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NONTRIVIAL SOLUTIONS FOR ASYMMETRIC PROBLEMS ON $\mathbb{R}^N$

RUICHANG PEI, JIHUI ZHANG

Abstract. We consider the elliptic equation

$$-\Delta u + V(x)u = f(x,u), \quad x \in \mathbb{R}^n, \quad u \in H^1(\mathbb{R}^N),\ N \geq 2,$$

where $V(x) \in C(\mathbb{R}^N)$ and $V(x) \geq V_0 > 0$ for all $x \in \mathbb{R}^N$. The nonlinear term f exhibits an asymmetric growth at $+\infty$ and $-\infty$ in $\mathbb{R}^N$ ($N \geq 2$). Namely, it is linear at $-\infty$ and superlinear at $+\infty$. However, it need not satisfy the Ambrosetti-Rabinowitz condition on the positive semiaxis. Some existence results for nontrivial solution are established by using the minimax methods combined with the improved Moser-Trudinger inequality.

1. Introduction

In this article we consider the semilinear elliptic equation

$$-\Delta u + V(x)u = f(x,u), \quad x \in \mathbb{R}^n,\ u \in H^1(\mathbb{R}^N),\ N \geq 2, \tag{1.1}$$

where $V(x) \in C(\mathbb{R}^N)$ and $V(x) \geq V_0 > 0$ for all $x \in \mathbb{R}^N$. This equation arises from many physical and chemical problems. By using the variational methods, there were many papers studying the existence and multiplicity of solutions for problem (1.1). Most of them treated the superlinear case (see [1, 2, 3]) and some deal with the asymptotically linear case (see [4, 5, 6, 7, 8, 9]).

The main difficulty in dealing with this class of problem is the lack of compactness due to the fact that the domain is unbounded. This was overcome in [10] by assuming that the potential V is coercive. Such condition was generalized in [2] by assuming

- (V1) for every $M > 0$, $\mu(\{x \in \mathbb{R}^N : V(x) \leq M\}) < \infty$, with μ denoting the Lebesgue measure in $\mathbb{R}^N$.

Actually, the above hypothesis implies that the eigenvalue problem

$$-\Delta u + V(x)u = \lambda u,\ x \in \mathbb{R}^N,$$

possesses a sequence of positive eigenvalues: $0 < \lambda_1 < \lambda_2 < \lambda_3 \cdots < \lambda_k < \cdots \to \infty$ with finite multiplicity for each λ_k. The principal eigenvalue λ_1 is simple with positive eigenfunction φ_1, and eigenfunction φ_k corresponding to λ_k ($k \geq 2$) is sign-changing.

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In the present paper, motivated by [11, 12, 13, 14, 15, 16], our main purpose is to establish existence results of nontrivial solution for problem (1.1) with $N \geq 2$ when the nonlinear term exhibits an asymmetric behavior as $t \in \mathbb{R}$ approaches $+\infty$ and $-\infty$. More precisely, we assume that for a.e. $x \in \mathbb{R}^N$, $f(x,.)$ grows superlinearly at $+\infty$, while at $-\infty$ it has a linear growth. To our knowledge, this asymmetric problem is rarely considered by other people.

In case of $N \geq 3$, we noticed that almost all of the above mentioned works involve the nonlinearity term $f(x,u)$ of a subcritical (polynomial) growth, say,

(SCP) There exist positive constants c_1 and c_2 and $q_0 \in (1, 2^* - 1)$ such that

$$|f(x,t)| \leq c_1 + c_2|t|^{q_0} \quad \text{for all } t \in \mathbb{R} \text{ and } x \in \mathbb{R}^N,$$

where $2^* = 2N/(N-2)$ denotes the critical Sobolev exponent.

One of the main reasons to assume this condition (SCP) is that they can use the Sobolev compact embedding theory.

Over the years, many researchers studied problem (1.1) by trying to drop the condition (AR) (see [2]), see for instance [5, 6, 7, 8, 9].

