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INVARIANT REGIONS AND GLOBAL SOLUTIONS FOR REACTION-DIFFUSION SYSTEMS WITH A TRIDIAGONAL SYMMETRIC TOEPLITZ MATRIX OF DIFFUSION COEFFICIENTS

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Abstract. In this article we construct the invariant regions for m-component reaction-diffusion systems with a tridiagonal symmetric Toeplitz matrix of diffusion coefficients and with nonhomogeneous boundary conditions. We establish the existence of global solutions, and use Lyapunov functional methods. The nonlinear reaction term is assumed to be of polynomial growth.

1. Introduction

In recent years, the existence of global solutions for nonlinear parabolic systems has received considerable attention. Among valuable works is the one by Morgan [12], where where all the components satisfy the same boundary conditions (Neumann or Dirichlet), and the reaction terms are polynomially bounded and satisfy certain inequalities. Hollis, later, completed the work of Morgan and established global existence in the presence of mixed boundary conditions subject to certain structure requirements of the system. In 2007, Abdelmalek and Kouachi [1] show that solutions of m-component reaction-diffusion systems with a diagonal diffusion matrix exist globally (for $m \geq 2$) and reaction terms of polynomial growth. In the case of 2×2-systems, Haraux and Youkana [3] using a judicious Lyapunov functional, succeeded in considering sub-exponential non-linearities. Kouachi and Youkana [10] generalized the results of Haraux and Youkana [3] to the triangular case. Then, Kanel and Kirane [8, 9] proved the global existence for a full matrix of diffusion coefficients under certain restrictions.

The results obtained in this work prove the existence of global solutions with nonhomogeneous Neumann, Dirichlet, or Robin conditions. The reaction terms are again assumed to be of polynomial growth and satisfy a mere single inequality. The diffusion matrix is a tri-diagonal symmetric Toeplitz matrix.

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In this article, we use the following notation and assumptions: we denote by $m \geq 2$ the number of equations of the system (i.e. an m-component system):

$$\begin{gathered}
\frac{\partial u_1}{\partial t} - a\Delta u_1 - b\Delta u_2 = f_1(U), \\
\frac{\partial u_\ell}{\partial t} - b\Delta u_{\ell-1} - a\Delta u_\ell - b\Delta u_{\ell+1} = f_\ell(U), \quad \ell = 2, \dots, m-1, \\
\frac{\partial u_m}{\partial t} - b\Delta u_{m-1} - a\Delta u_m = f_m(U),
\end{gathered} \tag{1.1}$$

with the boundary conditions

$$\alpha u_\ell + (1-\alpha)\partial_\eta u_\ell = \beta_\ell, \quad \ell = 1, \dots, m, \text{ on } \partial\Omega \times \{t > 0\}, \tag{1.2}$$

and the initial data

$$u_\ell(0, x) = u_\ell^0(x), \quad \ell = 1, \dots, m, \text{ on } \Omega, \tag{1.3}$$

where

(i) for nonhomogeneous Robin boundary conditions, we use $0 < \alpha < 1$, $\beta_\ell \in \mathbb{R}$, $\ell = 1, \dots, m$;
(ii) for homogeneous Neumann boundary conditions, we use $\alpha = \beta_\ell = 0$, $\ell = 1, \dots, m$;
(iii) for homogeneous Dirichlet boundary conditions, we use $1 - \alpha = \beta_\ell = 0$, $\ell = 1, \dots, m$.

Here Ω is an open bounded domain of class C^1 in $\mathbb{R}^n$ with boundary $\partial\Omega$, $\frac{\partial}{\partial\eta}$ denotes the outward normal derivative on $\partial\Omega$, and $U = (u_\ell)_{\ell=1}^m$. The constants a and b are supposed to be strictly positive and satisfy the condition

$$2b \cos \frac{\pi}{m+1} < a. \tag{1.4}$$

The initial data are assumed to be in the regions:

$$\begin{aligned}
\Sigma_{\mathfrak{L},\mathfrak{Z}} = \Big\{ &(u_1^0, \dots, u_m^0) \in \mathbb{R}^m : \sum_{k=1}^m u_k^0 \sin \frac{(m+1-\ell)k\pi}{m+1} \geq 0, \\
&\ell \in \mathfrak{L}, \ \sum_{k=1}^m u_k^0 \sin \frac{(m+1-z)k\pi}{m+1} \leq 0, \ z \in \mathfrak{Z} \Big\},
\end{aligned} \tag{1.5}$$

with

$$\begin{gathered}
\sum_{k=1}^m \beta_k \sin \frac{(m+1-\ell)k\pi}{m+1} \geq 0, \quad \ell \in \mathfrak{L}, \\
\sum_{k=1}^m \beta_k \sin \frac{(m+1-z)k\pi}{m+1} \leq 0, \quad z \in \mathfrak{Z},
\end{gathered}$$

where

$$\mathfrak{L} \cap \mathfrak{Z} = \emptyset, \quad \mathfrak{L} \cup \mathfrak{Z} = \{1, 2, \dots, m\}.$$

Hence, we can see that there are 2^m regions. The subsequent work is similar for all of these regions as will be shown at the end of the paper. Let us now examine the first region and then comment on the remaining cases. The chosen region is the case where $L = \{1, 2, \dots, m\}$ and $\mathfrak{Z} = \emptyset$: we have

$$\Sigma_{\mathfrak{L},\emptyset} = \Big\{(u_1^0, \dots, u_m^0) \in \mathbb{R}^m : \sum_{k=1}^{m} u_k^0 \sin \frac{(m+1-\ell)k\pi}{m+1} \geq 0,\ \ell \in \mathfrak{L}\Big\} \tag{1.6}$$

with

$$\sum_{k=1}^{m} \beta_k \sin \frac{(m+1-\ell)k\pi}{m+1} \geq 0, \quad \ell \in \mathfrak{L}.$$

