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LYAPUNOV FUNCTIONS FOR GENERAL NONUNIFORM TRICHOTOMY WITH DIFFERENT GROWTH RATES

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Abstract. In this article, we consider non-autonomous linear equations $x' = A(t)x$ that may exhibit stable, unstable and central behaviors in different directions. We give a complete characterization of nonuniform (μ,ν) trichotomies in terms of strict Lyapunov functions. In particular, we obtain an inverse theorem giving explicitly Lyapunov functions for each given trichotomy. The main novelty of our work is that we consider a very general type of nonuniform exponential trichotomy, which admits different growth rates in the uniform and nonuniform parts.

1. Introduction

We consider the non-autonomous linear equation

$$x' = A(t)x \tag{1.1}$$

where $A : \mathbb{R}_0^+ \to \mathcal{B}(X)$ is a continuous function with values in the space of bounded linear operators in a Banach space X. Our main aim is to characterize the existence of nonuniform (μ,ν) trichotomies behavior for the solutions of equation (1.1) in terms of strict Lyapunov functions.

We consider a very general type of nonuniform trichotomies, which generalizes the classical notion of exponential trichotomies in various ways: besides introducing a nonuniform term, we consider arbitrary rates that in particular may not be exponential, as well as different growth rates in the uniform and nonuniform parts. This includes for example the classical notion of uniform exponential trichotomies, as well as the notions of nonuniform exponential trichotomies and ρ-nonuniform exponential trichotomy.

Exponential trichotomy is the most complex asymptotic property of dynamical systems arising from the central manifold theory. The conception of trichotomy was first introduced by Sacker and Sell [22]. They described SS-trichotomy for linear differential systems by linear skew-product flows. Later, Elaydi and Hájek [11, 12] gave the notions of exponential trichotomy for differential systems and for nonlinear differential systems, respectively. These notions are stronger notions than SS-trichotomy. Recently, Barreira and Valls [6] considered a general concept

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of nonuniform exponential trichotomy, from which can see exponential trichotomy as a special case of the nonuniform exponential trichotomy. Jiang [17] consider a general case of nonuniform (μ, ν) trichotomy for an arbitrary non-autonomous linear dynamics, and establish the robustness of the nonuniform (μ, ν) trichotomy in Banach spaces, based on [1] for nonuniform (μ, ν) dichotomy. In [10], they proposed a ρ-nonuniform exponential trichotomy and considered the Lyapunov functions for the trichotomy.

The importance of Lyapunov functions is well-established, particularly in the study of the stability of trajectories both under linear and nonlinear perturbations. This study dates back to the seminal work of Lyapunov in his 1892 thesis [19]. Among the first accounts of the theory we have the works by LaSalle and Lefschetz [20], Hahn [16], and Bhatia and Szegö [14]. According to [15], the connection between Lyapunov functions and uniform exponential dichotomies was first considered by Mažel' in [21]. For recent works, we refer the reader to [1], [3]-[5], [13], for nonuniform dichotomies [2], [7]-[10], [18], for the corresponding characterization of nonuniform exponential contractions, dichotomies and trichotomies using Lyapunov functions. From there, we follow some of the ideas in the proofs of this articles. We give a complete characterization of nonuniform (μ, ν) trichotomies in terms of strict Lyapunov functions. In particular, we obtain an inverse theorem giving explicitly Lyapunov functions for each given trichotomy.

The remaining part of this paper is organized as follows. In Section 2, we introduce some basic definitions. In Section 3, we establish a criterion for the existence of partially hyperbolic behavior in terms of pairs of strict Lyapunov functions. A characterization of nonuniform (μ, ν) trichotomies in terms of quadratic Lyapunov functions is presented in Section 4.

2. Preliminaries

Let X be a Banach space and denote by $\mathcal{B}(X)$ the space of bounded linear operators acting on X. Given a continuous function $A : \mathbb{R}_0^+ \to \mathcal{B}(X)$, we assume that each solution of (1.1) is global and denote the evolution operator associated with (1.1) by $T(t,s)$, i.e., the linear operator such that

$$T(t,s)x(s) = x(t), \quad t,s > 0,$$

where $x(t)$ is any solution of (1.1). Clearly, $T(t,t) = \mathrm{Id}$ and

$$T(t,\tau)T(\tau,s) = T(t,s), \quad t,\tau,s > 0.$$

First we introduce the notion of nonuniform (μ, ν) trichotomy. We say that an increasing function $\mu : \mathbb{R}_0^+ \to [1, +\infty)$ is a growth rate if

$$\mu(0) = 1 \quad \text{and} \quad \lim_{t\to+\infty} \mu(t) = +\infty.$$

Definition 2.1. Given growth rates μ and ν, we say that (1.1) admits a nonuniform (μ, ν) trichotomy in I if there exist projections $P(t), Q(t), R(t)$ for each $t \in I$ such that

$$T(t,\tau)P(\tau) = P(t)T(t,\tau), T(t,\tau)Q(\tau) = Q(t)T(t,\tau), T(t,\tau)R(\tau) = R(t)T(t,\tau) \tag{2.1}$$

and

$$P(t) + Q(t) + R(t) = Id \tag{2.2}$$

for every $t, \tau \in I$, and there exist constants

$$0 \le \eta < \alpha, \quad 0 \le \xi < \beta, \quad \varepsilon \ge 0, \quad D \ge 1 \tag{2.3}$$

such that for every $t, \tau \in I$ with $t \ge \tau$, we have

$$\begin{gathered}
\|T(t,\tau)P(\tau)\| \le D\Big(\frac{\mu(t)}{\mu(\tau)}\Big)^{-\alpha}\nu^{\varepsilon}(\tau),\\
\|T(t,\tau)R(\tau)\| \le D\Big(\frac{\mu(t)}{\mu(\tau)}\Big)^{\xi}\nu^{\varepsilon}(\tau),
\end{gathered} \tag{2.4}$$

and

$$\begin{gathered}
\|T(t,\tau)^{-1}Q(t)\| \le D\Big(\frac{\mu(t)}{\mu(\tau)}\Big)^{-\beta}\nu^{\varepsilon}(t),\\
\|T(t,\tau)^{-1}R(t)\| \le D\Big(\frac{\mu(t)}{\mu(\tau)}\Big)^{\eta}\nu^{\varepsilon}(t).
\end{gathered} \tag{2.5}$$

When $\varepsilon = 0$, we say that (1.1) admits a uniform (μ, ν) trichotomy.

Setting $t = \tau$ in (2.4) and (2.5) we obtain

$$P(t) \le D\nu^{\varepsilon}(t), Q(t) \le D\nu^{\varepsilon}(t), R(t) \le D\nu^{\varepsilon}(t), \tag{2.6}$$

for every $t \in I$. We refer the reader to [17] for an example with a nonuniform (μ, ν) trichotomy which can not be uniform.

Now, we introduce the notion of Lyapunov function. Let $A : \mathbb{R}_0^+ \to M_p$ be a continuous function, where M_p is the set of $p \times p$ matrices. Given a function $V : \mathbb{R}^p \to \mathbb{R}$, we consider the cones

$$C^u(V) = \{0\} \cup V^{-1}(0, +\infty), \quad C^s(V) = \{0\} \cup V^{-1}(-\infty, 0).$$

Definition 2.2. We say that a continuous function $V : \mathbb{R}_0^+ \times \mathbb{R}^p \to \mathbb{R}$ is a Lyapunov function for (1.1) if there exist integers $r_u, r_s \in \mathbb{N} \cup \{0\}$ with $r_u + r_s = p$ such that the following properties hold:

(1) r_u and r_s are respectively the maximal dimensions of the linear subspaces inside the cones $C^u(V_t)$ and $C^s(V_t)$, for every $t \in \mathbb{R}_0^+$;
(2) for every $x \in \mathbb{R}^p$ and $t \ge \tau$ we have
$$T(t,\tau)\overline{C^u(V_\tau)} \subset \overline{C^u(V_t)}, \quad T(\tau,t)\overline{C^s(V_t)} \subset \overline{C^s(V_\tau)},$$
where $V_t = V(t, \cdot)$

In view of the compactness of the closed unit ball in $\mathbb{R}^p$, if $(V_t)_{t\in\mathbb{R}}$ is a Lyapunov function for (1.1), then for each $\tau \in \mathbb{R}_0^+$ the sets

$$H_\tau^u = \cap_{r\in\mathbb{R}_0^+} T(\tau, r)\overline{C^u(V_r)} \subset \overline{C^u(V_\tau)}, \tag{2.7}$$

$$H_\tau^s = \cap_{r\in\mathbb{R}_0^+} T(\tau, r)\overline{C^s(V_r)} \subset \overline{C^s(V_\tau)} \tag{2.8}$$

contain subspaces respectively of dimensions r_u and r_s. We note that for every $t, \tau \in \mathbb{R}_0^+$,

$$T(t,\tau)H_\tau^u = H_t^u \quad \text{and} \quad T(t,\tau)H_\tau^s = H_t^s \tag{2.9}$$

Next, we introduce the notion of strict Lyapunov functions. Given growth rates μ and ν, we denote by V the Lyapunov function.