In this paper, our first main results will be to study problem (1.1) in the improved subcritical polynomial growth

(SCPI) For $\varepsilon > 0$, there exists positive constant $C(\varepsilon)$ such that

$$|f(x,t)| \leq C(\varepsilon) + \varepsilon |t|^{2^*-1} \quad \text{for all } t \in \mathbb{R} \text{ and } x \in \mathbb{R}^N,$$

which is weaker than (SCP).

Note that in this case, we do not have the Sobolev compact embedding anymore. Our work is to study asymmetric problem (1.1) without the (AR)-condition in the positive semiaxis. In fact, this condition was studied by Liu and Wang in [17] in the case of Laplacian by the Nehari manifold approach. However, we will use the Mountain Pass Theorem and a suitable version of the Mountain Pass Theorem to get the nontrivial solution to problem (1.1) in the general case $N \geq 3$. Our proof of compactness condition is completely different from those in [14, 15, 16].

Let us now state our main results: Suppose that $f(x,t) \in C(\mathbb{R}^N \times \mathbb{R})$ and satisfies:

- (H1) $\lim_{t\to 0} \frac{f(x,t)}{t} = f_0$ uniformly for a.e. $x \in \mathbb{R}^N$, where $f_0 \in [0, +\infty)$;
- (H2) $\lim_{t\to -\infty} \frac{f(x,t)}{t} = l$ uniformly for a.e. $x \in \mathbb{R}^N$, where $l \in [0, +\infty]$;
- (H3) $\lim_{t\to +\infty} \frac{f(x,t)}{t} = +\infty$ uniformly for a.e. $x \in \mathbb{R}^N$;
- (H4) $\frac{f(x,t)}{t}$ is non-increasing with respect to $t \leq 0$, for a.e. $x \in \mathbb{R}^N$.

Let $H := H^1(\mathbb{R}^N)$ be the Sobolev space with the norm

$$\|u\|_H := \Big(\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)\,dx\Big)^{1/2}.$$

In our problem, the work space E is defined by

$$E := \big\{u \in H : \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\,dx < \infty\big\}.$$

Thus, E is a Hilbert space with the inner product

$$(u,v)_E := \int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv)\,dx$$

and is defined by $\|\cdot\|$ the associated norm.

We denote by $|\cdot|_p$ the usual L^p-norm. The condition (V_1) and the Sobolev Embedding Theorem imply that the immersion $E \hookrightarrow L^s(\mathbb{R}^N,\mathbb{R})$ $(N \geq 3)$ is continuous for $2 \leq s \leq 2^*$. Actually it is proved in [2] that this embedding is compact for $2 \leq s < 2^*$.

Recall that a function $u \in E$ is called a weak solution of (1.1) if

$$\int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv)\, dx = \int_{\mathbb{R}^N} f(x,u)v\, dx, \quad \forall v \in E.$$

Seeking a weak solution of problem (1.1) is equivalent to finding a critical point u^* of C^1 functional

$$I(u) := \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x,u)\, dx, \quad \forall u \in E, \tag{1.2}$$

where $F(x,u) = \int_0^u f(x,s)ds$. Then

$$\langle I'(u^*), v\rangle = \int_{\mathbb{R}^N} (\nabla u^* \nabla v + V(x)u^* v)\, dx - \int_{\mathbb{R}^N} f(x,u^*)v\, dx = 0, \quad \forall v \in E.$$

Definition 1.1. Let $(E,\|\cdot\|_E)$ be a real Banach space with its dual space $(E^*,\|\cdot\|_{E^*})$ and $I \in C^1(E,\mathbb{R})$. For $c \in \mathbb{R}$, we say that I satisfies the $(PS)_c$ condition if for any sequence $\{x_n\} \subset E$ with

$$I(x_n) \to c, \quad DI(x_n) \to 0 \quad \text{in } E^*,$$

there is a subsequence $\{x_{n_k}\}$ such that $\{x_{n_k}\}$ converges strongly in E. Also, we say that I satisfies the $(C)_c$ condition if for any sequence $\{x_n\} \subset E$ with

$$I(x_n) \to c, \quad \|DI(x_n)\|_{E^*}(1 + \|x_n\|_E) \to 0,$$

there is subsequence $\{x_{n_k}\}$ such that $\{x_{n_k}\}$ converges strongly in E.

