta \in (0, 1)$ satisfying $1/q^- = \theta/p^- + (1 - \theta)/q^+$

$$\rho_{q(\cdot)}(u_n - u) \leqslant \Big(\int_\Omega |u_n - u|^{p^-}\,dx\Big)^{\theta q^-/p^-} \Big(\int_\Omega |u_n - u|^{q^+}\,dx\Big)^{(1-\theta)q^-/q^+}.$$

Using the fact that $u_n \to u$ in $L^{p^-}(\Omega)$ and [1, Theorem 2.11] it follows that

$$\rho_{q(\cdot)}(u_n - u) \leqslant C\Big(\int_\Omega |u_n - u|^{p^-}\,dx\Big)^{\theta q^-/p^-} \to 0, \quad \text{as } n \to \infty, \tag{2.10}$$

and the proof is complete. □

3. Pohozaev-type inequalitiy

In this section, we state a Pohozaev-type inequality for weak solutions u (defined in (3.3) below) belonging to the class $\mathcal{P}$ defined as

$$\mathcal{P} = \big\{u \in \big(W_0^{1,p(\cdot)} \cap L^{q(\cdot)}\big)(\Omega) : x_i|u|^{q(x)-2}u \in L^{p'(\cdot)}(\Omega),\ i = 1, 2, \dots, N\big\} \tag{3.1}$$

where $p'(x) = p(x)/(p(x) - 1)$ and $p^+ < N$. To this aim, we employ the techniques introduced by Hashimoto and Ôtani in [11, 10, 16], but within the framework of spaces with variable exponent, which, as the reader may notice, require much more careful estimations than those in the constant case.

Let $g_n(\cdot) \in C^1(\mathbb{R})$ be the cutoff functions such that $0 \leqslant g_n'(s) \leqslant 1$, $s \in \mathbb{R}$ and

$$g_n(s) = \begin{cases} s, & |s| \leqslant n, \\ (n+1)\operatorname{sign} s, & |s| \geqslant n+1. \end{cases} \tag{3.2}$$

Let u be a weak solution of (1.2), i.e. a function $u \in \big(W_0^{1,p(\cdot)} \cap L^{q(\cdot)}\big)(\Omega)$, which satisfies

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla\phi\, dx = \int_\Omega |u|^{q(x)-2}u\,\phi\, dx \text{ for all } \phi \in \Big(W_0^{1,p(\cdot)} \cap L^{q(\cdot)}\Big)(\Omega), \tag{3.3}$$

and set $u_n = g_n(u)$ then $|u_n|^{r-2}u_n \in \big(W_0^{1,p(\cdot)} \cap L^\infty\big)(\Omega)$ for $r \in [2, \infty)$. Consider now the approximate problem

$$\begin{gathered} |w_n|^{q(x)-2}w_n - \Delta_{p(x)}w_n = 2|u_n|^{q(x)-2}u_n, \quad \text{in } \Omega, \\ w_n = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{3.4}$$

Since $u_n \in L^\infty(\Omega)$, there exists a sequence $\{v_n^\varepsilon\} \subset C_0^\infty(\Omega)$ satisfying

$$\|v_n^\varepsilon\|_{L^\infty(\Omega)} \leqslant C_o, \quad \text{for all } \varepsilon \in (0, 1), \tag{3.5}$$

$$v_n^\varepsilon \to 2|u_n|^{q(x)-2}u_n, \quad \text{strongly in } L^{r(\cdot)}(\Omega) \text{ as } \varepsilon \to 0, \text{ for all } r \in [1, \infty). \tag{3.6}$$

In turn, we require another approximate equation for $(E)_n$ given by

$$\begin{gathered} |w_n^\varepsilon|^{q(x)-2}w_n^\varepsilon + A_\varepsilon w_n^\varepsilon = v_n^\varepsilon, \quad \text{in } \Omega \\ w_n^\varepsilon = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.7}$$

where $A_\varepsilon u(x) = -\operatorname{div}\{(|\nabla u(x)|^2 + \varepsilon)^{(p(x)-2)/2}\nabla u(x)\}$ and $\varepsilon > 0$. It is possible to show that (3.4) and (3.7) have unique solutions and that (3.7) and (3.4) provide good approximations for (3.4) and (1.2), respectively. This fact is stated in the following lemma.

Lemma 3.1. *Let $p(\cdot)$ satisfying the log-Hölder condition on the open and bounded set $\Omega \subset \mathbb{R}^N$. Suppose that $\partial\Omega \in C^1$ or Ω satisfies the cone condition and $p^+ < N$. Then the following statements hold true:*

(i) *For each $\varepsilon \in (0, 1)$ and $n \in \mathbb{N}$, there exists a unique solution $w_n^\varepsilon \in C^2(\overline{\Omega})$ of (3.7).*

(ii) *For each $n \in \mathbb{N}$ there exists a unique solution $w_n \in C^{1,\alpha}(\overline{\Omega}) \cap W_0^{1,p(x)}(\Omega)$, $0 < \alpha < 1$, of (3.4).*

(iii) *w_n^ε converges to w_n as $\varepsilon \to 0$ in the following sense:*

$$\int_\Omega |\nabla w_n^\varepsilon|^{p(x)}\, dx \to \int_\Omega |\nabla w_n|^{p(x)}\, dx \quad \text{as } \varepsilon \to 0, \tag{3.8}$$

$$w_n^\varepsilon \to w_n \quad \textit{strongly in } L^{r(x)}(\Omega), \tag{3.9}$$

for $r(\cdot)$ *such that* $1 < r^- < r(x) < r^+$ *a.e. in* Ω *and* $p^+ < N$.

(iv) w_n *converges to* u *as* $n \to \infty$ *in the following sense:*

$$\int_\Omega |\nabla w_n|^{p(x)}\,dx \to \int_\Omega |\nabla u|^{p(x)}\,dx \quad \textit{as } n \to \infty, \tag{3.10}$$

$$\int_\Omega |w_n|^{q(x)}\,dx \to \int_\Omega |u|^{q(x)}\,dx, \quad \textit{as } n \to \infty, \tag{3.11}$$

Proof. (i) Since $u_n \in L^\infty(\Omega)$, there exists a sequence $\{v_n^\varepsilon\} \subset C_0^\infty(\Omega)$ satisfying

$$\|v_n^\varepsilon\|_{L^\infty(\Omega)} \leqslant C_o, \quad \text{for all } \varepsilon \in (0,1), \tag{3.12}$$

$$v_n^\varepsilon \to 2|u_n|^{q(x)-2}u_n, \quad \text{strongly in } L^r(\Omega) \text{ as } \varepsilon \to 0, \text{ for all } r \in [1,\infty). \tag{3.13}$$

Given that v_n^ε belongs to $C^2(\overline{\Omega})$ and since $A_\varepsilon u$ is elliptic, [19, Theorem 15.10] guarantees the existence of a unique solution $w_n^\varepsilon \in C^2(\overline{\Omega})$ of (3.7).