The aim is now to study the existence of global solutions for the reaction-diffusion system (1.1) in this region. To achieve this aim, we need to diagonalize the diffusion matrix, see formula (4.1). First, let us define the reaction diffusion functions as

$$F_\ell(w_1, w_2, \dots, w_m) = \sum_{k=1}^{m} f_k(U) \sin \frac{(m+1-\ell)k\pi}{m+1}, \tag{1.7}$$

where

$$w_\ell = \sum_{k=1}^{m} u_k \sin \frac{(m+1-\ell)k\pi}{m+1}. \tag{1.8}$$

The defined functions that satisfy the following three conditions:

(A1) The functions F_ℓ are continuously differentiable on $\mathbb{R}_+^m$ for all $\ell = 1, \dots, m$, and satisfy $F_\ell(w_1, \dots, w_{\ell-1}, 0, w_{\ell+1}, \dots, w_m) \geq 0$, for all $w_\ell \geq 0$, $\ell = 1, \dots, m$;

(A2) The functions F_ℓ are of polynomial growth (see Hollis and Morgan [5]), which means that for all $\ell = 1, \dots, m$ with integer $N \geq 1$,

$$|F_\ell(W)| \leq C_1\Big(1 + \sum_{\ell=1}^{m} w_\ell\Big)^N \quad \text{on } (0, +\infty)^m; \tag{1.9}$$

(A3) The inequality

$$\sum_{\ell=1}^{m-1} D_\ell F_\ell(W) + F_m(W) \leq C_2\Big(1 + \sum_{\ell=1}^{m} w_\ell\Big), \tag{1.10}$$

holds for all $w_\ell \geq 0$, $\ell = 1, \dots, m$, and all constants $D_\ell \geq \overline{D_\ell}$, $\ell = 1, \dots, m$ where $\overline{D_\ell}$, $\ell = 1, \dots, m$ are positive constants sufficiently large. Note that C_1 and C_2 are positive and uniformly bounded functions defined on $\mathbb{R}_+^m$.

2. Preliminary observations and notation

The usual norms in spaces $L^p(\Omega)$, $L^\infty(\Omega)$ and $C(\overline{\Omega})$ are denoted respectively by

$$\|u\|_p^p = \frac{1}{|\Omega|} \int_\Omega |u(x)|^p dx, \quad \|u\|_\infty = \text{ess}, \sup_{x \in \Omega} |u(x)|, \quad \|u\|_{C(\overline{\Omega})} = \max_{x \in \overline{\Omega}} |u(x)|. \tag{2.1}$$

It is well-known that to prove the existence of global solutions to a reaction-diffusion system (see Henry [4]), it suffices to derive a uniform estimate of the associated reaction term on $[0; T_{\max})$ in the space $L^p(\Omega)$ for some $p > n/2$. Our aim is to construct polynomial Lyapunov functionals allowing us to obtain L^p-bounds on the components, which leads to global existence. Since the reaction terms are continuously differentiable on $\mathbb{R}_+^m$, it follows that for any initial data in

$C(\overline{\Omega})$, it is easy to check directly their Lipschitz continuity on bounded subsets of the domain of a fractional power of the operator

$$\mathfrak{D} = -\begin{pmatrix} \lambda_1\Delta & 0 & \dots & 0 \\ 0 & \lambda_2\Delta & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_m\Delta \end{pmatrix}. \tag{2.2}$$

Under these assumptions, the local existence result is well known (see Friedman [2] and Pazy [13]).

Assumption (A1) contains smoothness and quasi-positivity conditions that guarantee local existence and nonnegativity of solutions as long as they exist, via the maximum principle (see Smoller [15]). Assumption (A3) is the usual polynomial growth condition necessary to obtain uniform bounds from p-dependent L^P estimates. (see Abdelmalek and Kouachi [1], and Hollis and Morgan [6]).

3. Some properties of the diffusion matrix

Lemma 3.1. *Considering the reaction-diffusion system in* (1.1)*, the resulting* $m\times m$ *diffusion matrix is given by*

$$A = \begin{pmatrix} a & b & 0 & \cdots & 0 & 0 \\ b & a & b & \ddots & 0 & 0 \\ 0 & b & a & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & b & 0 \\ 0 & \cdots & 0 & b & a & b \\ 0 & \cdots & 0 & 0 & b & a \end{pmatrix}. \tag{3.1}$$

This matrix is said to be positive definite if the condition in (1.4) *is satisfied.*

Proof. The proof of this lemma can be found in [7]. Note that if the matrix is positive definite, it follows that $\det A > 0$. □

Lemma 3.2. *The eigenvalues* ($\lambda_\ell < \lambda_{\ell-1}$; $\ell = 2,\dots,m$) *of* A *are positive and are given by*

$$\lambda_\ell = a + 2b\cos(\frac{\ell\pi}{m+1}), \tag{3.2}$$

with the corresponding eigenvectors

$$v_\ell = \big(\sin\frac{\ell\pi}{m+1}, \sin\frac{2\ell\pi}{m+1}, \dots, \sin\frac{m\ell\pi}{m+1}\big)^t,$$

for $\ell = 1,\dots,m$. *Hence, we conclude that* A *is diagonalizable. For simplicity, we write*

$$\bar\lambda_\ell = \lambda_{m+1-\ell} = a + 2b\cos(\frac{(m+1-\ell)\pi}{m+1}), \quad \ell = 1,\dots,m; \tag{3.3}$$

thus $\bar\lambda_\ell < \bar\lambda_{\ell+1}$, $\ell = 2,\dots,m$.

Proof. Recall that the diffusion matrix is positive definite, hence its eigenvalues are necessarily positive. For an eigenpair (λ, X), the components in $(A-\lambda I)X = 0$ are

$$bx_{k-1} + (a-\lambda)x_k + bx_{k+1} = 0, \quad k = 1,\dots,m,$$

with $x_0 = x_{m+1} = 0$, or equivalently,

$$x_{k+2} + (\frac{a-\lambda}{b})x_{k+1} + x_k = 0, \quad k = 0, \dots, m-1.$$

We seek solutions in the form $x_k = \xi r^k$ for constants ξ and r. This leads to the quadratic equation

$$r^2 + (\frac{a-\lambda}{b})r + 1 = 0,$$

with roots r_1 and r_2. The general solution of $x_{k+2} + (\frac{a-\lambda}{b})x_{k+1} + x_k = 0$ is

$$x_k = \begin{cases} \alpha r_1^k + \beta r^k, & \text{if } r_1 \neq r_2, \\ \alpha \rho^k + \beta k \rho^k, & \text{if } r_1 = r_2 = \rho, \end{cases}$$

where α and β are arbitrary constants.