Definition 2.3. Given $\lambda > \sigma > 0$ and $\gamma \geq 0$, we say that V is a (λ, σ)-strict Lyapunov function if there exist $C > 0$ and $\delta \geq 0$ such that

$$|V(t,x)| \leq C\nu^{\delta}(t)\|x\| \tag{2.10}$$

for each $t \in \mathbb{R}_0^+$ and $x \in \mathbb{R}^p$, the following properties hold:

(1) if $x \in H_\tau^u$ then

$$V(t, T(t,\tau)x) \geq \Big(\frac{\mu(t)}{\mu(\tau)}\Big)^{\log(\lambda+\sigma)} V(\tau, x), \quad t \geq \tau; \tag{2.11}$$

(2) if $x \in H_\tau^s$ then

$$|V(t, T(t,\tau)x)| \leq \Big(\frac{\mu(t)}{\mu(\tau)}\Big)^{\log(\lambda-\sigma)} V(\tau, x), \quad t \geq \tau; \tag{2.12}$$

(3) if $x \in H_\tau^u \cup H_\tau^s$ then

$$|V(\tau, x)| \geq \nu^{-\gamma}(\tau)\|x\|/C. \tag{2.13}$$

3. Criterion for nonuniform hyperbolic behavior

For each $\tau \in \mathbb{R}_0^+$, we set

$$\chi_\tau^+(x) = \limsup_{t\to+\infty} \frac{\log\|T(t,\tau)x\|}{\log\mu(t)}. \tag{3.1}$$

Lemma 3.1. *Let $\mu \geq \nu$ be growth rates with $\mu(t)/\mu(\tau) > \nu(t)/\nu(\tau)$ for every $t \geq \tau$. If there exist a (λ, σ)-strict Lyapunov function V for (1.1) with*

$$(\lambda+\sigma)/(\lambda-\sigma) > e^{\delta+\gamma}, \tag{3.2}$$

then for each $t, \tau \in \mathbb{R}_0^+$:

(1) *the sets H_τ^u and H_τ^s in* (2.7) *and* (2.8) *are linear subspaces respectively of dimensions r_u and r_s, with*

$$\mathbb{R}^p = H_\tau^u \oplus H_\tau^s, \tag{3.3}$$

$$T(t,\tau)H_\tau^s = H_t^s, \quad T(t,\tau)H_\tau^u = H_t^u; \tag{3.4}$$

(2)

$$\chi_\tau^+(x) \geq \log(\lambda+\sigma) - \delta \quad \textit{for } x \in H_\tau^u, \tag{3.5}$$

$$\chi_\tau^+(x) \leq \log(\lambda-\sigma) + \gamma \quad \textit{for } x \in H_\tau^s; \tag{3.6}$$

(3) *for each $t \geq \tau$ we have*

$$\|T(t,\tau)^{-1}|H_t^u\| \leq C^2\Big(\frac{\mu(t)}{\mu(\tau)}\Big)^{-\log(\lambda+\sigma)}\nu^{\delta+\gamma}(t), \tag{3.7}$$

$$\|T(t,\tau)|H_\tau^s\| \leq C^2\Big(\frac{\mu(t)}{\mu(\tau)}\Big)^{\log(\lambda-\sigma)+\gamma}\nu^{\delta+\gamma}(\tau). \tag{3.8}$$

Proof. It follows from (2.13) that (2.7) and (2.8) can be replaced by

$$H_\tau^u \subset C^u(V_\tau) \quad \text{and} \quad H_\tau^s \subset C^s(V_\tau). \tag{3.9}$$

Indeed, if $x \in H_\tau^u\backslash\{0\}$, then by (2.13) we have $V(\tau, x) > 0$. This establishes the first inclusion in (3.9). A similar argument establishes the second one. By (3.9), the function $V(\tau,\cdot)$ is positive in $H^u_\tau\backslash\{0\}$ and negative in $H^s_\tau\backslash\{0\}$. For each $x\in H^s_\tau$, it follows from (2.12) and (2.13) that for every $t\ge\tau$ we have

$$\begin{aligned} \|T(t,\tau)x\| &\le C\nu^\gamma(t)|V(t,T(t,\tau)x)| \\ &\le C\Big(\frac{\mu(t)}{\mu(\tau)}\Big)^{\log(\lambda-\sigma)}\nu^\gamma(t)|V(\tau,x)| \\ &\le C\Big(\frac{\mu(t)}{\mu(\tau)}\Big)^{\log(\lambda-\sigma)}\mu^\gamma(t)|V(\tau,x)|. \end{aligned} \tag{3.10}$$

Thus, (3.6) holds.

For each $x\in H^u_\tau$, it follows from (2.10) and (2.11) that for every $t\ge\tau$ we have

$$\begin{aligned} \|T(t,\tau)x\| &\ge \Big(\frac{\mu(t)}{\mu(\tau)}\Big)^{\log(\lambda+\sigma)}\frac{\nu^{-\delta}(t)}{C}|V(t,T(t,\tau)x)| \\ &\ge \Big(\frac{\mu(t)}{\mu(\tau)}\Big)^{\log(\lambda+\sigma)}\frac{\mu^{-\delta}(t)}{C}|V(\tau,x)|. \end{aligned} \tag{3.11}$$

Thus, (3.5) holds. For each $\tau\in\mathbb{R}^+_0$, let $D^u_\tau\subset H^u_\tau$ be any r_u-dimensional subspace, and let $D^s_\tau\subset H^s_\tau$ be any r_s-dimensional subspace. By (3.9), we have $H^u_\tau\cap H^s_\tau=\{0\}$, and hence $D^u_\tau\cap D^s_\tau=\{0\}$. So we have $\mathbb{R}^p=H^u_\tau\oplus H^s_\tau$. We want to show that

$$H^s_\tau=D^s_\tau \quad\text{and}\quad H^u_\tau=D^u_\tau.$$

If there exists $x\in H^s_\tau\backslash D^s_\tau$, then we write $x=y+z$ with $y\in D^s_\tau$ and $z\in D^u_\tau\backslash\{0\}$. By (3.2) we have

$$\log(\lambda+\sigma)-\delta>\log(\lambda-\sigma)+\gamma.$$

Hence, it follows from (3.5) and (3.6) that

$$\chi^+_\tau(x)=\max\{\chi^+_\tau(y),\chi^+_\tau(z)\}=\chi^+_\tau(z)\ge\log(\lambda+\sigma)-\delta,$$

which contradicts to (3.6). This implies that $H^s_\tau=D^s_\tau$ for each $\tau\in\mathbb{R}^+_0$. We can show in a similar manner that $H^u_\tau=D^u_\tau$ for each $\tau\in\mathbb{R}^+_0$. By (2.10) and (3.10), for every $x\in H^s_\tau$ and $t\ge\tau$ we have

$$\begin{aligned} \|T(t,\tau)x\| &\le C\Big(\frac{\mu(t)}{\mu(\tau)}\Big)^{\log(\lambda-\sigma)}\nu^\gamma(t)|V(\tau,x)| \\ &\le C\Big(\frac{\mu(t)}{\mu(\tau)}\Big)^{\log(\lambda-\sigma)}\Big(\frac{\nu(t)}{\nu(\tau)}\Big)^\gamma\nu^{\delta+\gamma}(\tau)\|x\| \\ &\le C\Big(\frac{\mu(t)}{\mu(\tau)}\Big)^{\log(\lambda-\sigma)+\gamma}\nu^{\delta+\gamma}(\tau)\|x\|. \end{aligned} \tag{3.12}$$