(ii) Set

$$F(z) = \int_\Omega \frac{|\nabla z|^{p(x)}}{p(x)}\,dx + \int_\Omega \frac{|z|^{q(x)}}{q(x)}\,dx - 2\int_\Omega |u_n|^{q(x)-2}u_n z\,dx,$$

so that $F(z)$ is strictly convex, coercive and Fréchet differentiable on

$$\Big(W_0^{1,p(x)} \cap L^{q(x)}\Big)(\Omega).$$

Now, if $z_n \rightharpoonup z_o$ weakly in $\big(W_0^{1,p(x)} \cap L^{q(x)}\big)(\Omega)$, then, since $p \in \mathcal{P}(\Omega,\mu)$ (for definitions see [3]), the modulars $\int_\Omega |\nabla z|^{p(x)}/p(x)\,dx$ and $\int_\Omega |z|^{q(x)}/q(x)\,dx$ are sequentially weakly lower semicontinuous [3, Theorem 3.2.9] and $\int_\Omega |u_n|^{q(x)-2}u_n z\,dx \in (L^{q(x)}(\Omega))^*$. We conclude that $\liminf_{n\to\infty} F(z_n) \geqslant F(z_o)$. Since F is bounded below, there exists $w_n \in \big(W_0^{1,p(x)} \cap L^{q(x)}\big)(\Omega)$ where F attains its minimum, and since F is Fréchet differentiable $\langle F'(w_n), \phi\rangle = 0$ for all $\phi \in \big(W_0^{1,p(x)} \cap L^{q(x)}\big)(\Omega)$, i.e. w_n solves (3.7) in the weak sense and the uniqueness follows from the strict convexity of $F(z)$. Multiplying (3.7) by $|w_n|^{r-2}w_n$ ($r \geqslant 2$ constant), using Young's ε-inequality with $\varepsilon = 1/2$, and considering that $|u_n|^{q(x)-2}u_n$ belongs to $L^\infty(\Omega)$, we obtain

$$\begin{aligned}
&\int_\Omega |w_n|^{q(x)+r-2}\,dx + (r-1)\int_\Omega |w_n|^{p(x)}|w_n|^{r-2}\,dx \\
&\leqslant \int_\Omega 2(n+1)^{q(x)-1}|w_n|^{r-1}\,dx \\
&\leqslant \frac{1}{2}\int_\Omega |w_n|^{q(x)+r-2}\,dx + 2^{(q^+ + 2r-3)/(q^- -1)}(n+1)^{q^+ + r-2}|\Omega|.
\end{aligned} \tag{3.14}$$

So, by [8, Theorem 1.3, p. 427]

$$\|w_n\|_{L^{q(x)+r-2}}^{q^\pm + r-2} \leqslant 2\cdot 2^{(q^+ + 2r-3)/(q^- -1)}(n+1)^{q^+ + r-2}|\Omega|,$$

where

$$q^\pm = \begin{cases} q^+, & \text{if } \|w_n\|_{L^{q(x)+r-2}} < 1,\\ q^-, & \text{if } \|w_n\|_{L^{q(x)+r-2}} > 1.\end{cases}$$

Hence we can obtain an a priori bound for $\|w_n\|_{L^{q(x)+r-2}}$ independent of r. Letting $r \to \infty$ we get an L^∞-estimate for w_n. Therefore, using [5, Theorem 1.2, p. 400], we conclude $w_n \in C^{1,\alpha}(\overline{\Omega})$.

(iii) With a similar argumentation as in (ii) we obtain

$$\|w_n^\varepsilon\|_{L^\infty(\Omega)} \leqslant C_n \quad \text{for all } \varepsilon > 0. \tag{3.15}$$

Multiply (3.7) by w_n^ϵ to obtain

$$\int_\Omega |w_n^\varepsilon|^{q(x)}\,dx + \int_\Omega (|\nabla w_n^\varepsilon|^2 + \varepsilon)^{(p(x)-2)/2} |\nabla w_n^\varepsilon|^2\,dx = \int_\Omega v_n^\varepsilon w_n^\varepsilon\,dx.$$

On the other hand, note that

$$\begin{aligned}\int_\Omega |\nabla w_n^\varepsilon|^{p(x)}\,dx &= \int_\Omega (|\nabla w_n^\varepsilon|^2)^{(p(x)-2)/2} |\nabla w_n^\varepsilon|^2\,dx \\ &\leqslant \int_\Omega (|\nabla w_n^\varepsilon|^2 + \varepsilon)^{(p(x)-2)/2} |\nabla w_n^\varepsilon|^2\,dx.\end{aligned}$$

Hence, it follows that

$$\int_\Omega |\nabla w_n^\varepsilon|^{p(x)}\,dx \leqslant \int_\Omega v_n^\varepsilon w_n^\varepsilon\,dx.$$

Next, use Young's inequality and the fact that $q(x), q'(x) > 1$ to obtain

$$\int_\Omega |\nabla w_n^\varepsilon|^{p(x)}\,dx \leqslant \int_\Omega |v_n^\varepsilon|^{q'(x)}\,dx + \int_\Omega |w_n^\varepsilon|^{q(x)}\,dx.$$

Therefore, by (3.15) and the fact that $v_n \in C_0^\infty(\Omega)$, we obtain

$$\|\nabla w_n^\varepsilon\|_{L^{p(x)}(\Omega)} \leqslant C_n \quad \text{for all} \quad \varepsilon > 0. \tag{3.16}$$

Combining (3.15), (3.16), Proposition 2.1, and Lemma, 2.2 it follows that there exists a sequence $\{w_n^{\varepsilon_k}\}$ such that for $p^+ < N$

$$w_n^{\varepsilon_k} \to w \quad \text{strongly in } L^r(\Omega), \text{ with } 1 \leqslant r^- < r(x) < r^+ < \infty, \tag{3.17}$$

$$\nabla w_n^{\varepsilon_k} \rightharpoonup \nabla w \quad \text{weakly in} \quad L^{p(x)}(\Omega), \tag{3.18}$$

$$\int_\Omega |w_n^{\varepsilon_k}|^{q(x)-2} w_n^{\varepsilon_k} v \to \int_\Omega |w|^{q(x)-2} w v \quad \text{as } \varepsilon_k \to 0, \text{ for all } v \in W_0^{p(x)}(\Omega). \tag{3.19}$$

Weak convergence holds since $L^{p(x)}$ spaces are uniformly convex [3, Theorem 3.4.9], and hence reflexive.