For the eigenvalue problem at hand, r_1 and r_2 must be distinct -otherwise $x_k = \alpha\rho^k + \beta k\rho^k$, and $x_0 = x_{m+1} = 0$ implies that each $x_k = 0$, which is impossible because X is an eigenvector. Hence, $x_k = \alpha r_1^k + \beta r^k$, and $x_0 = x_{m+1} = 0$ yields

$$\left\{ \begin{array}{l} 0 = \alpha + \beta \\ 0 = \alpha r_1^{m+1} + \beta r_2^{m+1} \end{array} \right\} \Rightarrow (\frac{r_1}{r_2})^{m+1} = \frac{-\beta}{\alpha} = 1 \Rightarrow \frac{r_1}{r_2} = e^{\frac{2i\pi\ell}{m+1}};$$

therefore, $r_1 = r_2 e^{\frac{2i\pi\ell}{m+1}}$ for some $1 \leq \ell \leq m$. This coupled with

$$r^2 + (\frac{a-\lambda}{b})r + 1 = (r - r_1)(r - r_2) \Rightarrow \left\{ \begin{array}{l} r_1 r_2 = 1, \\ r_1 + r_2 = -(\frac{a-\lambda}{b}), \end{array} \right.$$

leads to $r_1 = e^{\frac{i\pi\ell}{m+1}}$, $r_2 = e^{-\frac{i\pi\ell}{m+1}}$, and

$$\lambda = a + b\big(e^{\frac{i\pi\ell}{m+1}} + e^{-\frac{i\pi\ell}{m+1}}\big) = a + 2b\cos(\frac{\ell\pi}{m+1}).$$

The eigenvalues of A can, therefore, be given by

$$\lambda_\ell = a + 2b\cos(\frac{\ell\pi}{m+1}), \text{ for } \ell = 1, \dots, m.$$

Since these λ_ℓ's are all distinct ($\cos\theta$ is a strictly decreasing function of θ on $(0,\pi)$, and $b \neq 0$), A is necessarily diagonalizable.

Finally, the ℓ^{th} component of any eigenvector associated with λ_ℓ satisfies $x_k = \alpha r_1^k + \beta r_2^k$ with $\alpha + \beta = 0$, thus

$$x_k = \alpha\big(e^{\frac{2i\pi k}{m+1}} - e^{-\frac{2i\pi k}{m+1}}\big) = 2i\alpha\sin\big(\frac{k}{m+1}\pi\big).$$

Setting $\alpha = 1/(2i)$ yields a particular eigenvector associated with λ_ℓ given by

$$v_\ell = \Big(\sin(\frac{1\ell\pi}{m+1}), \sin(\frac{2\ell\pi}{m+1}), \dots, \sin(\frac{m\ell\pi}{m+1})\Big)^t.$$

Because the λ_ℓ's are distinct, $\{v_1, v_2, \dots, v_m\}$, is a complete linearly independent set, so $(v_1|v_2|\dots|v_m)$ is the diagonal form of A.

Now, let us prove that

$$\lambda_\ell < \lambda_{\ell-1}, \ \ell = 2, \dots, m.$$

We have $\ell > \ell - 1$, whereupon

$$\frac{\ell\pi}{m+1} > \frac{(\ell-1)\pi}{m+1},$$

The function $\cos\theta$ is strictly decreasing in θ on $(0,\pi)$, thus we have

$$\cos\big(\frac{\ell\pi}{m+1}\big) < \cos\big(\frac{(\ell-1)\pi}{m+1}\big).$$

Finally, multiplying both sides of the inequality by $2b$ and adding a yields

$$\lambda_\ell = a + 2b\cos\big(\frac{\ell\pi}{m+1}\big) < a + 2b\cos\big(\frac{(\ell-1)\pi}{m+1}\big) = \lambda_{\ell-1}.$$

□

4. Main results

Proposition 4.1. *The eigenvectors of the diffusion matrix associated with the eigenvalues $\overline{\lambda}_\ell$ are defined as $\overline{v}_\ell = (\overline{v}_{\ell 1}, \overline{v}_{\ell 2}, \dots, \overline{v}_{\ell m})^T$. They satisfy the equations:*

$$\frac{\partial w_\ell}{\partial t} - \bar{\lambda}_\ell \Delta w_\ell = F_\ell(w_1, w_2, \dots, w_m), \tag{4.1}$$

$$\alpha w_\ell + (1-\alpha)\partial_\eta w_\ell = \rho_\ell \quad \text{on } \partial\Omega \times \{t > 0\}, \tag{4.2}$$

where the reaction term F_ℓ, and w_ℓ are given in (1.8) *and* (1.7)*, respectively.*

Note that condition (1.4) guarantees the parabolicity of the proposed reaction-diffusion system in (1.1)–(1.3), which implies it is equivalent to (4.1)–(4.2) in the region

$$\Sigma_{\mathfrak{L},\emptyset} = \big\{(u_1^0, \dots, u_m^0) \in \mathbb{R}^m : w_\ell^0 = \sum_{k=1}^{m} u_k^0 \sin\frac{(m+1-\ell)k\pi}{m+1} \ge 0,\ \ell \in \mathfrak{L}\big\}$$

with

$$\rho_\ell^0 = \sum_{k=1}^{m} \beta_k \sin\frac{(m+1-\ell)k\pi}{m+1} \ge 0, \quad \ell \in \mathfrak{L}.$$

This implies that the components w_ℓ are necessarily positive.