Moreover, by (2.13) and (3.11), for every $x\in H^u_\tau$ and $t\ge\tau$ we have

$$\begin{aligned} \|T(t,\tau)x\| &\ge \Big(\frac{\mu(t)}{\mu(\tau)}\Big)^{\log(\lambda+\sigma)}\frac{\nu^{-\delta}(t)}{C}|V(\tau,x)| \\ &\ge \frac{\nu^{-\delta}(t)}{C}\Big(\frac{\mu(t)}{\mu(\tau)}\Big)^{\log(\lambda+\sigma)}\frac{\nu^{-\gamma}(\tau)}{C}\|x\| \\ &\ge \frac{1}{C^2}\Big(\frac{\mu(t)}{\mu(\tau)}\Big)^{\log(\lambda+\sigma)}\Big(\frac{\nu(t)}{\nu(\tau)}\Big)^\gamma\nu^{-(\delta+\gamma)}(t)\|x\| \\ &\ge \frac{1}{C^2}\Big(\frac{\mu(t)}{\mu(\tau)}\Big)^{\log(\lambda+\sigma)}\nu^{-(\delta+\gamma)}(t)\|x\|. \end{aligned} \tag{3.13}$$

Hence, it gives

$$\|T(t,\tau)^{-1}x\| \le C^2\Big(\frac{\mu(t)}{\mu(\tau)}\Big)^{-\log(\lambda+\sigma)}\nu^{\delta+\gamma}(t)\|x\| \tag{3.14}$$

for every $x \in H^u_\tau$ and $t \ge \tau$. This completes the proof. □

Now we establish a criterion for the existence of partially hyperbolic behavior in terms of pairs of strict Lyapunov functions. Without loss of generality we consider the same constants δ and γ for the two functions in each pair.

Theorem 3.2. *Let $\lambda_1 > \lambda_2 > 0$. If there exist a (λ_1,σ_1)-strict Lyapunov function V and a (λ_2,σ_2)-strict Lyapunov function W for (1.1) with*

$$(\lambda_i+\sigma_i)/(\lambda_i-\sigma_i) > e^{\delta+\gamma}, \quad i=1,2; \tag{3.15}$$

$$\lambda_1-\lambda_2 \ge |\sigma_1-\sigma_2|, \tag{3.16}$$

then for each $t,\tau \in \mathbb{R}^+_0$, the following statements hold.

(1) *The sets*

$$F^s_\tau = H^s_\tau(V), \quad F^u_\tau = H^u_\tau(W), \quad F^c_\tau = H^s_\tau(W)\cap H^u_\tau(V) \tag{3.17}$$

are linear subspaces, with

$$\mathbb{R}^p = F^u_\tau \oplus F^s_\tau \oplus F^c_\tau, \tag{3.18}$$

$$T(t,\tau)F^u_\tau = F^u_t, \quad T(t,\tau)F^s_\tau = F^s_t, \quad T(t,\tau)F^c_\tau = F^c_t; \tag{3.19}$$

(2) *for each $t \ge \tau$ we have*

$$\begin{gathered}\|T(t,\tau)^{-1}|F^u_t\| \le C^2\Big(\frac{\mu(t)}{\mu(\tau)}\Big)^{-(\log(\lambda_2+\sigma_2)-\gamma)}\nu^{\delta+\gamma}(t),\\ \|T(t,\tau)|F^s_\tau\| \le C^2\Big(\frac{\mu(t)}{\mu(\tau)}\Big)^{(\log(\lambda_1-\sigma_1)+\gamma)}\nu^{\delta+\gamma}(\tau),\end{gathered} \tag{3.20}$$

and

$$\begin{gathered}\|T(t,\tau)|F^c_\tau\| \le C^2\Big(\frac{\mu(t)}{\mu(\tau)}\Big)^{(\log(\lambda_2-\sigma_2)+\gamma)}\nu^{\delta+\gamma}(\tau),\\ \|T(t,\tau)^{-1}|F^c_t\| \le C^2\Big(\frac{\mu(t)}{\mu(\tau)}\Big)^{-(\log(\lambda_1+\sigma_1)-\gamma)}\nu^{\delta+\gamma}(t).\end{gathered} \tag{3.21}$$

Proof. According to Lemma 3.1, it is easy to prove Part 2. Using a similar argument as in [10], it can be obtained for every $\tau \in \mathbb{R}^+_0$ that

$$\begin{gathered}H^s_\tau(V) \subset H^s_\tau(W), \quad H^u_\tau(W) \subset H^s_\tau(V),\\ H^s_\tau(W)\cap H^u_\tau(V) \oplus H^s_\tau(V) \oplus H^u_\tau(W) = \mathbb{R}^p.\end{gathered}$$

So Part 1 holds. This completes the proof. □

Now we consider the particular case of differentiable Lyapunov functions. Set

$$\begin{aligned}\dot V(t,x) &= \frac{d}{dh}V(t+h,T(t+h,\tau)x)_{|h=0}\\ &= \frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x)A(t)x.\end{aligned}$$

Proposition 3.3. *Let V and μ be C^1 functions.*

(1) *For each* $x \in H_\tau^u$, *property* (2.11) *is equivalent to*

$$\dot V(t,T(t,\tau)x) \geq V(t,T(t,\tau)x)\frac{\mu'(t)}{\mu(t)}\ln(\lambda+\sigma), \quad t>\tau. \tag{3.22}$$

(2) *For each* $x \in H_\tau^s$, *property* (2.12) *is equivalent to*

$$\dot V(t,T(t,\tau)x) \geq V(t,T(t,\tau)x)\frac{\mu'(t)}{\mu(t)}\ln(\lambda-\sigma), \quad t>\tau. \tag{3.23}$$

Proof. Let $x \in H_\tau^u$, by (3.19) we have $T(t,\tau)x \in H_t^u$ for every $t \in \mathbb{R}_0^+$. We assume that (2.11) holds. If $t>\tau$ and $h>0$ then

$$V(t+h,T(t+h,\tau)x) \geq \Big(\frac{\mu(t+h)}{\mu(t)}\Big)^{\ln(\lambda+\sigma)} V(t,T(t,\tau)x),$$

and

$$\begin{aligned}
&\lim_{h\to 0^+}\frac{V(t+h,T(t+h,\tau)x)-V(t,T(t,\tau)x)}{h}\\
&\geq V(t,T(t,\tau)x)\lim_{h\to 0^+}\frac{\Big(\frac{\mu(t+h)}{\mu(t)}\Big)^{\ln(\lambda+\sigma)}-1}{h}\\
&= V(t,T(t,\tau)x)\frac{\mu'(t)}{\mu(\tau)}\ln(\lambda+\sigma).
\end{aligned}$$

Similarly, If $h<0$ is such that $t+h>\tau$, then

$$V(t+h,T(t+h,\tau)x) \leq \Big(\frac{\mu(t+h)}{\mu(t)}\Big)^{\ln(\lambda+\sigma)} V(t,T(t,\tau)x),$$

and

$$\begin{aligned}
&\lim_{h\to 0^-}\frac{V(t+h,T(t+h,\tau)x)-V(t,T(t,\tau)x)}{h}\\
&\geq V(t,T(t,\tau)x)\lim_{h\to 0^-}\frac{\Big(\frac{\mu(t+h)}{\mu(t)}\Big)^{\ln(\lambda+\sigma)}-1}{h}\\
&= V(t,T(t,\tau)x)\frac{\mu'(t)}{\mu(\tau)}\ln(\lambda+\sigma).
\end{aligned}$$

This establishes (3.22). Now we assume that (3.22) holds. Given $x \in H_\tau^u\backslash\{0\}$, it follows from (2.13) that $V(\tau,x)>0$, and thus, by (2.9), $V(t,T(t,\tau)x)>0$ for every $t \in \mathbb{R}_0^+$. Thus, we can rewrite (3.22) in the form

$$\frac{\dot V(t,T(t,\tau)x)}{V(t,T(t,\tau)x} \geq \frac{\mu'(t)}{\mu(t)}\ln(\lambda+\sigma),$$

which implies that

$$\ln\frac{V(t,T(t,\tau)x)}{V(\tau,x)} = \int_\tau^t \frac{\dot V(s,T(s,\tau)x)ds}{V(s,T(s,\tau)x)} \geq \ln(\lambda+\sigma)\ln\frac{\mu(t)}{\mu(\tau)}.$$