From this point we refer to [13] for all the notations and results concerning to subdifferentials. Set

$$\phi_\varepsilon(z) := \int_\Omega \frac{1}{p(x)} (|\nabla z|^2 + \varepsilon)^{p(x)/2}\,dx$$

with $D(\phi_\varepsilon) = W_0^{1,p(x)}(\Omega)$ so that ϕ_ε is a convex operator according to in [13, Definition in section 1.3.3, p. 24]. Next, since ϕ_ε is Fréchet differentiable, and since

$$\phi'_\varepsilon(z)v = \langle A_\varepsilon z, v\rangle = \int_\Omega (|\nabla z|^2 + \varepsilon)^{p(x)/2} \nabla z \cdot \nabla v\,dx.$$

According to [13, Section 4.2.2], $A_\varepsilon \in \partial\phi_\varepsilon$ where $\partial\phi_\varepsilon$ is the subdifferential of ϕ_ε. Hence w_n^ε satisfies

$$\phi_\varepsilon(v) - \phi_\varepsilon(w_n^\varepsilon) \geqslant \int_\Omega (|\nabla w_n^\varepsilon|^2 + \varepsilon)^{p(x)/2} \nabla w_n^\varepsilon \cdot \nabla(v - w_n^\varepsilon)\,dx, \quad \forall v \in W_0^{1,p(x)}(\Omega).$$

Now by (3.7),

$$\phi_\varepsilon(v) - \phi_\varepsilon(w_n^\varepsilon) \geqslant \int_\Omega (-|w_n^\varepsilon|^{q(x)-2} w_n^\varepsilon + v_n^\varepsilon) \cdot (v - w_n^\varepsilon)\, dx. \tag{3.20}$$

On the other hand, given strong convergence of $w_n^\varepsilon \to w_n$ as $\varepsilon \to 0$ and strong convergence of $v_n \to 2|u_n|^{q(x)-2}u_n$ in $L^1(\Omega)$, we have that $v_n^\varepsilon w_n^\varepsilon \to 2|u_n|^{q(x)-2}u_n w_n$ as $\varepsilon \to 0$ in $L^1(\Omega)$ since

$$\begin{aligned}
&\int_\Omega |v_n^\varepsilon w_n^\varepsilon - 2|u_n|^{q(x)-2}u_n w_n|\, dx \\
&\leqslant \int_\Omega |v_n^\varepsilon||w_n^\varepsilon - w_n|\, dx + \int_\Omega |w_n||v_n^\varepsilon - 2|u_n|^{q(x)-2}u_n|\, dx \\
&\leqslant C_o \int_\Omega |w_n^\varepsilon - w_n|\, dx + \int_\Omega |w_n||v_n^\varepsilon - 2|u_n|^{q(x)-2}u_n|\, dx,
\end{aligned} \tag{3.21}$$

given that (3.5) holds. Note that the last integral approaches zero as $\varepsilon \to 0$. It follows from Hölder's inequality for spaces with variable exponent, $w_n \in L^r(\Omega)$, and (3.6).

Taking into account that $\phi_\varepsilon(v) \to \phi_0(v)$, as $\varepsilon \to 0$ for all $v \in W^{1,p(x)}(\Omega)$, and that

$$\liminf_{k\to\infty} \phi_{\varepsilon_k}(w_n^{\varepsilon_k}) \geqslant \phi_{\varepsilon_k}(w) \geqslant \phi_0(w) \tag{3.22}$$

holds (since modulars are weakly lower semicontinuous [3, Theorem 2.2.8]), we can take limits as $\varepsilon \to 0$ in (3.20), and after that, we can use (3.6), (3.17), and (3.19) to obtain

$$\phi_0(v) - \phi_0(w) \geqslant \int_\Omega \big(- |w|^{q(x)-2}w + 2|u_n|^{q(x)-2}u_n\big) \cdot (v - w)\, dx, \tag{3.23}$$

holds for all $v \in W_0^{1,p(x)}(\Omega)$. The last inequality implies, by the definition of subdifferential [13], that

$$\int_\Omega |\nabla w|^{p(x)-2}\nabla w \cdot \nabla\varphi\, dx = \int_\Omega (-|w|^{q(x)-2}w + 2|u_n|^{q(x)-2}u_n) \cdot \varphi\, dx, \tag{3.24}$$

for all $\varphi \in W_0^{1,p(x)}(\Omega)$. We conclude that $w = w_n$, since the argument above does not depend on the choice of $\{\varepsilon_k\}$.

Multiply equation (3.4) by w_n and equation (3.7) by w_n^ε, and integrate by parts to obtain

$$\int_\Omega |\nabla w_n|^{p(x)}\, dx = -\int_\Omega |w_n|^{q(x)}\, dx + 2\int_\Omega |u_n|^{q(x)-2}u_n w_n\, dx,$$

$$\int_\Omega (|\nabla w_n^\varepsilon|^2 + \varepsilon)^{(p(x)-2)/2}|\nabla w_n^\varepsilon|^2\, dx = -\int_\Omega |w_n^\varepsilon|^{q(x)}\, dx + \int_\Omega v_n^\varepsilon w_n^\varepsilon\, dx.$$

So that, (3.6) and (3.17) imply

$$\int_\Omega (|\nabla w_n^\varepsilon|^2 + \varepsilon)^{(p(x)-2)/2}|\nabla w_n^\varepsilon|^2\, dx \to \int_\Omega |\nabla w_n|^{p(x)}\, dx \quad \text{as } \varepsilon \to 0. \tag{3.25}$$

Take $v = w = w_n$ in (3.20) and let $\varepsilon \to 0$ in (3.20) to obtain

$$\limsup_{\varepsilon\to 0} \phi_\varepsilon(w_n^\varepsilon) \leqslant \phi_0(w_n). \tag{3.26}$$

Inequality (3.26) and (3.22) imply

$$\int_\Omega (|\nabla w_n^\varepsilon|^2+\varepsilon)^{p(x)/2}\,dx \to \int_\Omega |\nabla w_n|^{p(x)}\,dx \quad \text{as } \varepsilon\to 0. \tag{3.27}$$

Moreover, since (3.18) holds, we have

$$\liminf_{\varepsilon} \int_\Omega |\nabla w_n^\varepsilon|^{p(x)}\,dx \geqslant \int_\Omega |\nabla w_n|^{p(x)}\,dx$$

since modulars are weakly lower semicontinuous.