Proposition 4.2. *System* (4.1)–(4.2) *admits a unique classical solution* $(w_1, w_2, \dots, w_m)$ *on* $(0, T_{\max}) \times \Omega$*.*

$$\text{If } T_{\max} < \infty \text{ then } \lim_{t \nearrow T_{\max}} \sum_{\ell=1}^{m} \|w_\ell(t,.)\|_\infty = \infty, \tag{4.3}$$

where $T_{\max}\ (\|w_1^0\|_\infty, \|w_2^0\|_\infty, \dots, \|w_m^0\|_\infty)$ *denotes the eventual blow-up time.*

Before we present the main result of this paper, let us define

$$K_l^r = K_{r-1}^{r-1} \times K_l^{r-1} - [H_l^{r-1}]^2, \quad r = 3, \dots, l, \tag{4.4}$$

where

$$H_l^r = \det_{1\le\ell,\kappa\le l}\Big((a_{\ell,\kappa})_{\substack{\ell\ne l,\dots r+1\\ \kappa\ne l-1,\dots,r}}\Big) \prod_{k=1}^{k=r-2}\big(\det[k]\big)^{2^{(r-k-2)}}, \quad r = 3, \dots, l-1,$$

$$K_l^2 = \underbrace{\bar{\lambda}_1\bar{\lambda}_l \prod_{k=1}^{l-1}\theta_k^{2(p_k+1)^2} \prod_{k=l}^{m-1}\theta_k^{2(p_k+2)^2}}_{\text{positive value}} \Big[\prod_{k=1}^{l-1}\theta_k^2 - A_{1l}^2\Big],$$

$$H_l^2 = \underbrace{\bar{\lambda}_1\sqrt{\bar{\lambda}_2\bar{\lambda}_l}\theta_1^{2(p_1+1)^2}\prod_{k=2}^{l-1}\theta_k^{(p_k+2)^2+(p_k+1)^2}\prod_{k=l}^{m-1}\theta_k^{2(p_k+2)^2}}_{\text{positive value}}\big[\theta_1^2A_{2l}-A_{12}A_{1l}\big].$$

The expression $\det_{1\le\ell,\kappa\le l}((a_{\ell,\kappa})_{\substack{\ell\neq l,\dots r+1\\ \kappa\neq l-1,\dots r}})$ denotes the determinant of r square symmetric matrix obtained from $(a_{\ell,\kappa})_{1\le\ell,\kappa\le m}$ by removing the $(r+1)^{\text{th}},(r+2)^{\text{th}},\dots,l^{\text{th}}$ rows and the $r^{\text{th}},(r+1)^{\text{th}},\dots,(l-1)^{\text{th}}$ columns. where $\det[1],\dots,\det[m]$ are the minors of the matrix $(a_{\ell,\kappa})_{1\le\ell,\kappa\le m}$. The elements of the matrix are:

$$a_{\ell\kappa} = \frac{\bar{\lambda}_\ell+\bar{\lambda}_\kappa}{2}\theta_1^{p_1^2}\dots\theta_{(\ell-1)}^{p_{(\ell-1)}^2}\theta_\ell^{(p_\ell+1)^2}\dots\theta_{\kappa-1}^{(p_{(\kappa-1)}+1)^2}\theta_\kappa^{(p_\kappa+2)^2}\dots\theta_{(m-1)}^{(p_{(m-1)}+2)^2}. \tag{4.5}$$

where $\bar{\lambda}_\ell$ in (3.2)–(3.3). Note that $A_{\ell\kappa} = \frac{\bar{\lambda}_\ell+\bar{\lambda}_\kappa}{2\sqrt{\bar{\lambda}_\ell\bar{\lambda}_\kappa}}$ for all $\ell,\kappa = 1,\dots,m$. and $\theta_\ell;\ell = 1,\dots,(m-1)$ are positive constants.

Theorem 4.3. *Suppose that the functions F_ℓ, $\ell = 1,\dots,m$, are of polynomial growth and satisfy condition* (1.10) *for some positive constants D_ℓ, $\ell = 1,\dots,m$, sufficiently large. Let $(w_1(t,\cdot),w_2(t,\cdot),\dots,w_m(t,\cdot))$ be the solution of* (4.1)–(4.2) *and*

$$L(t) = \int_\Omega H_{p_m}(w_1(t,x),w_2(t,x),\dots,w_m(t,x))dx, \tag{4.6}$$

where

$$\begin{aligned} &H_{p_m}(w_1,\dots,w_m)\\ &= \sum_{p_{m-1}=0}^{p_m}\dots\sum_{p_1=0}^{p_2}C_{p_m}^{p_{m-1}}\dots C_{p_2}^{p_1}\theta_1^{p_1^2}\dots\theta_{(m-1)}^{p_{(m-1)}^2}w_1^{p_1}w_2^{p_2-p_1}\dots w_m^{p_m-p_{m-1}}, \end{aligned}$$

with p_m a positive integer and $C_{p_\kappa}^{p_\ell} = \frac{p_\kappa!}{p_\ell!(p_\kappa-p_\ell)!}$. Also suppose that the following condition is satisfied

$$K_l^l > 0; \quad l = 2,\dots,m, \tag{4.7}$$

where K_l^l was defined in (4.4). *Then, the functional L is uniformly bounded on the interval $[0,T^*], T^* < T_{\max}$.*

Corollary 4.4. *Under the assumptions of Theorem 4.3, all solutions of* (4.1)–(4.2) *with positive initial data in $L^\infty(\Omega)$ are in $L^\infty(0,T^*;L^p(\Omega))$ for some $p\ge 1$.*

Proposition 4.5. *Under the assumptions of Theorem 4.3, and assuming the condition* (1.4) *is satisfied, all solutions of* (4.1)–(4.2) *with positive initial data in $L^\infty(\Omega)$ are global for some $p > \frac{Nn}{2}$.*

5. Proofs of main results

For the proof of Theorem 4.3, we first need to define some preparatory Lemmas.