Hence, (2.11) holds. Part 2 is true. □

4. Quadratic Lyapunov functions

In this section, we give a complete characterization of nonuniform (μ,ν) trichotomies by using quadratic Lyapunov functions. For each $t\in\mathbb{R}_0^+$, let $S(t)$ and $T(t)$ be symmetric invertible $p\times p$ matrices. We consider the functions

$$G(t,x)=\langle S(t)x,x\rangle,\quad V(t,x)=-\operatorname{sign}G(t,x)\sqrt{|G(t,x)|}, \tag{4.1}$$

$$H(t,x)=\langle T(t)x,x\rangle,\quad W(t,x)=-\operatorname{sign}H(t,x)\sqrt{|H(t,x)|}. \tag{4.2}$$

Any Lyapunov functions V and W as in (4.1) and (4.2) are called quadratic Lyapunov functions. Notice that when $t\mapsto S(t)$ is differentiable we have

$$\begin{aligned}\dot V(t,x)&=\frac{\partial V}{\partial t}(t,x)+\frac{\partial V}{\partial x}(t,x)A(t)x\\&=\langle S'(t)x,x\rangle+2\langle S(t)x,A(t)x\rangle\end{aligned}$$

with similar identities for $\dot W$. We present in two theorems a characterization of nonuniform (μ,ν) trichotomies in terms of quadratic Lyapunov functions.

Theorem 4.1. *Assume that* (1.1) *admits a nonuniform (μ,ν) trichotomy. Then there exist symmetric invertible $p\times p$ matrices $S(t)$ and $T(t)$ for $t\in\mathbb{R}_0^+$ such that:*

(1) *$t\mapsto S(t)$ and $t\mapsto T(t)$ are of class C^1, and*

$$\limsup_{t\to\pm\infty}\frac{\log\|S(t)\|}{\log\nu(t)}<\infty, \tag{4.3}$$

$$\limsup_{t\to\pm\infty}\frac{\log\|T(t)\|}{\log\nu(t)}<\infty; \tag{4.4}$$

(2) *there exist $K_1>K_2>0$ and $L_1>L_2>0$ such that for every $t\in\mathbb{R}_0^+$ and $x\in\mathbb{R}^p$ we have*

$$\dot G(t,x)\le\begin{cases}-K_1\frac{\mu'(t)}{\mu(t)}G(t,x)-\frac12\frac{\mu'(t)}{\mu(t)}\|x\|^2 & \text{if } G(t,x)\ge0,\\ -K_2\frac{\mu'(t)}{\mu(\tau)}G(t,x)-\frac12\frac{\mu'(t)}{\mu(\tau)}\|x\|^2 & \text{if } G(t,x)\le0,\end{cases} \tag{4.5}$$

$$\dot H(t,x)\le\begin{cases}L_2\frac{\mu'(t)}{\mu(t)}H(t,x)-\frac12\frac{\mu'(t)}{\mu(t)}\|x\|^2 & \text{if } H(t,x)\ge0,\\ L_1\frac{\mu'(t)}{\mu(\tau)}H(t,x)-\frac12\frac{\mu'(t)}{\mu(\tau)}\|x\|^2 & \text{if } H(t,x)\le0.\end{cases} \tag{4.6}$$

Proof. Take $\varrho_1\in(0,(\alpha-\eta)/2)$. Let $\Gamma_1(t)=Q(t)\oplus R(t)$. Consider the matrices

$$\begin{aligned}S(t)&=\int_t^\infty T(\tau,t)^*P(\tau)^*P(\tau)T(\tau,t)\Big(\frac{\mu(\tau)}{\mu(t)}\Big)^{2(\alpha-\varrho_1)}\frac{\mu'(\tau)}{\mu(\tau)}d\tau\\&\quad-\int_{-\infty}^t T(\tau,t)^*\Gamma_1(\tau)^*\Gamma_1(\tau)T(\tau,t)\Big(\frac{\mu(t)}{\mu(\tau)}\Big)^{-2(\eta+\varrho_1)}\frac{\mu'(\tau)}{\mu(\tau)}d\tau.\end{aligned} \tag{4.7}$$

Similarly, Take $\varrho_2\in(0,(\beta-\xi)/2)$, and let $\Gamma_2(t)=P(t)\oplus R(t)$. Consider also the matrices

$$\begin{aligned}T(t)&=\int_t^\infty T(\tau,t)^*\Gamma_2(\tau)^*\Gamma_2(\tau)T(\tau,t)\Big(\frac{\mu(\tau)}{\mu(t)}\Big)^{-2(\xi+\varrho_2)}\frac{\mu'(\tau)}{\mu(\tau)}d\tau\\&\quad-\int_{-\infty}^t T(\tau,t)^*Q(\tau)^*Q(\tau)T(\tau,t)\Big(\frac{\mu(t)}{\mu(\tau)}\Big)^{2(\beta-\varrho_2)}\frac{\mu'(\tau)}{\mu(\tau)}d\tau.\end{aligned} \tag{4.8}$$

The matrices $S(t)$ and $T(t)$ are symmetric and invertible for each $t \in \mathbb{R}_0^+$. We define the functions G and H by (4.1) and (4.2). By (2.4) and (2.5), since μ is an increasing function we have

$$\begin{aligned}
|G(t,x)| &\le \int_t^\infty \|T(\tau,t)P(t)x\|^2 \Big(\frac{\mu(\tau)}{\mu(t)}\Big)^{2(\alpha-\varrho_1)} \frac{\mu'(\tau)}{\mu(\tau)} d\tau \\
&\quad + \int_{-\infty}^t \|T(\tau,t)\Gamma_1(t)x\|^2 \Big(\frac{\mu(t)}{\mu(\tau)}\Big)^{-2(\eta+\varrho_1)} \frac{\mu'(\tau)}{\mu(\tau)} d\tau \\
&\le D^2\|x\|^2\nu^{2\epsilon}(t) \int_t^\infty \Big(\frac{\mu(\tau)}{\mu(t)}\Big)^{-2\varrho_1} \frac{\mu'(\tau)}{\mu(\tau)} d\tau \\
&\quad + 4D^2\|x\|^2\nu^{2\epsilon}(t) \int_{-\infty}^t \Big(\frac{\mu(t)}{\mu(\tau)}\Big)^{-2\varrho_1} \frac{\mu'(\tau)}{\mu(\tau)} d\tau \\
&= \frac{D^2}{2\varrho_1}\|x\|^2\nu^{2\epsilon}(t) + \frac{4D^2}{2\varrho_1}\|x\|^2\nu^{2\epsilon}(t)\Big(1 - \lim_{\tau\mapsto-\infty}\Big(\frac{\mu(t)}{\mu(\tau)}\Big)^{-2\varrho_1}\Big) \\
&\le \frac{5D^2}{2\varrho_1}\|x\|^2\nu^{2\epsilon}(t).
\end{aligned}$$

Since

$$\frac{\partial}{\partial t}T(\tau,t) = -T(\tau,t)A(t) \quad \text{and} \quad \frac{\partial}{\partial t}T(\tau,t)^* = -A(t)^*T(\tau,t)^*,$$

one can easily verify that $S(t)$ and $T(t)$ are of class C^1 in t. Moreover, since $S(t)$ is symmetric we obtain

$$\|S(t)\| = \sup_{x\neq 0}\frac{|G(t,x)|}{\|x\|} \le \frac{5D^2}{2\varrho_1}\nu^{2\epsilon}(t) \tag{4.9}$$

and (4.3) holds. Similar arguments apply to $T(t)$ to obtain (4.4). Furthermore, taking derivatives in (4.7) we obtain