On the other hand, since $(|\nabla w_n^\varepsilon|^2)^{p(x)/2} \leqslant (|\nabla w_n^\varepsilon|^2+\varepsilon)^{p(x)/2}$ we have

$$\limsup_{\varepsilon} \int_\Omega |\nabla w_n^\varepsilon|^{p(x)}\,dx \leqslant \limsup_{\varepsilon} \int_\Omega (|\nabla w_n^\varepsilon|^2+\varepsilon)^{p(x)/2}\,dx \leqslant \int_\Omega |\nabla w_n|^{p(x)}\,dx.$$

Therefore, we conclude (3.8).

(iv) We proceed first by noticing that

$$|u_n|^{q(x)-2}u_n \to |u|^{q(x)-2}u \quad \text{strongly in } L^{q'(x)}(\Omega) \text{ as } n\to\infty, \tag{3.28}$$

by the uniform convexity of $L^{q'(x)}(\Omega)$. Multiply (3.4) by w_n and integrate by parts to obtain

$$\begin{aligned} \int_\Omega |w_n|^{q(x)}\,dx + \int_\Omega |\nabla w_n|^{p(x)}\,dx &= 2\int_\Omega |u_n|^{q(x)-2}u_n w_n\,dx \\ &\leqslant 4\||u_n|^{q(x)-1}\|_{L^{q'(x)}(\Omega)}\|w_n\|_{L^{q(x)}(\Omega)}, \end{aligned} \tag{3.29}$$

by Hölder's inequality for Sobolev Spaces with Variable Exponents [3, lemma 2.6.5]. Now, using [8, Theorem 1.3] and (3.29) we obtain

$$\|w_n\|^{q^\pm}_{L^{q(x)}(\Omega)} + \|\nabla w_n\|^{p^\pm}_{L^{p(x)}(\Omega)} \leqslant C\|w_n\|_{L^{q(x)}(\Omega)}, \tag{3.30}$$

where

$$q^\pm = \begin{cases} q^+, & \text{if } \|w_n\|_{L^{q(x)}(\Omega)}<1 \\ q^-, & \text{if } \|w_n\|_{L^{q(x)}(\Omega)}\geqslant 1, \end{cases} \qquad p^\pm = \begin{cases} p^+, & \text{if } \|\nabla w_n\|_{L^{q(x)}(\Omega)}<1 \\ p^-, & \text{if } \|\nabla w_n\|_{L^{q(x)}(\Omega)}\geqslant 1. \end{cases}$$

The fact that $p^\pm, q^\pm > 1$ imply that $\|w_n\|^{q^\pm}_{L^{q(x)}(\Omega)}, \|\nabla w_n\|^{p^\pm}_{L^{p(x)}(\Omega)} \leqslant C$. We use again Proposition 2.1 and Lemma 2.2 to obtain that, up to a subsequence $\{n_k\}$,

$$\nabla w_{n_k} \rightharpoonup \nabla w \quad \text{weakly in } L^{p(x)}(\Omega), \tag{3.31}$$

$$w_{n_k} \rightharpoonup w \quad \text{weakly in } L^{q(x)}(\Omega). \tag{3.32}$$

And, moreover, $w_{n_k}\to w$ strongly in $L^{q(x)}(\Omega)$ for all q such that $1\leqslant q^- < q(x) < q^+ < \infty$, and

$$\int_\Omega |w_{n_k}|^{q(x)-2}w_{n_k}\cdot v\,dx \to \int_\Omega |w|^{q(x)-2}w\cdot v\,dx \quad \text{for all } v\in L^{q'(x)}(\Omega) \tag{3.33}$$

as $k \to \infty$. Given that w_n is a solution of (3.4), the definition of subdifferential leads to

$$
\begin{aligned}
&\int_\Omega \frac{1}{p(x)}|\nabla v|^{p(x)}\,dx - \int_\Omega \frac{1}{p(x)}|\nabla w_n|^{p(x)}\,dx\\
&= \int_\Omega \frac{1}{p(x)}|\nabla v|^{p(x)}\,dx - \int_\Omega \frac{1}{p(x)}|\nabla w_n|^{p(x)}\,dx\\
&\geqslant \int_\Omega (-|w_n|^{q(x)-2}w_n + 2|u_n|^{q(x)-2}u_n)(v-w_n)\,dx\\
&\geqslant \int_\Omega |w_n|^{q(x)}\,dx - \int_\Omega |w_n|^{q(x)-2}w_n v\,dx + 2\int_\Omega |u_n|^{q(x)-2}u_n(v-w_n)\,dx,
\end{aligned}
\tag{3.34}
$$

for all $v \in C_0^\infty(\Omega)$ and for n such that $supp\, v \subset \Omega$. Let $n = n_k \to \infty$ in (3.34) and recall (3.28), (3.31), (3.32), and (3.33) to obtain

$$
\begin{aligned}
&\int_\Omega \frac{1}{p(x)}|\nabla v|^{p(x)}\,dx - \int_\Omega \frac{1}{p(x)}|\nabla w|^{p(x)}\,dx\\
&\geqslant \int_\Omega (-|w|^{q(x)-2}w + 2|u|^{q(x)-2}u)(v-w)\,dx,
\end{aligned}
\tag{3.35}
$$

for all $v \in C_0^\infty(\Omega)$. Now put $v = w + tz$ with $z \in C_o^\infty(\Omega)$ and let $t \to 0^+$, $t \to 0^-$ in (3.35) and use the definition of Fréchet derivative to see that w satisfies

$$
\int_\Omega |\nabla w|^{p(x)-2}\nabla w \cdot \nabla z\,dx + \int_\Omega |w|^{q(x)-2}wz\,dx = 2\int_\Omega |u|^{q(x)-2}uz\,dx
$$

for all $z \in C_o^\infty(\Omega)$. Hence

$$
|w|^{q(x)-2}w - \Delta_{p(x)}w = |u|^{q(x)-2}u - \Delta_{p(x)}u
$$

in the sense of distributions. That $w = u$ follows from well-known inequality

$$
|a-b|^p \leqslant C_p\big\{(|a|^{p-2}a - |b|^{p-2}b)\cdot(a-b)\big\}^{s/2}(|a|^p + |b|^p)^{1-s/2}
$$

which holds for all $a, b \in \mathbb{R}^N$, where $s = p$ if $p \in (1,2)$ and $s = 2$ if $p \geqslant 2$, and $C_p > 0$ does not depend on a, b (a proof of this inequality is in [17, Lemma A.0.5, p. 80]). Since the above argument does not depend on the choice of subsequences, then (3.31), (3.32) and (3.33) hold for $n_k = n$.