Lemma 5.1. *With H_{p_m} being the homogeneous polynomial defined by* (4.6), *differentiating in w_1 yields*

$$\begin{aligned} \partial_{w_1}H_{p_m} = p_m&\sum_{p_{m-1}=0}^{p_m-1}\dots\sum_{p_1=0}^{p_2}C_{p_m-1}^{p_{m-1}}\dots C_{p_2}^{p_1}\theta_1^{(p_1+1)^2}\dots\theta_{(m-1)}^{(p_{(m-1)}+1)^2}\\ &\times w_1^{p_1}w_2^{p_2-p_1}w_3^{p_3-p_2}\dots w_m^{(p_m-1)-p_{m-1}}. \end{aligned} \tag{5.1}$$

Similarly for $\ell = 2, \dots, m-1$, we have

$$
\begin{aligned}
\partial_{w_\ell} H_{p_m} = p_m \sum_{p_{m-1}=0}^{p_m-1} \cdots \sum_{p_1=0}^{p_2} & C_{p_m-1}^{p_{m-1}} \dots C_{p_2}^{p_1} \theta_1^{p_1^2} \dots \theta_{\ell-1}^{p_{(\ell-1)}^2} \theta_\ell^{(p_\ell+1)2} \dots \theta_{(m-1)}^{(p_{(m-1)}+1)^2} \\
& \times w_1^{p_1} w_2^{p_2-p_1} w_3^{p_3-p_2} \dots w_m^{(p_m-1)-p_{m-1}}.
\end{aligned}
\tag{5.2}
$$

Finally, differentiating in w_m yields

$$
\begin{aligned}
\partial_{w_m} H_{p_m} = p_m \sum_{p_{m-1}=0}^{p_m-1} \cdots \sum_{p_1=0}^{p_2} & C_{p_m-1}^{p_{m-1}} \dots C_{p_3}^{p_2} C_{p_2}^{p_1} \theta_1^{p_1^2} \theta_2^{p_2^2} \dots \theta_{(m-1)}^{p_{(m-1)}^2} \\
& \times w_1^{p_1} w_2^{p_2-p_1} w_3^{p_3-p_2} \dots w_m^{(p_m-1)-p_{m-1}}.
\end{aligned}
\tag{5.3}
$$

Lemma 5.2. *The second partial derivative of H_{p_m} in w_1 is*

$$
\begin{aligned}
\partial_{w_1^2} H_n = p_m(p_m-1) \sum_{p_{m-1}=0}^{p_m-2} \cdots \sum_{p_2=0}^{p_3} \sum_{p_1=0}^{p_2} & C_{p_m-2}^{p_{m-1}} \dots C_{p_2}^{p_1} \\
& \times \theta_1^{(p_1+2)^2} \dots \theta_{(m-1)}^{(p_{(m-1)}+2)^2} w_1^{p_1} w_2^{p_2-p_1} \dots w_m^{(p_m-2)-p_{m-1}}.
\end{aligned}
\tag{5.4}
$$

Similarly, we obtain

$$
\begin{aligned}
\partial_{w_\ell^2} H_n = p_m(p_m-1) \sum_{p_{m-1}=0}^{p_m-2} \cdots \sum_{p_1=0}^{p_2} & C_{p_m-2}^{p_{m-1}} \dots C_{p_2}^{p_1} \\
& \times \theta_1^{p_1^2} \theta_2^{p_2^2} \dots \theta_{\ell-1}^{p_{\ell-1}^2} \theta_\ell^{(p_\ell+2)^2} \dots \theta_{(m-1)}^{(p_{(m-1)}+2)^2} w_1^{p_1} w_2^{p_2-p_1} \dots w_m^{(p_m-2)-p_{m-1}}.
\end{aligned}
\tag{5.5}
$$

for all $\ell = 2, \dots, m-1$,

$$
\begin{aligned}
\partial_{w_\ell w_\kappa} H_n = p_m(p_m-1) \sum_{p_{m-1}=0}^{p_m-2} \cdots \sum_{p_1=0}^{p_2} & C_{p_m-2}^{p_{m-1}} \dots C_{p_2}^{p_1} \\
& \times \theta_1^{p_1^2} \dots \theta_{\ell-1}^{p_{\ell-1}^2} \theta_\ell^{(p_\ell+1)^2} \dots \theta_{\kappa-1}^{(p_{\kappa-1}+1)^2} \theta_\kappa^{(p_\kappa+2)^2} \dots \theta_{(m-1)}^{(p_{(m-1)}+2)^2} \\
& \times w_1^{p_1} w_2^{p_2-p_1} \dots w_m^{(p_m-2)-p_{m-1}}
\end{aligned}
\tag{5.6}
$$

for all $1 \le \ell < \kappa \le m$. Finally, the second derivative in w_m is

$$
\begin{aligned}
\partial_{w_m^2} H_n = p_m(p_m-1) \sum_{p_{m-1}=0}^{p_m-2} \cdots \sum_{p_1=0}^{p_2} & C_{p_m-2}^{p_{m-1}} \dots C_{p_2}^{p_1} \theta_1^{p_1^2} \dots \theta_{(m-1)}^{p_{(m-1)}^2} \\
& w_1^{p_1} w_2^{p_2-p_1} \dots w_m^{(p_m-2)-p_{m-1}}.
\end{aligned}
\tag{5.7}
$$

Lemma 5.3 ([1]). *Let A be the m-square symmetric matrix defined by $A = (a_{\ell\kappa})$, with $1 \le \ell, \kappa \le m$, then*

$$
\begin{gathered}
K_m^m = \det[m] \prod_{k=1}^{k=m-2} (\det[k])^{2^{(m-k-2)}}, \quad m > 2, \\
K_2^2 = \det[2],
\end{gathered}
\tag{5.8}
$$

where

$$
K_m^l = K_{l-1}^{l-1} K_m^{l-1} - (H_m^{l-1})^2, \quad l = 3, \dots, m,
$$

$$H_m^l = \det_{1\le \ell,\kappa\le m}\Big((a_{\ell,\kappa})_{\substack{\ell\neq m,\dots l+1\\ \kappa\neq m-1,\dots l}}\Big)\prod_{k=1}^{k=l-2}\Big(\det[k]\Big)^{2^{(l-k-2)}}, \quad l=3,\dots,m-1,$$

$$K_m^2 = a_{11}a_{mm}-(a_{1m})^2,\ H_m^2 = a_{11}a_{2m}-a_{12}a_{1m}.$$

Proof of Theorem 4.3. The aim is to prove that $L(t)$ is uniformly bounded on the interval $[0,T^*], T^* < T_{\max}$. Let us start by differentiating L with respect to t:

$$\begin{aligned} L'(t) &= \int_\Omega \partial_t H_{p_m}dx = \int_\Omega \sum_{\ell=1}^m \partial_{w_\ell}H_{p_m}\frac{\partial w_\ell}{\partial t}dx \\ &= \int_\Omega \sum_{\ell=1}^m \partial_{w_\ell}H_{p_m}(\bar\lambda_\ell \Delta w_\ell + F_\ell)dx \\ &= \int_\Omega \sum_{\ell=1}^m \bar\lambda_\ell \partial_{w_\ell}H_{p_m}\Delta w_\ell dx + \int_\Omega \sum_{\ell=1}^m \partial_{w_\ell}H_{p_m}F_\ell dx \\ &= I+J, \end{aligned}$$

where

$$I = \int_\Omega \sum_{\ell=1}^m \bar\lambda_\ell \partial_{w_\ell}H_{p_m}\Delta w_\ell dx, \tag{5.9}$$

$$J = \int_\Omega \sum_{\ell=1}^m \partial_{w_\ell}H_{p_m}F_\ell dx. \tag{5.10}$$

Using Green's formula, we can divide I into two parts I_1 and I_2 where

$$I_1 = \int_{\partial\Omega} \sum_{\ell=1}^m \bar\lambda_\ell \partial_{w_\ell}H_{p_m}\partial_\eta w_\ell dx, \tag{5.11}$$

$$I_2 = -\int_\Omega \Big[\Big(\Big(\frac{\bar\lambda_\ell+\bar\lambda_\kappa}{2}\partial_{w_\kappa w_\ell}H_{p_m}\Big)_{1\le\ell,\kappa\le m}\Big)T\Big]T\,dx, \tag{5.12}$$

for $p_1 = 0,\dots,p_2$, $p_2=0,\dots,p_3 \dots p_{m-1}=0,\dots,p_m-2$ and

$$T = (\nabla w_1, \nabla w_2, \dots, \nabla w_m)^t.$$

Applying Lemmas 5.1 and 5.2 yields

$$\begin{aligned} &\Big(\frac{\bar\lambda_\ell+\bar\lambda_\kappa}{2}\partial_{w_\kappa w_\ell}H_{p_m}\Big)_{1\le\ell,\kappa\le m} \\ &= p_m(p_m-1)\sum_{p_{m-1}=0}^{p_m-2}\cdots\sum_{p_1=0}^{p_2} C_{p_m-2}^{p_{m-1}}\dots C_{p_2}^{p_1}\Big((a_{\ell\kappa})_{1\le\ell,\kappa\le m}\Big)w_1^{p_1}\dots w_m^{(p_m-2)-p_{m-1}}, \end{aligned} \tag{5.13}$$

where $(a_{\ell\kappa})_{1\le\ell,\kappa\le m}$ is the matrix defined in formula (4.5).

Now, the proof of positivity for I is reduced to proving that there exists a positive constant C_4 independent of $t\in[0,T_{\max})$ such that

$$I_1 \le C_4 \quad \text{for all } t\in[0,T_{\max}), \tag{5.14}$$

and that

$$I_2 \le 0, \tag{5.15}$$

for several boundary conditions. First, let us prove the formula in (5.14):

(i) If $0 < \alpha < 1$, then using the boundary conditions (1.2) we obtain

$$I_1 = \int_{\partial\Omega} \sum_{\ell=1}^{m} \bar{\lambda}_\ell \partial_{w_\ell} H_{p_m}(\gamma_\ell - \sigma w_\ell) dx,$$

where $\sigma = \frac{\alpha}{1-\alpha}$ and $\gamma_\ell = \frac{\beta_\ell}{1-\alpha}$, for $\ell = 1, \dots m$. Since

$$H(W) = \sum_{\ell=1}^{m} \overline{\lambda}_\ell \partial_{w_\ell} H_{p_m}(\gamma_\ell - \sigma w_\ell) = P_{n-1}(W) - Q_n(W),$$

where P_{n-1} and Q_n, are polynomials with positive coefficients and degrees $n-1$ and n, respectively, and since the solution is positive, it follows that

$$\limsup_{\sum_{\ell=1}^{m} |w_\ell| \to +\infty} H(W) = -\infty, \tag{5.16}$$

which proves that H is uniformly bounded on $\mathbb{R}_+^m$ and consequently (5.14).

(ii) If $\alpha = 0$, then $I_1 = 0$ on $[0, T_{\max})$.

(iii) The case of homogeneous Dirichlet conditions is trivial, since in this case the positivity of the solution on $[0, T_{\max}) \times \Omega$ implies $\partial_\eta w_\ell \leq 0, \forall \ell = 1, \dots m$ on $[0, T_{\max}) \times \partial\Omega$. Consequently, one obtains the same result in (5.14) with $C_4 = 0$. Hence, the proof of (5.14) is complete.

Now, we pass to the proof of (5.15): Recall the matrix $(a_{\ell\kappa})_{1 \leq \ell, \kappa \leq m}$ which was defined in formula (4.5). The quadratic forms (with respect to ∇w_ℓ, $\ell = 1, \dots, m$) associated with the matrix $(a_{\ell\kappa})_{1 \leq \ell, \kappa \leq m}$, with $p_1 = 0, \dots, p_2$, $p_2 = 0, \dots, p_3$ $\dots p_{m-1} = 0, \dots, p_m - 2$, is positive definite since its minors $\det[1]$, $\det[2], \dots \det[m]$ are all positive. Let us examine these minors and prove their positivity by induction.

The first minor

$$\det[1] = \lambda_1 \theta_1^{(p_1+2)^2} \theta_2^{(p_2+2)^2} \dots \theta_{(m-1)}^{(p_{(m-1)}+2)^2} > 0$$

is trivial for $p_1 = 0, \dots, p_2$, $p_2 = 0, \dots, p_3$ $\dots p_{m-1} = 0, \dots, p_m - 2$.

For the second minor $\det[2]$, according to Lemma 5.3, we obtain

$$\det[2] = K_2^2 = \lambda_1 \lambda_2 \theta_1^{2(p_1+1)^2} \prod_{k=2}^{m-1} \theta_k^{2(p_k+2)^2} [\theta_1^2 - A_{12}^2].$$

Using (4.7) for $l = 2$ we get $\det[2] > 0$.