$$\begin{aligned}
S'(t) &= -P(t)^*P(t)\frac{\mu'(t)}{\mu(t)} \\
&\quad - \int_t^\infty A(t)^*T(\tau,t)^*P(\tau)^*P(\tau)T(\tau,t)\Big(\frac{\mu(\tau)}{\mu(t)}\Big)^{2(\alpha-\varrho_1)}\frac{\mu'(\tau)}{\mu(\tau)}d\tau \\
&\quad - \int_t^\infty T(\tau,t)^*P(\tau)^*P(\tau)T(\tau,t)A(t)\Big(\frac{\mu(\tau)}{\mu(t)}\Big)^{2(\alpha-\varrho_1)}\frac{\mu'(\tau)}{\mu(\tau)}d\tau \\
&\quad - 2(\alpha-\varrho_1)\frac{\mu'(t)}{\mu(t)}\int_t^\infty T(\tau,t)^*P(\tau)^*P(\tau)T(\tau,t)\Big(\frac{\mu(\tau)}{\mu(t)}\Big)^{2(\alpha-\varrho_1)}\frac{\mu'(\tau)}{\mu(\tau)}d\tau \\
&\quad - \Gamma_1(t)^*\Gamma_1(t)\frac{\mu'(t)}{\mu(t)} \\
&\quad + \int_{-\infty}^t A(t)^*T(\tau,t)^*\Gamma_1(\tau)^*\Gamma_1(\tau)T(\tau,t)\Big(\frac{\mu(t)}{\mu(\tau)}\Big)^{-2(\eta+\varrho_1)}\frac{\mu'(\tau)}{\mu(\tau)}d\tau. \\
&\quad + \int_{-\infty}^t T(\tau,t)^*\Gamma_1(\tau)^*\Gamma_1(\tau)T(\tau,t)A(t)\Big(\frac{\mu(t)}{\mu(\tau)}\Big)^{-2(\eta+\varrho_1)}\frac{\mu'(\tau)}{\mu(\tau)}d\tau. \\
&\quad + 2(\eta+\varrho_1)\frac{\mu'(t)}{\mu(t)}\int_{-\infty}^t T(\tau,t)^*\Gamma_1(\tau)^*\Gamma_1(\tau)T(\tau,t)\Big(\frac{\mu(t)}{\mu(\tau)}\Big)^{-2(\eta+\varrho_1)}\frac{\mu'(\tau)}{\mu(\tau)}d\tau \\
&= -[P(t)^*P(t) + \Gamma_1(t)^*\Gamma_1(t)]\frac{\mu'(t)}{\mu(t)} - A(t)^*S(t) - S(t)A(t)
\end{aligned}$$

$$- 2(\alpha-\varrho_1)\frac{\mu'(t)}{\mu(t)}\int_t^\infty T(\tau,t)^*P(\tau)^*P(\tau)T(\tau,t)\Big(\frac{\mu(\tau)}{\mu(t)}\Big)^{2(\alpha-\varrho_1)}\frac{\mu'(\tau)}{\mu(\tau)}d\tau$$
$$+ 2(\eta+\varrho_1)\frac{\mu'(t)}{\mu(t)}\int_{-\infty}^t T(\tau,t)^*\Gamma_1(\tau)^*\Gamma_1(\tau)T(\tau,t)\Big(\frac{\mu(t)}{\mu(\tau)}\Big)^{-2(\eta+\varrho_1)}\frac{\mu'(\tau)}{\mu(\tau)}d\tau.$$

Thus, we have

$$\begin{aligned}
&S'(t)+A(t)^*S(t)+S(t)A(t)+[P(t)^*P(t)+\Gamma_1(t)^*\Gamma_1(t)]\frac{\mu'(t)}{\mu(t)}\\
&= -2(\alpha-\varrho_1)\frac{\mu'(t)}{\mu(t)}\int_t^\infty T(\tau,t)^*P(\tau)^*P(\tau)T(\tau,t)\Big(\frac{\mu(\tau)}{\mu(t)}\Big)^{2(\alpha-\varrho_1)}\frac{\mu'(\tau)}{\mu(\tau)}d\tau\\
&\quad + 2(\eta+\varrho_1)\frac{\mu'(t)}{\mu(t)}\int_{-\infty}^t T(\tau,t)^*\Gamma_1(\tau)^*\Gamma_1(\tau)T(\tau,t)\Big(\frac{\mu(t)}{\mu(\tau)}\Big)^{-2(\eta+\varrho_1)}\frac{\mu'(\tau)}{\mu(\tau)}d\tau\\
&= -2(\alpha-\varrho_1)\frac{\mu'(t)}{\mu(t)}\int_t^\infty (T(\tau,t)P(t))^*T(\tau,t)P(t)\Big(\frac{\mu(\tau)}{\mu(t)}\Big)^{2(\alpha-\varrho_1)}\frac{\mu'(\tau)}{\mu(\tau)}d\tau\\
&\quad + 2(\eta+\varrho_1)\frac{\mu'(t)}{\mu(t)}\int_{-\infty}^t (T(\tau,t)\Gamma_1(t))^*T(\tau,t)\Gamma_1(t)\Big(\frac{\mu(t)}{\mu(\tau)}\Big)^{-2(\eta+\varrho_1)}\frac{\mu'(\tau)}{\mu(\tau)}d\tau.
\end{aligned} \tag{4.10}$$

Since

$$\begin{aligned}
2\langle (P(t)^*P(t)+\Gamma_1(t)^*\Gamma_1(t))x,x\rangle &\ge (\|P(t)x\|+\|\Gamma_1(t)x\|)^2\\
&\ge \|(P(t)+\Gamma_1(t))x\|^2=\|x\|,
\end{aligned} \tag{4.11}$$

we have

$$P(t)^*P(t)+\Gamma_1(t)^*\Gamma_1(t)\ge \frac{1}{2}\mathrm{Id}. \tag{4.12}$$

Given two $p\times p$ matrices A and B, we say that $A\ge B$ if $\langle Ax,x\rangle \ge \langle Bx,x\rangle$ for every $x\in\mathbb{R}^p$. Furthermore, if $x(t)$ is a solution of (1.1), then we obain

$$\begin{aligned}
\frac{d}{dt}G(t,x(t)) &= \langle S'(t)x(t),x(t)\rangle + \langle S(t)x'(t),x(t)\rangle + \langle S(t)x(t),x'(t)\rangle\\
&= \langle (S'(t)+A(t)^*S(t)+S(t)A(t))x(t),x(t)\rangle.
\end{aligned} \tag{4.13}$$

We note that

$$\begin{aligned}
G(t,x(t)) &= \int_t^\infty \|T(\tau,t)P(t)\|^2\Big(\frac{\mu(\tau)}{\mu(t)}\Big)^{2(\alpha-\varrho_1)}\frac{\mu'(\tau)}{\mu(\tau)}d\tau\\
&\quad -\int_{-\infty}^t \|T(\tau,t)\Gamma_1(t)\|^2\Big(\frac{\mu(t)}{\mu(\tau)}\Big)^{-2(\eta+\varrho_1)}\frac{\mu'(\tau)}{\mu(\tau)}d\tau\,.
\end{aligned}$$

If $G(t,x(t))\ge 0$, then by (4.10), (4.12) and (4.13), since $\eta+\varrho_1<\alpha-\varrho_1$ and μ is an increasing function, we obtain

$$\begin{aligned}
\frac{d}{dt}G(t,x(t)) &\le -\frac{1}{2}\|x(t)\|^2\frac{\mu'(t)}{\mu(t)}\\
&\quad - 2(\alpha-\varrho_1)\frac{\mu'(t)}{\mu(t)}\int_t^\infty \|T(\tau,t)P(t)\|^2\Big(\frac{\mu(\tau)}{\mu(t)}\Big)^{2(\alpha-\varrho_1)}\frac{\mu'(\tau)}{\mu(\tau)}d\tau\\
&\quad + 2(\alpha-\varrho_1)\frac{\mu'(t)}{\mu(t)}\int_{-\infty}^t \|T(\tau,t)\Gamma_1(t)\|^2\Big(\frac{\mu(t)}{\mu(\tau)}\Big)^{-2(\eta+\varrho_1)}\frac{\mu'(\tau)}{\mu(\tau)}d\tau\\
&= -\frac{1}{2}\|x(t)\|^2\frac{\mu'(t)}{\mu(t)} - 2(\alpha-\varrho_1)\frac{\mu'(t)}{\mu(t)}G(t,x(t)).
\end{aligned}$$

Thus, we can take $K_1 = 2(\alpha - \varrho_1) > 0$. On the other hand, if $G(t, x(t)) \le 0$, then by (4.10), (4.12) and (4.13) again, since $\eta + \varrho_1 < \alpha - \varrho_1$ and μ is an increasing function, we obtain

$$\begin{aligned}
&\frac{d}{dt}G(t,x(t))\\
&\le -\frac{1}{2}\|x(t)\|^2\frac{\mu'(t)}{\mu(t)} - 2(\eta+\varrho_1)\frac{\mu'(t)}{\mu(t)}\int_t^\infty \|T(\tau,t)P(t)\|^2\Big(\frac{\mu(\tau)}{\mu(t)}\Big)^{2(\alpha-\varrho_1)}\frac{\mu'(\tau)}{\mu(\tau)}d\tau\\
&\quad + 2(\eta+\varrho_1)\frac{\mu'(t)}{\mu(t)}\int_{-\infty}^t \|T(\tau,t)\Gamma_1(t)\|^2\Big(\frac{\mu(t)}{\mu(\tau)}\Big)^{-2(\eta+\varrho_1)}\frac{\mu'(\tau)}{\mu(\tau)}d\tau\\
&= -\frac{1}{2}\|x(t)\|^2\frac{\mu'(t)}{\mu(t)} - 2(\eta+\varrho_1)\frac{\mu'(t)}{\mu(t)}G(t,x(t)).
\end{aligned}$$