Taking into account (3.28), (3.29), (3.31) and (3.32) we obtain

$$
\begin{aligned}
2\int_\Omega |u|^{q(x)}\,dx &= \int_\Omega |u|^{q(x)}\,dx + \int_\Omega |\nabla u|^{p(x)}\,dx\\
&\leqslant \liminf_{n\to\infty}\Big(\int_\Omega |w_n|^{q(x)}\,dx + \int_\Omega |\nabla w_n|^{p(x)}\,dx\Big)\\
&= \lim_{n\to\infty}\Big(\int_\Omega |w_n|^{q(x)}\,dx + \int_\Omega |\nabla w_n|^{p(x)}\,dx\Big)\\
&\leqslant 2\int_\Omega |u|^{q(x)}\,dx.
\end{aligned}
$$

Consequently,

$$
\lim_{n\to\infty}\Big(\int_\Omega |w_n|^{q(x)}\,dx + \int_\Omega |\nabla w_n|^{p(x)}\,dx\Big) = \int_\Omega |u|^{q(x)}\,dx + \int_\Omega |\nabla u|^{p(x)}\,dx
$$

Moreover, notice that

$$\begin{aligned}
&\int_\Omega |u|^{q(x)}\,dx\\
&\leqslant \liminf_{n\to\infty}\int_\Omega |w_n|^{q(x)}\,dx \leqslant \limsup_{n\to\infty}\int_\Omega |w_n|^{q(x)}\,dx\\
&= \limsup_{n\to\infty}\Big(\int_\Omega |w_n|^{q(x)}\,dx + \int_\Omega \frac{|\nabla w_n|^{p(x)}}{p(x)}\,dx - \int_\Omega \frac{|\nabla w_n|^{p(x)}}{p(x)}\,dx\Big)\\
&\leqslant \limsup_{n\to\infty}\Big(\int_\Omega |w_n|^{q(x)}\,dx + \int_\Omega \frac{|\nabla w_n|^{p(x)}}{p(x)}\,dx\Big) - \liminf_{n\to\infty}\int_\Omega \frac{|\nabla w_n|^{p(x)}}{p(x)}\,dx\\
&\leqslant \int_\Omega |u|^{q(x)}\,dx.
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\int_\Omega |w_n|^{q(x)}\,dx = \int_\Omega |u|^{q(x)}\,dx,$$
$$\lim_{n\to\infty}\int_\Omega |\nabla w_n|^{p(x)}\,dx = \int_\Omega |\nabla u|^{p(x)}\,dx.$$

This completes the proof. □

To obtain a Pohozaev-type inequality, we introduce the function

$$\mathcal{F}(x,u,s) := \frac{|u(x)|^{q(x)}}{q(x)} + \frac{(|s|^2+\varepsilon)^{p(x)/2}}{p(x)} - v_n^\varepsilon(x)u(x) \tag{3.36}$$

where $s=(s_1,\dots,s_N)$, which will be used in the context of a Pucci-Serrin formula in [18].

Theorem 3.2 (Pohozaev-type inequality)**.** *Let u be a weak solution of* (1.2) *belonging to $\mathcal{P}$. Then u satisfies*

$$\begin{aligned}
&-\int_\Omega \frac{N}{q(x)}|u|^{q(x)}\,dx + \int_\Omega \frac{N-p(x)}{p(x)}|\nabla u|^{p(x)}\,dx\\
&+\int_\Omega x\cdot\nabla p(x)\frac{|\nabla u|^{p(x)}}{p(x)^2}\log\Big(e^{-1}|\nabla u|^{p(x)}\Big)\,dx\\
&-\int_\Omega x\cdot\nabla q(x)\frac{|u|^{q(x)}}{q(x)^2}\log\Big(e^{-1}|u|^{q(x)}\Big)\,dx + R \le 0,
\end{aligned}\tag{3.37}$$

where

$$R = \frac{p^\dagger-1}{p^+}\limsup_{n\to\infty}\limsup_{\varepsilon\to0}\int_{\partial\Omega}\big(|\nabla w_n^\varepsilon|^2+\varepsilon\big)^{p(x)/2}(x\cdot\nu(x))\,dS,$$

$p^\dagger = \min_{x\in\Omega}\{2,p(x)\}$*, and w_n^ε is the solution of* (3.7) *uniquely determined by u.*

Proof. Denote by $\mathcal{F}_s(x,u,s) = (\partial_{s_1}\mathcal{F},\dots,\partial_{s_N}\mathcal{F})$, where $\mathcal{F}$ is defined in (3.36). Then

$$\partial_{s_i}\mathcal{F}(x,u,s) = (|s|^2+\varepsilon)^{p(x)/2-1}s_i \quad \text{for } i=1,2,\dots,N. \tag{3.38}$$

Hence, we denote

$$\partial_{s_i}\mathcal{F}(x,u,\nabla u) = (|\nabla u|^2+\varepsilon)^{p(x)/2-1}\partial_i u \quad \text{for } i=1,2,\dots,N, \tag{3.39}$$

and

$$\mathcal{F}_s(x,u,\nabla u) = (|\nabla u|^2+\varepsilon)^{(p(x)-2)/2}\nabla u.$$

It follows from (3.38) and (3.39) that

$$\operatorname{div}\,\mathcal{F}(x,u,\nabla u) = -A_\varepsilon u.$$

Finally, we denote by

$$\nabla\mathcal{F}(x,u,\nabla u) = (\partial_{x_1}\mathcal{F},\dots,\partial_{x_N}\mathcal{F}) = (\partial_1\mathcal{F},\dots,\partial_N\mathcal{F})$$

with

$$\begin{aligned}
\partial_i\mathcal{F} &= \partial_i\Big(\frac{|u(x)|^{q(x)}}{q(x)} + \frac{(|s|^2+\varepsilon)^{p(x)/2}}{p(x)} - v_n^\varepsilon(x)u(x)\Big)\\
&= \frac{|u|^{q(x)}}{(q(x))^2}\big(\log|u|^{q(x)}-1\big)\partial_i q(x) + |u|^{q(x)-2}u\partial_i u\\
&\quad + \frac{(|\nabla u|^2+\varepsilon)^{p(x)/2}}{2(p(x))^2}\big(\log(|\nabla u|^2+\varepsilon)^{p(x)}-1\big)\partial_i p(x)\\
&\quad + (|\nabla u|^2+\varepsilon)^{p(x)/2-1}\partial_i(|\nabla u|^2) - \big[(\partial_i v_n^\varepsilon)u + v_n^\varepsilon\partial_i u\big] \quad \text{for } i=1,\dots,N.
\end{aligned}$$