Similarly, for the third minor $\det[3]$, and again using Lemma 5.3, we have

$$K_3^3 = \det[3] \det[1].$$

Since $\det[1] > 0$, we conclude that

$$\operatorname{sign}(K_3^3) = \operatorname{sign}(\det[3]).$$

Again, using (4.7) for $l = 3$, we obtain $\det[3] > 0$.

To conclude the proof, let us suppose $\det[k] > 0$ for $k = 1, 2, \dots, l-1$ and show that $\det[l]$ is necessarily positive. We have

$$\det[k] > 0,\ k = 1, \dots, (l-1), \Rightarrow \prod_{k=1}^{k=l-2} (\det[k])^{2^{(l-k-2)}} > 0. \tag{5.17}$$

From Lemma 5.3, we obtain

$$K_l^l = \det[l] \prod_{k=1}^{k=l-2} (\det[k])^{2^{(l-k-2)}},$$

and from (5.17), we obtain $\operatorname{sign}(K_l^l) = \operatorname{sign}(\det[l])$. Since $K_l^l > 0$ according to (4.7) then $\det[l] > 0$ and the proof of (5.15) is finished. It follows from (5.14) and (5.15) that I is bounded. Now, let us prove that J in (5.10) is bounded. Substituting the expressions of the partial derivatives given by 5.1 in the second integral of (5.10) yields

$$\begin{aligned}
J &= \int_\Omega \Big[p_m \sum_{p_{m-1}=0}^{p_m-1} \cdots \sum_{p_1=0}^{p_2} C_{p_m-1}^{p_{m-1}} \dots C_{p_2}^{p_1} w_1^{p_1} w_2^{p_2-p_1} \dots w_m^{p_m-1-p_{m-1}}\Big] \\
&\quad \times \Big(\prod_{\ell=1}^{m-1} \theta_\ell^{(p_\ell+1)^2} F_1 + \sum_{\kappa=2}^{m-1} \prod_{k=1}^{\kappa-1} \theta_k^{p_k^2} \prod_{\ell=\kappa}^{m-1} \theta_\ell^{(p_\ell+1)^2} F_\kappa + \prod_{\ell=1}^{m-1} \theta_\ell^{p_\ell^2} F_m\Big) dx \\
&= \int_\Omega \Big[p_m \sum_{p_{m-1}=0}^{p_m-1} \cdots \sum_{p_1=0}^{p_2} C_{p_m-1}^{p_{m-1}} \dots C_{p_2}^{p_1} w_1^{p_1} w_2^{p_2-p_1} \dots w_m^{p_m-1-p_{m-1}}\Big] \\
&\quad \times \Big(\frac{\prod_{\ell=1}^{m-1} \theta_\ell^{(p_\ell+1)^2}}{\prod_{\ell=1}^{m-1} \theta_\ell^{p_\ell^2}} F_1 + \sum_{\kappa=2}^{m-1} \frac{\prod_{k=1}^{\kappa-1} \theta_k^{p_k^2} \prod_{\ell=\kappa}^{m-1} \theta_\ell^{(p_\ell+1)^2}}{\prod_{\ell=1}^{m-1} \theta_\ell^{p_\ell^2}} F_\kappa + F_m\Big) \prod_{\ell=1}^{m-1} \theta_\ell^{p_\ell^2} dx \\
&= \int_\Omega \Big[p_m \sum_{p_{m-1}=0}^{p_m-1} \cdots \sum_{p_1=0}^{p_2} C_{p_m-1}^{p_{m-1}} \dots C_{p_2}^{p_1} w_1^{p_1} w_2^{p_2-p_1} \dots w_m^{p_m-1-p_{m-1}}\Big] \\
&\quad \times \Big(\prod_{\ell=1}^{m-1} \frac{\theta_\ell^{(p_\ell+1)^2}}{\theta_\ell^{p_\ell^2}} F_1 + \sum_{\kappa=2}^{m-1} \prod_{k=1}^{\kappa-1} \theta_k^{p_k^2} \prod_{\ell=\kappa}^{m-1} \frac{\theta_\ell^{(p_\ell+1)^2}}{\theta_\ell^{p_\ell^2}} F_\kappa + F_m\Big) \prod_{\ell=1}^{m-1} \theta_\ell^{p_\ell^2} dx.
\end{aligned}$$

Hence, using condition (1.10), we deduce that

$$J \le C_5 \int_\Omega \Big[\sum_{p_{m-1}=0}^{p_m-1} \cdots \sum_{p_1=0}^{p_2} C_{p_2}^{p_1} \dots C_{p_m-1}^{p_{m-1}} w_1^{p_1} w_2^{p_2-p_1} \dots w_m^{p_m-1-p_{m-1}} (1+\sum_{\ell=1}^m w_\ell)\Big] dx.$$

To prove that the functional L is uniformly bounded on the interval $[0, T^*]$, let us first write

$$\begin{aligned}
&\sum_{p_{m-1}=0}^{p_m-1} \cdots \sum_{p_1=0}^{p_2} C_{p_2}^{p_1} \dots C_{p_m-1}^{p_{m-1}} w_1^{p_1} w_2^{p_2-p_1} \dots w_m^{p_m-1-p_{m-1}} \Big(1 + \sum_{\ell=1}^m w_\ell\Big) \\
&= R_{p_m}(W) + S_{p_m-1}(W),
\end{aligned}$$

where $R_{p_m}(W)$ and $S_{p_m-1}(W)$ are two homogeneous polynomials of degrees p_m and $p_m - 1$, respectively. Since all of the polynomials H_{p_m} and R_{p_m} are of degree p_m, there exists a positive constant C_6 such that

$$\int_\Omega R_{p_m}(W) dx \le C_6 \int_\Omega H_{p_m}(W) dx. \tag{5.18}$$

Applying Hölder's inequality to the following integral one obtains

$$\int_\Omega S_{p_m-1}(W) dx \le (\operatorname{meas}\Omega)^{\frac{1}{p_m}} \Big(\int_\Omega (S_{p_m-1}(W))^{\frac{p_m}{p_m-1}} dx\Big)^{\frac{p_m-1}{p_m}}.$$