Take $K_2 = 2(\eta + \varrho_1) > 0$. Furthermore, we have

$$K_1 - K_2 = 2(\alpha - \eta - 2\varrho_1) > 0.$$

Proceeding in a similar manner with $T(t)$ we deduce that

$$\begin{aligned}
&T'(t) + A(t)^*T(t) + T(t)A(t) + [\Gamma_2(t)^*\Gamma_2(t) + Q(t)^*Q(t)]\frac{\mu'(t)}{\mu(t)}\\
&= 2(\xi+\varrho_2)\frac{\mu'(t)}{\mu(t)}\int_t^\infty (T(\tau,t)\Gamma_2(t))^*T(\tau,t)\Gamma_2(t)\Big(\frac{\mu(\tau)}{\mu(t)}\Big)^{-2(\xi+\varrho_2)}\frac{\mu'(\tau)}{\mu(\tau)}d\tau\\
&\quad - 2(\beta-\varrho_2)\frac{\mu'(t)}{\mu(t)}\int_{-\infty}^t (T(\tau,t)Q(t))^*T(\tau,t)Q(t)\Big(\frac{\mu(t)}{\mu(\tau)}\Big)^{2(\beta-\varrho_2)}\frac{\mu'(\tau)}{\mu(\tau)}d\tau
\end{aligned}$$

Note that inequalities (4.6) hold with $L_1 = 2(\beta - \varrho_2) > 0$ and $L_2 = 2(\xi + \varrho_2) > 0$. Moreover, by the choice of ϱ_2 we have

$$L_1 - L_2 = 2(\beta - \xi - 2\varrho_2) > 0.$$

This completes the proof. ☐

Theorem 4.2. *Assume that there exist constants $\gamma, a, \kappa > 0$ such that*

$$\|T(t,s)\| \le \kappa\nu^a(t) \quad \text{whenever } |t-s| \le \gamma. \tag{4.14}$$

Also assume that μ, ν is of class C^1, and that there exist symmetric invertible $p \times p$ matrices $S(t)$ and $T(t)$ for $t \in R_0^+$, satisfying conditions 1 and 2 in Theorem 4.1 with $K_1 - K_2 > 2a$ and $L_1 - L_2 > 2a$. Then (1.1) admits a nonuniform (μ, ν) trichotomy with

$$\alpha = \frac{K_1}{2} - a, \quad \beta = \frac{L_1}{2} - a, \quad \xi = \frac{L_2}{2} + a, \quad \eta = \frac{K_2}{2} + a. \tag{4.15}$$

Proof. We start with some auxiliary results. Set

$$\begin{aligned}
I_\tau^s &= \{0\} \cup \{x \in \mathbb{R}^p : G(t, T(t,\tau)x) > 0 \text{ for every } t \ge \tau\},\\
I_\tau^u &= \{0\} \cup \{x \in \mathbb{R}^p : G(t, T(t,\tau)x) < 0 \text{ for every } t \ge \tau\}.
\end{aligned}$$

Lemma 4.3. *If $x \in I_\tau^s$ then*

$$G(t, T(t,\tau)x) \le \Big(\frac{\mu(t)}{\mu(\tau)}\Big)^{-K_1} G(\tau, x), \quad t \ge \tau, \tag{4.16}$$

and if $x \in I_\tau^u$ then

$$|G(t, T(t,\tau)x)| \ge \Big(\frac{\mu(t)}{\mu(\tau)}\Big)^{-K_2} |G(\tau, x)|, \quad t \ge \tau. \tag{4.17}$$

Proof. Given $x \in I_\tau^s \setminus \{0\}$, since $G(t, T(t,\tau)x) > 0$ for every $t \geq \tau$, it follows from (4.5) that

$$\frac{\dot{G}(t, T(t,\tau)x)}{G(t, T(t,\tau)x)} \leq -K_1 \frac{\mu'(t)}{\mu(t)}, \quad t > \tau.$$

This implies that

$$\ln \frac{G(t, T(t,\tau)x)}{G(\tau, x)} \leq -K_1 \int_\tau^t \frac{\mu'(s)}{\mu(s)} ds = -K_1(\ln \mu(t) - \ln \mu(\tau)).$$

Hence, (4.16) holds. Similarly, given $x \in I_\tau^u \setminus \{0\}$, since $G(t, T(t,\tau)x) < 0$ for every $t \geq \tau$, it follows from (4.5) that

$$\frac{\dot{G}(t, T(t,\tau)x)}{G(t, T(t,\tau)x)} \geq -K_2 \frac{\mu'(t)}{\mu(t)}, \ t > \tau.$$

This implies that

$$\ln \Big| \frac{G(t, T(t,\tau)x)}{G(\tau, x)} \Big| \geq -K_2 \int_\tau^t \frac{\mu'(s)}{\mu(s)} ds = -K_2(\ln \mu(t) - \ln \mu(\tau)).$$

Thus, (4.17) holds. □

Lemma 4.4. *If $x \in I_\tau^u \cup I_\tau^s$ then there holds*

$$|G(\tau, x)| \geq \frac{1}{2\kappa^2} \max\{\gamma, (1 - e^{-K_2\gamma})/K_2\} \nu^{-2a}(\tau) \|x\|^2.$$

Proof. Set $x(t) = T(t,\tau)x$. It follows from (4.5) that if $x \in I_\tau^s$, then it holds

$$\frac{d}{dt} G(t, x(t)) \leq -\frac{1}{2} \|x(t)\|^2 \frac{\mu'(t)}{\mu(t)}.$$

Since μ is a growth rate, given $\tau \in \mathbb{R}_0^+$, we take $t > \tau$ such that $\ln \mu(t) = \ln \mu(\tau) + \gamma$ (with γ as in (4.14)). Then

$$\begin{aligned}
G(t, x(t)) - G(\tau, x) &= \int_\tau^t \frac{d}{ds} G(s, x(s)) ds \leq -\frac{1}{2} \int_\tau^t \|x(s)\|^2 \frac{\mu'(s)}{\mu(s)} ds \\
&= -\frac{1}{2} \int_\tau^t \|T(s,\tau)x\|^2 \frac{\mu'(s)}{\mu(s)} ds \\
&\leq -\frac{1}{2} \|x\|^2 \int_\tau^t \frac{1}{\|T(\tau, s)\|^2} \frac{\mu'(s)}{\mu(s)} ds.
\end{aligned}$$

It follows from (4.14) that

$$\begin{aligned}
G(t, x(t)) - G(\tau, x) &\leq -\frac{1}{2} \|x\|^2 \int_\tau^t \frac{\nu^{-2a}(\tau)}{\kappa^2} \frac{\mu'(s)}{\mu(s)} ds \\
&= -\frac{1}{2\kappa^2} \nu^{-2a}(\tau) \|x\|^2 (\ln \mu(t) - \ln \mu(\tau)) \\
&= -\frac{\gamma}{2\kappa^2} \nu^{-2a}(\tau) \|x\|^2.
\end{aligned}$$

So we have

$$G(\tau, x) \geq G(\tau, x) - G(t, x(t)) \geq \frac{\gamma}{2\kappa^2} \nu^{-2a}(\tau) \|x\|^2.$$