We shall use the Pucci-Serrin formula [18, Proposition 1, p. 683] in the form

$$\begin{aligned}
&\int_{\partial\Omega}\Big[\mathcal{F}(x,0,\nabla u) - \nabla u\cdot\mathcal{F}_s(x,0,\nabla u)\Big](h\cdot\nu)\,dS\\
&= \int_\Omega\Big[\mathcal{F}(x,u,\nabla u)\operatorname{div}h + h\cdot\nabla\mathcal{F}(x,u,\nabla u) - (h\cdot\nabla u)\operatorname{div}\mathcal{F}_s(x,u,\nabla u)\\
&\quad - \mathcal{F}_s(x,u,\nabla u)\cdot\nabla(h\cdot\nabla u) - au\operatorname{div}\mathcal{F}_s(x,u,\nabla u)\\
&\quad - \nabla(au)\cdot\mathcal{F}_s(x,u,\nabla u)\Big]\,dx,
\end{aligned}\tag{3.40}$$

where a and h are respectively scalar and vector-valued functions of class $C^1(\Omega)$. Taking a constant, $h = x = (x_1,\dots,x_n)$, and $u = w_n^\varepsilon$, equation (3.40) becomes

$$\begin{aligned}
&\int_{\partial\Omega}\frac{(|\nabla w_n^\varepsilon|^2+\varepsilon)^{p(x)/2}}{p(x)}(x\cdot\nu)\,dS\\
&- \int_{\partial\Omega}(|\nabla w_n^\varepsilon|^2+\varepsilon)^{p(x)/2-1}|\nabla w_n^\varepsilon|^2(x\cdot\nu)\,dS\\
&= \int_\Omega N\Big(\frac{|w_n^\varepsilon|^{q(x)}}{q(x)} + \frac{(|\nabla w_n^\varepsilon|^2+\varepsilon)^{p(x)/2}}{p(x)} - v_n^\varepsilon w_n^\varepsilon\Big)\,dx\\
&\quad + \int_\Omega(x\cdot\nabla q(x))\frac{|w_n^\varepsilon|^{q(x)}}{(q(x))^2}\big(\log|w_n^\varepsilon|^{q(x)}-1\big)\,dx\\
&\quad + \int_\Omega(x\cdot\nabla p(x))\frac{(|\nabla w_n^\varepsilon|^2+\varepsilon)^{p(x)/2}}{(p(x))^2}\big(\log(|\nabla w_n^\varepsilon|^2+\varepsilon)^{p(x)/2}-1\big)\,dx\\
&\quad - \int_\Omega w_n^\varepsilon(x\cdot\nabla v_n^\varepsilon)\,dx - \int_\Omega(|\nabla w_n^\varepsilon|^2+\varepsilon)^{(p(x)-2)/2}|\nabla w_n^\varepsilon|^2\,dx\\
&\quad + \int_\Omega aw_n^\varepsilon A_\varepsilon w_n^\varepsilon\,dx - \int_\Omega(\nabla(aw_n^\varepsilon)\cdot\nabla w_n^\varepsilon)(|\nabla w_n^\varepsilon|^2+\varepsilon)^{(p(x)-2)/2}\,dx.
\end{aligned}\tag{3.41}$$

For the surface integrals in (3.41), by adding and subtracting $\varepsilon\int_{\partial\Omega}(|\nabla w_n^\varepsilon|^2+\varepsilon)^{p(x)/2-1}(x\cdot\nu)\,dS$ we have

$$\begin{aligned}&\int_{\partial\Omega}\frac{(|\nabla w_n^\varepsilon|^2+\varepsilon)^{p(x)/2}}{p(x)}(x\cdot\nu)\,dS-\int_{\partial\Omega}(|\nabla w_n^\varepsilon|^2+\varepsilon)^{p(x)/2-1}|\nabla w_n^\varepsilon|^2(x\cdot\nu)\,dS\\&=\int_{\partial\Omega}\big(\frac{1}{p(x)}-1\big)\big(|\nabla w_n^\varepsilon|^2+\varepsilon\big)^{p(x)/2}(x\cdot\nu)\,dS\\&\quad+\varepsilon\int_{\partial\Omega}(|\nabla w_n^\varepsilon|^2+\varepsilon)^{p(x)/2-1}(x\cdot\nu)\,dS.\end{aligned}\tag{3.42}$$

On the other hand, since $(x\cdot\nu(x))\geqslant 0$ for all $x\in\partial\Omega$, it follows that

$$\begin{aligned}&\varepsilon\int_{\partial\Omega}(|\nabla w_n^\varepsilon|^2+\varepsilon)^{p(x)/2-1}(x\cdot\nu)\,dS\\&\leqslant\begin{cases}\int_{\partial\Omega}\varepsilon^{p(x)/2}(x\cdot\nu(x))\,dS, & \text{if } 1<p(x)\leqslant 2,\\ \int_{\partial\Omega}\frac{p(x)-2}{p(x)}(|\nabla w_n^\varepsilon|^2+\varepsilon)^{p(x)/2}(x\cdot\nu)\,dS & \\ +\int_{\partial\Omega}\frac{2}{p(x)}\varepsilon^{p(x)/2}(x\cdot\nu(x))\,dS, & \text{if } 2<p(x).\end{cases}\end{aligned}\tag{3.43}$$

Next, we analyze the behavior of each term in (3.41) as $\varepsilon\to 0$. We begin the analysis with the last term in the right hand side of the equation and we end with the first term:

$$-\int_\Omega(\nabla(aw_n^\varepsilon)\cdot\nabla w_n^\varepsilon)(|\nabla w_n^\varepsilon|^2+\varepsilon)^{(p(x)-2)/2}\,dx\to -a\int_\Omega|\nabla w_n|^{p(x)}\,dx\tag{3.44}$$

by (3.25).

$$\int_\Omega aw_n^\varepsilon A_\varepsilon w_n^\varepsilon\,dx\to a\Big(\int_\Omega 2|u_n|^{q(x)-2}u_nw_n\,dx-\int_\Omega|w_n|^{q(x)}\,dx\Big)\tag{3.45}$$

by (3.7) and (3.21).

$$-\int_\Omega(|\nabla w_n^\varepsilon|^2+\varepsilon)^{(p(x)-2)/2}|\nabla w_n^\varepsilon|^2\,dx\to-\int_\Omega|\nabla w_n|^{p(x)}\,dx\tag{3.46}$$

by (3.25).

For the term $-\int_\Omega w_n^\varepsilon(x\cdot\nabla v_n^\varepsilon)\,dx$, since $\nabla(w_n^\varepsilon v_n^\varepsilon)=v_n^\varepsilon\nabla w_n^\varepsilon+w_n^\varepsilon\nabla v_n^\varepsilon$, we have

$$-\int_\Omega w_n^\varepsilon(x\cdot\nabla v_n^\varepsilon)\,dx=-\int_\Omega x\cdot\nabla(w_n^\varepsilon v_n^\varepsilon)\,dx+\int_\Omega v_n^\varepsilon x\cdot\nabla w_n^\varepsilon\,dx.\tag{3.47}$$

Note that

$$\int_\Omega v_n^\varepsilon x\cdot\nabla w_n^\varepsilon\,dx\to 2\int_\Omega|u_n|^{q(x)-2}u_nx\cdot\nabla w_n\,dx$$

as $\varepsilon\to 0$, by a similar proof as in (3.21).