Since for all $w_1, w_{2,\dots}, w_{m-1} \ge 0$ and $w_m > 0$,

$$\frac{(S_{p_m-1}(W))^{\frac{p_m}{p_m-1}}}{H_{p_m}(W)} = \frac{(S_{p_m-1}(x_1, x_2, \dots, x_{m-1}, 1))^{\frac{p_m}{p_m-1}}}{H_{p_m}(x_1, x_2, \dots, x_{m-1}, 1)},$$

where for all $\ell \in \{1, 2, \dots, m-1\}$, $x_\ell = \frac{w_\ell}{w_{\ell+1}}$ and

$$\lim_{x_\ell \to +\infty} \frac{(S_{p_m-1}(x_1, x_2, \dots, x_{m-1}, 1))^{\frac{p_m}{p_m-1}}}{H_{p_m}(x_1, x_2, \dots, x_{m-1}, 1)} < +\infty,$$

one asserts that there exists a positive constant C_7 such that

$$\frac{(S_{p_m-1}(W))^{\frac{p_m}{p_m-1}}}{H_{p_m}(W)} \le C_7, \quad \text{for all } w_1, w_2, \dots, w_m \ge 0. \tag{5.19}$$

Hence, the functional L satisfies the differential inequality

$$L'(t) \le C_6 L(t) + C_8 L^{\frac{p_m-1}{p_m}}(t),$$

which for $Z = L^{\frac{1}{p_m}}$ can be written as

$$p_m Z' \le C_6 Z + C_8. \tag{5.20}$$

A simple integration gives the uniform bound of the functional L on the interval $[0, T^*]$. This completes the proof. $\square$

Proof of Corollary 4.4. The proof of is an immediate consequence of Theorem 4.3 and the inequality

$$\int_\Omega \Big(\sum_{\ell=1}^m w_\ell(t,x) \Big)^p dx \le C_9 L(t) \quad \text{on } [0, T^*]. \tag{5.21}$$

for some $p \ge 1$. $\square$

Proof of Proposition 4.2. From Corollary 4.4, there exists a positive constant C_{10} such that

$$\int_\Omega \Big(\sum_{\ell=1}^m w_\ell(t,x) + 1 \Big)^p dx \le C_{10} \quad \text{on } [0, T_{\max}). \tag{5.22}$$

From (1.9), we have that for all $\ell \in \{1, 2, \dots, m\}$,

$$|F_\ell(W)|^{\frac{p}{N}} \le C_{11}(W) \Big(\sum_{\ell=1}^m W_\ell(t,x) \Big)^p \quad \text{on } [0, T_{\max}) \times \Omega. \tag{5.23}$$

Since $w_1, w_2, \dots, w_m$ are in $L^\infty(0, T^*; L^p(\Omega))$ and $\frac{p}{N} > \frac{n}{2}$, the solution is global. $\square$

6. Final remarks

Recall that the eigenvectors of the diffusion matrix associated with the eigenvalue $\overline{\lambda}_\ell$ is defined as $\overline{v}_\ell = (\overline{v}_{\ell 1}, \overline{v}_{\ell 2}, \dots, \overline{v}_{\ell m})^t$. It is important to note that if $\overline{v}_\ell$ is an eigenvector then so is $(-1)\overline{v}_\ell$. In the region considered in previous sections, we only used the positive $\overline{v}_\ell$. The remainder of the 2^m regions can be formed using negative versions of the eigenvectors. In each region, the reaction-diffusion system with a diagonalized diffusion matrix is formed by multiplying each of the m equations in (1.1) by the corresponding element of either $\overline{v}_\ell$ or $(-1)\overline{v}_\ell$ and then adding the m equations together. The equations multiplied by elements of $\overline{v}_\ell$ form a set $\mathfrak{L}$, whereas the equations multiplied by elements of $(-1)\overline{v}_\ell$ form a set $\mathfrak{Z}$. Hence, we can define the region in the form

$$\Sigma_{\mathfrak{L},\mathfrak{Z}} = \Big\{ (u_1^0, u_2^0, \dots, u_m^0) \in \mathbb{R}^m : w_\ell^0 = \sum_{k=1}^m u_k^0 \overline{v}_{\ell k} \ge 0,$$

$$\ell \in \mathfrak{L},\ w_z^0 = (-1)\sum_{k=1}^{m} u_k^0 \overline{v}_{zk} \geq 0,\ z \in \mathfrak{Z}\Big\}$$

with

$$\rho_\ell^0 = \sum_{k=1}^{m} \beta_k v_{(m+1-\ell)k} \geq 0, \quad \ell \in \mathfrak{L},$$

$$\rho_\ell^0 = (-1)\sum_{k=1}^{m} \beta_k v_{(m+1-z)k} \geq 0, \quad z \in \mathfrak{Z}.$$

Using Lemma 3.2 we obtain

$$\Sigma_{\mathfrak{L},\mathfrak{Z}} = \Big\{(u_1^0, u_2^0, \dots, u_m^0) \in \mathbb{R}^m : w_\ell^0 = \sum_{k=1}^{m} u_k^0 \sin\frac{(m+1-\ell)k\pi}{m+1} \geq 0,$$

$$\ell \in \mathfrak{L},\ w_z^0 = (-1)\sum_{k=1}^{m} u_k^0 \sin\frac{(m+1-z)k\pi}{m+1} \geq 0,\ z \in \mathfrak{Z}\Big\},$$

with

$$\rho_\ell^0 = \sum_{k=1}^{m} \beta_k \sin\frac{(m+1-\ell)k\pi}{m+1} \geq 0, \quad \ell \in \mathfrak{L}$$

$$\rho_z^0 = (-1)\sum_{k=1}^{m} \beta_k \sin\frac{(m+1-z)k\pi}{m+1} \geq 0, \quad z \in \mathfrak{Z},$$

$$\mathfrak{L} \cap \mathfrak{Z} = \emptyset, \quad \mathfrak{L} \cup \mathfrak{Z} = \{1, 2, \dots, m\}.$$

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