On the other hand, it follows from (4.5) that if $x \in I_\tau^u$, then it has

$$\frac{d}{dt}(\mu^{K_2}(t) G(t, x(t)) = \mu^{K_2}(t) \Big(\frac{d}{dt} G(t, x(t)) + K_2 \frac{\mu'(t)}{\mu(t)} G(t, x(t)) \Big)$$

$$\leq -\frac{1}{2}\frac{\mu'(t)}{\mu(t)}\mu^{K_2}(t)\|x(t)\|^2.$$

Hence, given $\tau \in \mathbb{R}_0^+$ and $t < \tau$ such that $\ln \mu(\tau) = \ln \mu(t) + \gamma$, using (4.14) again we obtain

$$\begin{aligned}\mu^{K_2}(\tau)G(\tau, x) - \mu^{K_2}(t)G(t, x(t)) &\leq -\frac{1}{2\kappa^2}\nu^{-2a}(\tau)\|x\|^2 \int_t^\tau \frac{\mu'(s)}{\mu(s)}\mu^{K_2}(s)ds \\ &= -\frac{1}{2\kappa^2 K_2}\nu^{-2a}(\tau)(\mu^{K_2}(\tau) - \mu^{K_2}(t))\|x\|^2.\end{aligned}$$

Since $G(t, x(t)) < 0$ and $G(\tau, x) < 0$, we have

$$\begin{aligned}\mu^{K_2}(\tau)|G(\tau, x)| &\geq -\frac{1}{2\kappa^2 K_2}\nu^{-2a}(\tau)(\mu^{K_2}(\tau) - \mu^{K_2}(t))\|x\|^2, \\ |G(\tau, x)| &\geq -\frac{1}{2\kappa^2 K_2}\nu^{-2a}(\tau)\Big[1 - \Big(\frac{\mu(t)}{\mu(\tau)}\Big)^{K_2}\Big]\|x\|^2 \\ &= -\frac{1}{2\kappa^2 K_2}\nu^{-2a}(\tau)[1 - e^{-K_2\gamma}]\|x\|^2.\end{aligned}$$

This completes the proof. □

Now we set

$$\begin{aligned}J_\tau^s &= \{0\} \cup \{x \in \mathbb{R}^p : H(t, T(t, \tau)x) > 0 \text{ for every } t \geq \tau\}, \\ J_\tau^u &= \{0\} \cup \{x \in \mathbb{R}^p : H(t, T(t, \tau)x) < 0 \text{ for every } t \geq \tau\}.\end{aligned}$$

Proceeding in a similar manner to that in the proofs of Lemmas 4.3 and 4.4 we obtain the following statements.

Lemma 4.5. *If $x \in J_\tau^s$, then*

$$H(t, T(t, \tau)x) \leq \Big(\frac{\mu(t)}{\mu(\tau)}\Big)^{L_2} H(\tau, x), \quad t \geq \tau, \tag{4.18}$$

and if $x \in J_\tau^u$, then

$$|H(t, T(t, \tau)x)| \geq \Big(\frac{\mu(t)}{\mu(\tau)}\Big)^{L_1} |H(\tau, x)|, \quad t \geq \tau. \tag{4.19}$$

Lemma 4.6. *If $x \in J_\tau^u \cup J_\tau^s$, then*

$$|H(\tau, x)| \geq \frac{1}{2\kappa^2} \max\{\gamma, (1 - e^{-L_2\gamma})/L_2\}\nu^{-2a}(\tau)\|x\|^2.$$

Lemma 4.7. *The function V in (4.1) is a (λ_1, σ_1) strict Lyapunov function for (1.1) with*

$$\lambda_1 = \frac{e^{-K_1/2} + e^{-K_2/2}}{2}, \quad \sigma_1 = \frac{e^{-K_2/2} - e^{-K_1/2}}{2}.$$

Proof. By (4.3), for each $\delta > 0$ there exists $d > 0$ such that

$$\|S(t)\| \leq d\nu^\delta(t),$$

for every $t \in \mathbb{R}_0^+$. So we have

$$|G(t, x)| \leq d\nu^\delta(t)\|x\|^2.$$

That is,

$$|V(t, x)| \leq \sqrt{d}\nu^{\delta/2}(t)\|x\|^2,$$

and (2.10) holds. Furthermore, by Lemma 4.4, for $x \in I_\tau^u \cup I_\tau^s = H_\tau^u \cup H_\tau^s$ we have

$$|V(\tau,x)| \ge \frac{1}{\sqrt{2}\kappa} \max\{\gamma, (1-e^{-L_2\gamma})/L_2\}^{1/2} \nu^{-a}(\tau)\|x\|^2. \tag{4.20}$$

and (2.13) holds. Finally, by Lemma 4.3, if $x \in I_\tau^s = H_\tau^s$ then

$$|V(t,T(t,\tau)x)| \le \Big(\frac{\mu(t)}{\mu(\tau)}\Big)^{-K_1/2} |V(\tau,x)|, \quad t \ge \tau.$$

That is, (2.12) holds with $\lambda_1 - \sigma_1 = e^{-K_1/2}$. Moreover, if $x \in I_\tau^u = H_\tau^u$ then

$$V(t,T(t,\tau)x) \ge \Big(\frac{\mu(t)}{\mu(\tau)}\Big)^{-K_2/2} V(\tau,x), \quad t \ge \tau.$$

That is, (2.11) holds with $\lambda_1 + \sigma_1 = e^{-K_2/2}$. This completes the proof. □

In an analogous manner we can prove the following result.

Lemma 4.8. *The function W in (4.2) is a (λ_2,σ_2) strict Lyapunov function for (1.1) with*

$$\lambda_2 = \frac{e^{L_1/2}+e^{L_2/2}}{2}, \quad \sigma_2 = \frac{e^{L_1/2}-e^{L_2/2}}{2}.$$

Since V and W is a strict Lyapunov function, by Lemma 3.1, there exist subspaces $H_t^u(V)$ and $H_t^s(V)$, $H_t^u(W)$ and $H_t^s(W)$ such that $\mathbb{R}^p = H_t^u(V) \oplus H_t^s(V)$, $\mathbb{R}^p = H_t^u(W) \oplus H_t^s(W)$ for each $t \in \mathbb{R}_0^+$. We consider the associated projections

$$P_V(t): \mathbb{R}^p \to H_t^s(V), \quad Q_V(t): \mathbb{R}^p \to H_t^u(V),$$
$$P_W(t): \mathbb{R}^p \to H_t^s(W), \quad Q_W(t): \mathbb{R}^p \to H_t^u(W).$$

Lemma 4.9. *There exists $K>0$ such that for each $t \in \mathbb{R}_0^+$ we have*

$$\|P_V(t)\| = \|Q_V(t)\| \le K\nu^{2a}(t)\|S(t)\|,$$
$$\|P_W(t)\| = \|Q_W(t)\| \le K\nu^{2a}(t)\|T(t)\|.$$

Proof. We prove only the statement for the Lyapunov function V. The proof for W is completely analogous. We note that

$$V(t,P_V(t)x)^2 = \langle S(t)P_V(t)x, P_V(t)x\rangle, \tag{4.21}$$
$$V(t,Q_V(t)x)^2 = -\langle S(t)Q_V(t)x, Q_V(t)x\rangle. \tag{4.22}$$

Given $x \in \mathbb{R}^p$ we write $x = y+z$ with

$$y = P_V(t)x \in H_t^s(V) \quad \text{and} \quad z = Q_V(t)x \in H_t^u(V).$$

Now take $b(t)>0$, and set

$$V^s(t,y) = -V(t,y)^2 + b(t)\|y\|^2 = -\langle S(t)y,y\rangle + b(t)\|y\|^2.$$

By (4.20), there exists $K>0$ such that

$$V^s(t,y) \le -K\nu^{-2a}(t)\|y\|^2 + b(t)\|y\|^2 = (b(t) - K\nu^{-2a}(t))\|y\|^2.$$

Similarly, for each $t \in \mathbb{R}_0^+$ we set

$$V^u(t,z) = V(t,z)^2 - b(t)\|z\|^2 = -\langle S(t)z,z\rangle - b(t)\|z\|^2.$$

By (4.20), we have

$$V^u(t,z) \ge (K\nu^{-2a}(t) - b(t))\|z\|^2.$$

We conclude that if $b(t) \le K\nu^{-2a}(t)$, then

$$-V(t,y)^2 + b(t)\|y\|^2 \le 0, \quad V(t,z)^2 - b(t)\|z\|^2 \ge 0.$$

Thus, from (4.21) and (4.22) it follows that

$$\begin{gathered}
-\langle S(t)P_V(t)x, P_V(t)x\rangle + b(t)\|P_V(t)x\|^2 \le 0,\\
-\langle S(t)Q_V(t)x, Q_V(t)x\rangle - b(t)\|Q_V(t)x\|^2 \ge 0.
\end{gathered}$$