On the other hand, calculating the first term in the right-hand side of (3.47), we obtain

$$\begin{aligned}-\int_\Omega x\cdot\nabla(w_n^\varepsilon v_n^\varepsilon)\,dx&=\int_\Omega v_n^\varepsilon w_n^\varepsilon\operatorname{div}x\,dx-\int_{\partial\Omega}v_n^\varepsilon w_n^\varepsilon(x\cdot\nu)\,dS\\&=N\int_\Omega v_n^\varepsilon w_n^\varepsilon\,dx.\end{aligned}\tag{3.48}$$

We claim that

$$\begin{aligned} I_1 &:= \int_\Omega (x\cdot\nabla q(x))\frac{|w_n^\varepsilon|^{q(x)}}{(q(x))^2}\big(\log|w_n^\varepsilon|^{q(x)}-1\big)\,dx \\ &\to \int_\Omega (x\cdot\nabla q(x))\frac{|w_n|^{q(x)}}{(q(x))^2}\big(\log|w_n|^{q(x)}-1\big)\,dx \end{aligned} \tag{3.49}$$

and

$$\begin{aligned} I_2 &:= \int_\Omega (x\cdot\nabla p(x))\frac{(|\nabla w_n^\varepsilon|^2+\varepsilon)^{p(x)/2}}{(p(x))^2}\big(\log(|\nabla w_n^\varepsilon|^2+\varepsilon)^{p(x)/2}-1\big)\,dx \\ &\to \int_\Omega (x\cdot\nabla p(x))\frac{|\nabla w_n|^{p(x)}}{(p(x))^2}\big(\log|\nabla w_n|^{p(x)}-1\big)\,dx. \end{aligned} \tag{3.50}$$

To prove (3.49) and (3.50), we estimate I_1 by distinguishing the two cases $|w_n^\varepsilon|\le 1$ and $|w_n^\varepsilon|>1$. Notice that the relations

$$\sup_{0\le t\le 1} t^\eta|\log t|<\infty, \tag{3.51}$$

$$\sup_{t>1} t^{-\eta}\log t<\infty \tag{3.52}$$

hold for $\eta>0$.

Set $\Omega_1:=\{x\in\Omega:|w_n^\varepsilon(x)|\le 1\}$ and $\Omega_2:=\{x\in\Omega:|w_n^\varepsilon(x)|>1\}$. We can choose $k\in\mathbb{N}$ such that $p(x)-1/k\ge p^-$. Since $w_n^\varepsilon\in L^{p^-}(\Omega)$ and $|w_n^\varepsilon(x)|\le 1$, in Ω_1, we have

$$\Big|(x\cdot\nabla q(x))\frac{|w_n^\varepsilon|^{q(x)}}{(q(x))^2}\log|w_n^\varepsilon|^{q(x)}\Big|\le C|w_n^\varepsilon(x)|^{p(x)-1/m}\le C|w_n^\varepsilon(x)|^{p^-}, \tag{3.53}$$

for $m>k$. For $x\in\Omega_2$, we can choose k' such that $p(x)+1/k'\le (p(x))^*=Np(x)/(N-p(x))$. So

$$\Big|(x\cdot\nabla q(x))\frac{|w_n^\varepsilon|^{q(x)}}{(q(x))^2}\log|w_n^\varepsilon|^{q(x)}\Big|\le C|w_n^\varepsilon(x)|^{p(x)+1/m}\le C|w_n^\varepsilon(x)|^{(p(x))^*}, \tag{3.54}$$

for $m>k'$, and $x\in\Omega_2$. Therefore (3.53), (3.54), and the convergence of w_n^ε in Lemma 3.1 imply that there exists $h(x)\in L^1(\Omega)$ such that

$$\Big|(x\cdot\nabla q(x))\frac{|w_n^\varepsilon|^{q(x)}}{(q(x))^2}\log|w_n^\varepsilon|^{q(x)}\Big|\le h(x). \tag{3.55}$$

On the other hand, given the convergence Lemma 3.1, assertion (3.9) and the continuity of the log function, we conclude that

$$(x\cdot\nabla q(x))\frac{|w_n^\varepsilon|^{q(x)}}{(q(x))^2}\log|w_n^\varepsilon|^{q(x)}\to (x\cdot\nabla q(x))\frac{|w_n|^{q(x)}}{(q(x))^2}\log|w_n|^{q(x)} \tag{3.56}$$

a.e. in Ω as $\varepsilon\to 0$. With (3.55), (3.56), and the Lebesgue Convergence Theorem the claims (3.49) and loggrad follow.

Finally,

$$\int_\Omega N\Big(\frac{|w_n^\varepsilon|^{q(x)}}{q(x)}+\frac{(|\nabla w_n^\varepsilon|^2+\varepsilon)^{p(x)/2}}{p(x)}\Big)\,dx\to\int_\Omega N\Big(\frac{|w_n|^{q(x)}}{q(x)}+\frac{|\nabla w_n|^{p(x)}}{p(x)}\Big)\,dx \tag{3.57}$$

as $\varepsilon\to 0$ by (3.25) and (3.9).

Considering items (3.44)–(3.57), identities (3.41), (3.42), and inequality (3.43), we obtain

$$\begin{aligned} &N\int_\Omega \frac{|w_n|^{q(x)}}{q(x)}\,dx+\int_\Omega \frac{N-p(x)}{p(x)}|\nabla w_n|^{p(x)}\,dx\\ &+\int_\Omega x\cdot\nabla p(x)\frac{|\nabla w_n|^{p(x)}}{p(x)^2}\big(\log|\nabla w_n|^{p(x)}-1\big)\,dx\\ &+\int_\Omega x\cdot\nabla q(x)\frac{|w_n|^{q(x)}}{q(x)^2}\big(\log|w_n|^{q(x)}-1\big)\,dx+2\int_\Omega |u_n|^{q(x)-2}u_n x\cdot\nabla w_n\,dx\\ &+a\Big(\int_\Omega 2|u_n|^{q(x)-2}u_n w_n\,dx-\int_\Omega |w_n|^{q(x)}\,dx-\int_\Omega |\nabla w_n|^{p(x)}\,dx\Big)+R_n\\ &\le 0, \end{aligned} \tag{3.58}$$

where

$$R_n=\frac{p^\dagger-1}{p^+}\limsup_{\varepsilon\to 0}\int_{\partial\Omega}\big(|\nabla w_n^\varepsilon|^2+\varepsilon\big)^{p(x)/2}(x\cdot\nu(x))\,dS,$$

and $p^\dagger=\min_{x\in\Omega}\{2,p(x)\}$.