Since $S(t)$ is symmetric, subtracting the two inequalities we obtain

$$\begin{aligned}
0 &\ge b(t)\|P_V(t)x\|^2 + b(t)\|Q_V(t)x\|^2\\
&\quad - \langle S(t)P_V(t)x, P_V(t)x\rangle + \langle S(t)Q_V(t)x, Q_V(t)x\rangle\\
&= b(t)\|P_V(t)x\|^2 + b(t)\|Q_V(t)x\|^2 + \langle S(t)x, x\rangle - 2\langle S(t)P_V(t)x, x\rangle.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&b(t)\|P_V(t)x - \frac{1}{2b(t)}S(t)x\|^2 + b(t)\|Q_V(t)x + \frac{1}{2b(t)}S(t)x\|^2\\
&= b(t)\|P_V(t)x\|^2 + b(t)\|Q_V(t)x\|^2 + \frac{\|S(t)x\|}{2b(t)} + \langle S(t)x, x\rangle - 2\langle S(t)P_V(t)x, x\rangle\\
&\ge \frac{\|S(t)x\|}{2b(t)},
\end{aligned}$$

and

$$\|P_V(t)x - \frac{1}{2b(t)}S(t)x\|^2 + \|Q_V(t)x + \frac{1}{2b(t)}S(t)x\|^2 \le \frac{\|S(t)x\|}{2b(t)^2}.$$

This implies

$$\begin{aligned}
\|P_V(t)x\|^2 &= \|P_V(t)x - \frac{1}{2b(t)}S(t)x + \frac{1}{2b(t)}S(t)x\|\\
&\le \|P_V(t)x - \frac{1}{2b(t)}S(t)x\| + \frac{1}{2b(t)}\|S(t)x\|\\
&\le \frac{1}{\sqrt{2}b(t)}\|S(t)x\| + \frac{1}{\sqrt{2}b(t)}\|S(t)x\|\\
&\le \frac{\sqrt{2}}{b(t)}\|S(t)x\|.
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
\|Q_V(t)x\|^2 &\le \|Q_V(t)x - \frac{1}{2b(t)}S(t)x\| + \frac{1}{2b(t)}\|S(t)x\|\\
&\le \frac{1}{\sqrt{2}b(t)}\|S(t)x\| + \frac{1}{\sqrt{2}b(t)}\|S(t)x\|\\
&\le \frac{\sqrt{2}}{b(t)}\|S(t)x\|.
\end{aligned}$$

Taking $b(t) = \frac{\sqrt{2}}{K}\nu^{-2a}(t)$ we obtain the desired statement. □

Note that by taking δ sufficiently small we have

$$\frac{\lambda_1 + \sigma_1}{\lambda_1 - \sigma_1} = e^{(K_1 - K_2)/2} > e^{a+\delta/2},$$

$$\frac{\lambda_2+\sigma_2}{\lambda_2-\sigma_2} = e^{(L_1-L_2)/2} > e^{a+\delta/2}.$$

Moreover, we can easily verify that $\lambda_2 - \lambda_1 > |\sigma_2 - \sigma_1|$. This allows us to apply Theorem 3.2 (with V and W interchanged). If we set

$$\begin{gathered} P(\tau) = P_W(\tau) : \mathbb{R} \to F_\tau^s = H_\tau^s(W), \\ Q(\tau) = Q_V(\tau) : \mathbb{R} \to F_\tau^u = H_\tau^u(V), \\ R(\tau) = P_V(\tau) \oplus Q_W(\tau) : \mathbb{R} \to F_\tau^c = H_\tau^s(V) \cap H_\tau^u(W). \end{gathered}$$

The subspaces F_τ^s, F_τ^u and F_τ^c satisfy the properties in Theorem 3.2. Moreover, for every $t \geq \tau$ we have

$$\begin{gathered} \|T(t,\tau)P(\tau)\| \leq \|T(t,\tau)|F_\tau^s\| \, \|P(\tau)\|, \\ \|T(t,\tau)^{-1}Q(\tau)\| \leq \|T(t,\tau)^{-1}|F_\tau^u\| \, \|Q(\tau)\|, \\ \|T(t,\tau)R(\tau)\| \leq \|T(t,\tau)|F_\tau^c\| \, \|R(\tau)\|, \\ \|T(t,\tau)^{-1}R(\tau)\| \leq \|T(t,\tau)^{-1}|F_\tau^c\| \, \|R(\tau)\|. \end{gathered}$$

Hence, by property 2 in Theorem 3.2 and Lemma 4.9 there exist constants as in (2.3) satisfying (2.4) and (2.5). In other words, (1.1) admits a nonuniform (μ,ν) trichotomy. By (3.20), (3.21), and Lemma 4.9 we can take the constants α, β, ξ, η in (2.3) as in (4.15). This completes the proof. □

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References

[1] L. Barreira, J. Chu, C. Valls; *Robustness of a nonuniform dichotomy with different growth rates*, São Paulo J. Math. Sci. 5, 2 (2011), 1-29.
[2] L. Barreira, J. Chu, C. Valls; *Lyapunov Functions for General Nonuniform Dichotomies*, Milan J. Math. 81 (2013) 153-169.
[3] L. Barreira, Y. Pesin; *Nonuniform Hyperbolicity*, Encyclopedia Math. Appl., vol. 115, Cambridge Univ. Press, 2007.
[4] L. Barreira, C. Valls; *Robustness of nonuniform exponential dichotomies in Banach spaces*, J. Differential Equations, 244 (2008), 2407-2447.
[5] L. Barreira, C. Valls; *Robustness of general dichotomies*, J. Funct. Anal. 257 (2009), 464-484.
[6] L. Barreira, C. Valls; *Robustness of nonuniform exponential trichotomies in Banach spaces*, J. Math. Anal. Appl. 351 (2009), 373-381.
[7] L. Barreira, C. Valls; *Quadratic Lyapunov functions and nonuniform exponential dichotomies*, J. Differential Equations, 246 (2009), 1235-1263.
[8] L. Barreira, C. Valls; *Robustness via Lyapunov functions*, J. Differential Equations, 246 (2009), 2891-2907.
[9] L. Barreira, C. Valls; *Lyapunov functions versus exponential contractions*, Math. Z. 268 (2011), 187-196.
[10] L. Barreira, C. Valls; *Lyapunov functions for trichotomies with growth rates*, J. Differential Equations, 248 (2010) 151-183.
[11] S. Elaydi, O. Hájek; *Exponential trichotomy of differential systems*, J. Math. Anal. Appl. 129 (1988), 362-374.
[12] S. Elaydi, O. Hájek; *Exponential dichotomy and trichotomy of nonlinear differential equations*, Differential Integral Equations, 3, (1990), 1201-1224.
[13] A. J. G. Bento, C. Silva; *Nonuniform (μ,ν)-dichotomies and local dynamics of difference equations*, Nonlinear Anal. 75 (2012), 78-90.

[14] N. Bhatia, G. Szegio; *Stability Theory of Dynamical Systems*, Grundlehren der mathematischen Wissenschaften 161, Springer, 1970.
[15] W. A. Coppel; *Dichotomy in Stability Theory*, Lecture Notes in Mathematics, Vol. 629, Springer-Verlag, New York/Berlin, 1978.
[16] W. Hahn; *Stability of Motion*, Grundlehren der mathematischen Wissenschaften 138, Springer, 1967.
[17] Y. Jiang; *Robustness of a nonuniform (μ,ν) trichotomy in Banach spaces.* Electron. J. Differential Equations (2012), No. 154, 11 pp.
[18] F. F. Liao, Y. Jiang, Z. Xie; *A generalized nonuniform contraction and Lyapunov function*, Abstr. Appl. Anal., Vol. 2012 (2012), Article ID 613038, 14 pp.
[19] A. Lyapunov; *The General Problem of the Stability of Motion*, Taylor and Francis, 1992.
[20] J. LaSalle, S. Lefschetz; *Stability by Liapunovs Direct Method, with Applications*, Mathematics in Science and Engineering 4, Academic Press, 1961.
[21] A. Mažel'; *On stability of solutions of systems of differential equations*, Ural. Politehn. Inst. Trudy 51 (1954), 20-50.
[22] R. J. Sacker, G. R. Sell; *Existence of dichotomies and invariant splittings for linear differential systems III*, Journal of Differential Equations, 22, (1976) 497-522.

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