Next let $n\to\infty$ in (3.58) and take into account (3.10), and (3.11) to obtain

$$\begin{aligned} &N\int_\Omega \frac{|u|^{q(x)}}{q(x)}\,dx\\ &+\int_\Omega \frac{N-p(x)}{p(x)}|\nabla u|^{p(x)}\,dx+\int_\Omega x\cdot\nabla p(x)\frac{|\nabla u|^{p(x)}}{p(x)^2}\big(\log|\nabla u|^{p(x)}-1\big)\,dx\\ &+\int_\Omega x\cdot\nabla q(x)\frac{|u|^{q(x)}}{q(x)^2}\big(\log|u|^{q(x)}-1\big)\,dx+2\int_\Omega |u|^{q(x)-2}u(x\cdot\nabla u)\,dx\\ &+a\Big(\int_\Omega |u|^{q(x)}\,dx-\int_\Omega |\nabla u|^{p(x)}\,dx\Big)+R\le 0, \end{aligned} \tag{3.59}$$

where

$$R=\frac{p^\dagger-1}{p^+}\limsup_{n\to\infty}\limsup_{\varepsilon\to 0}\int_{\partial\Omega}\big(|\nabla w_n^\varepsilon|^2+\varepsilon\big)^{p(x)/2}(x\cdot\nu(x))\,dS.$$

Further, notice that since u is a weak solution of (1.2),

$$\int_\Omega |u|^{q(x)}\,dx-\int_\Omega |\nabla u|^{p(x)}\,dx=0. \tag{3.60}$$

In fact, multiplying (1.2) by $\varphi\in W_0^{1,p(\cdot)}(\Omega)$, and integrating by parts, we have

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u\,dx=\int_\Omega |u|^{q(x)-2}u\varphi\,dx.$$

Taking $\varphi=u$ we obtain (3.60), as wanted. On the other hand,

$$\begin{aligned} \int_\Omega \frac{x\cdot\nabla|u|^{q(x)}}{q(x)}\,dx&=\int_\Omega |u|^{q(x)-2}u(x\cdot\nabla u)\,dx\\ &\quad+\int_\Omega \frac{1}{q(x)^2}|u|^{q(x)}\log|u|^{q(x)}(x\cdot\nabla q(x))\,dx, \end{aligned} \tag{3.61}$$

so that

$$\begin{aligned}\int_\Omega \frac{x\cdot\nabla|u|^{q(x)}}{q(x)}\,dx &= -\int_\Omega \operatorname{div}\big(\frac{x}{q(x)}\big)|u|^{q(x)}\,dx + \int_{\partial\Omega}|u|^{q(x)}\frac{\partial}{\partial\nu}\big(\frac{x}{q(x)}\big)\,dS\\ &\quad - N\int_\Omega \frac{|u|^{q(x)}}{q(x)}\,dx + \int_\Omega \frac{|u|^{q(x)}x\cdot\nabla q(x)}{q(x)^2}\,dx.\end{aligned} \tag{3.62}$$

Hence, from (3.61), and (3.62), we obtain

$$\begin{aligned}&\int_\Omega |u|^{q(x)-2}u(x\cdot\nabla u)\,dx\\ &= -N\int_\Omega \frac{|u|^{q(x)}}{q(x)}\,dx + \int_\Omega \frac{|u|^{q(x)}x\cdot\nabla q(x)}{q(x)^2}\big(1-\log|u|^{q(x)}\big)\,dx\end{aligned} \tag{3.63}$$

We obtain (3.37) by substituting (3.60) and (3.63) in (3.59). □

4. Nonexistence of nontrivial solutions

Now we can state a nonexistence theorem which is a generalization to the case of Sobolev Spaces with variable exponents of [16, Theorem III, p. 142]. The proofs are similar to those in [16], but are included here for the reader's convenience.

Theorem 4.1. *Consider Problem* (1.2)*, where* $\Omega\subset\mathbb{R}^N$ *is a bounded domain of class* C^1*,* $p(\cdot)$ *is a log-Hölder exponent with* $1<p^-\leqslant p(x)\leqslant p^+<N$*. Let* $\mathcal{P}$ *be as defined in* (3.1)*. Then we have:*

(i) *If* Ω *is star-shaped and* $q^->(p^+)^*$ *then Problem* (1.2) *has no nontrivial weak solution belonging to* $\mathcal{P}\cap\mathcal{E}$ *where*

$$\mathcal{E}=\Big\{u:\int_\Omega \log\Big(\frac{\big(|\nabla u|^{p(x)}e^{-1}\big)^{\frac{x\cdot\nabla p}{p^2}|\nabla u|^{p(x)}}}{\big(|u|^{q(x)}e^{-1}\big)^{\frac{x\cdot\nabla q}{q^2}|u|^{q(x)}}}\Big)\,dx\geqslant 0\Big\}.$$

(ii) *If* Ω *is strictly star-shaped and* $q^-=(p^+)^*$ *then Problem* (1.2) *has no nontrivial weak solution of definite sign belonging to* $\mathcal{P}\cap\mathcal{E}$*.*

Proof. (i) If Ω is star-shaped, then $R\geqslant 0$ in (3.37). Then it follows that

$$\Big(\frac{N-p^+}{p^+}-\frac{N}{q^-}\Big)\int_\Omega |u|^{q(x)}\,dx\leqslant 0.$$

So $u\equiv 0$.

(ii) If Ω is strictly star-shaped, then $R=0$ in (3.37). It follows that

$$0=R\geqslant \rho\limsup_{n\to\infty}\limsup_{\varepsilon\to 0}\int_{\partial\Omega}\big(|\nabla w_n^\varepsilon|^2+\varepsilon\big)^{p(x)/2}\,dS.$$

Since $\rho>0$ we have

$$0=\limsup_{n\to\infty}\limsup_{\varepsilon\to 0}\int_{\partial\Omega}\big(|\nabla w_n^\varepsilon|^2+\varepsilon\big)^{p(x)/2}\,dS.$$

Multiplying (3.7) by $v(x)\equiv 1$, integrating by parts, and taking lim sup as $\varepsilon\to 0$ and $n\to\infty$ we obtain

$$\Big|\int_\Omega |u|^{q(x)-2}u\,dx\Big|\leqslant C\limsup_{n\to\infty}\limsup_{\varepsilon\to 0}\int_{\partial\Omega}\big(|\nabla w_n^\varepsilon|^2+\varepsilon\big)^{p(x)/2}\,dS=0,\quad C\geqslant 0.$$

Therefore, $\int_\Omega |u|^{q(x)-2}u\,dx=0$. □

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