We have the following version of the Mountain Pass Theorem (see [18, 19]).

Proposition 1.2. *Let E be a real Banach space and suppose that $I \in C^1(E,R)$ satisfies the condition*

$$\max\{I(0), I(u_1)\} \leq \alpha < \beta \leq \inf_{\|u\|=\rho} I(u),$$

for some $\alpha < \beta$, $\rho > 0$ and $u_1 \in E$ with $\|u_1\| > \rho$. Let $c \geq \beta$ be characterized by

$$c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1],E), \gamma(0) = 0, \gamma(1) = u_1\}$ is the set of continuous paths joining 0 and u_1. Then, there exists a sequence $\{u_n\} \subset E$ such that

$$I(u_n) \to c \geq \beta \text{ and } (1 + \|u_n\|)\|I'(u_n)\|_{E^*} \to 0 \quad \text{as } n \to \infty.$$

Theorem 1.3. *Let $N \geq 3$ and assume that f has the improved subcritical polynomial growth on $\mathbb{R}^N$ (condition* (SCPI)*) and satisfies* (H1)–(H3)*. If $f_0 < \lambda_1 < l < \infty$, then problem* (1.1) *has at least one nontrivial solution.*

Remark 1.4. In view of conditions (SCPI), (H2) and (H3), problem (1.1) with the improved subcritical polynomial growth is called asymmetric. Hence, Theorem 1.3 is completely different from results contained in [4, 5, 6, 7, 8, 9].

Theorem 1.5. *Let $N \geq 3$ and assume that f has the improved subcritical polynomial growth on $\mathbb{R}^N$ (condition* (SCPI)*) and satisfies* (H1)–(H3)*. If $f_0 < \lambda_1 = l$ and $\lim_{t\to-\infty}[f(x,t)t - 2F(x,t)] = -\infty$ uniformly for a.e. $x \in \mathbb{R}^N$, then problem* (1.1) *has at least one nontrivial solution.*

Remark 1.6. When $l = \lambda_1$, problem (1.1) is called resonant at negative infinity. This case is completely new. Here, we also give an example for $f(x,t)$. It satisfies our conditions (H1)–(H3) and (SCPI).

Example Define

$$f(x,t) = \begin{cases} g(t)t, & t \le 0, \\ g(t)t + h(t), & t > 0, \end{cases}$$

where $g(t) \in C(R)$, $g(0) = 0$; $g(t) \ge 0$, $t \in \mathbb{R}$; $h(t) \in C[0,+\infty)$; $\lim_{t\to+0} \frac{h(t)}{t} = 0$; $\lim_{t\to+\infty} \frac{h(t)}{t^{2^*-1}} = 0$; $\lim_{t\to+\infty} \frac{h(t)}{t} = +\infty$. Moreover, there exists $t_0 > 0$ such that $g(t) \equiv \lambda_1$ for all $|t| \ge t_0$.

Theorem 1.7. *Let $N \ge 3$ and assume that f has the improved subcritical polynomial growth on $\mathbb{R}^N$ (condition (SCPI)) and satisfies (H1)–(H4). If $f_0 < \lambda_1$ and $l = +\infty$, then problem (1.1) has at least one nontrivial solution.*

Remark 1.8. *When $l = +\infty$, problem (1.1) is generalized superlinearity at negative infinity.*

In the case $N = 2$, we have $2^* = +\infty$. In this case, every polynomial growth is admitted. Hence, one is led to look for a function $g(s) : \mathbb{R} \to R^+$ with maximal growth such that

$$\sup_{u\in H, \|u\|_H \le 1} \int_{\mathbb{R}^N} g(u)\, dx < \infty.$$

It was shown by Trudinger [20], Moser [21] and Ruf [22] that the maximal growth is of exponential type. So, we must redefine the subcritical (exponential) growth in this case as follows:

(SCE): f has subcritical (exponential) growth on $\mathbb{R}^N$, i.e., For $\varepsilon > 0$, there exists positive constant $C^*(\varepsilon)$ such that

$$|f(x,t)| \le C^*(\varepsilon) + \varepsilon \exp(\alpha|t|^2) \quad \text{for all } t \in \mathbb{R},\ x \in \mathbb{R}^N \text{ and } \alpha > 0.$$

When $N = 2$ and f has the subcritical (exponential) growth (SCE), our work is still to study asymmetric problem (1.1) without the (AR)-condition in the positive semiaxis. To our knowledge, this problem is rarely studied by other people. Hence, our results are completely new and our methods are skillful since we skillfully combined Mountain Pass Theorem with Moser-Trudinger inequality. Our results are as follows:

Theorem 1.9. *Let $N = 2$ and assume that f has the subcritical exponential growth on $\mathbb{R}^N$ (condition (SCE)) and satisfies (H1)–(H3). If $f_0 < \lambda_1 < l < \infty$, then problem (1.1) has at least one nontrivial solution.*

Remark 1.10. In view of the conditions (H2), (H3) and (SCE), problem (1.1) is called asymmetric subcritical exponential problem. Hence, Theorem 1.9 is completely different from results contained in [1, 2, 3, 4, 5, 6, 7, 8, 9].

Theorem 1.11. *Let $N = 2$ and assume that f has the subcritical exponential growth on $\mathbb{R}^N$ (condition (SCE)) and satisfies (H1)–(H3). If $f_0 < \lambda_1 = l$ and $\lim_{t\to-\infty}[f(x,t)t - 2F(x,t)] = -\infty$ uniformly for a. e. $x \in \mathbb{R}^N$, then problem (1.1) has at least one nontrivial solution.*

Remark 1.12. *When $l = \lambda_1$, problem (1.1) is called resonant at negative infinity. This case is completely new.*

Theorem 1.13. *Let $N = 2$ and assume that f has the subcritical exponential growth on $\mathbb{R}^N$ (condition* (SCE)*) and satisfies* (H1)–(H4)*. If $f_0 < \lambda_1$ and $l = +\infty$, then problem* (1.1) *has at least one nontrivial solution.*

2. Preliminaries

Lemma 2.1. *Let $N \geq 3$ and $\varphi_1 > 0$ be a λ_1-eigenfunction with $\|\varphi_1\| = 1$ and assume that* (H1)–(H3) *and* (SCPI) *hold. If $f_0 < \lambda_1 < l \leq +\infty$, then*

(i) *There exist $\rho, \alpha > 0$ such that $I(u) \geq \alpha$ for all $u \in E$ with $\|u\| = \rho$,*
(ii) *$I(t\varphi_1) \to -\infty$ as $t \to +\infty$.*

Proof. By (SCPI) and (H1)–(H3), for any $\varepsilon > 0$, there exist $A_1 = A_1(\varepsilon)$, $B_1 = B_1(\varepsilon)$ such that for all $(x, s) \in \mathbb{R}^N \times \mathbb{R}$,

$$F(x,s) \leq \frac{1}{2}(f_0 + \varepsilon)|s|^2 + A_1|s|^{2^*}. \tag{2.1}$$

Choose $\varepsilon > 0$ such that $(f_0 + \varepsilon) < \lambda_1$. By (2.1), the continuous imbedding and the Sobolev inequality: $|u|_{2^*}^{2^*} \leq K\|u\|^{2^*}$, we obtain

$$\begin{aligned} I(u) &\geq \frac{1}{2}\|u\|^2 - \frac{f_0+\varepsilon}{2}|u|_2^2 - A_1|u|_{2^*}^{2^*} \\ &\geq \frac{1}{2}(1 - \frac{f_0+\varepsilon}{\lambda_1})\|u\|^2 - A_1K\|u\|^{2^*}. \end{aligned}$$

So, part (i) is proved if we choose $\|u\| = \rho > 0$ small enough.

On the other hand, by the definition of I and (H2) with $l > \lambda_1$, we have

$$\lim_{t\to-\infty} \frac{I(t\varphi_1)}{t^2} \leq \frac{1}{2}(\lambda_1 - l)|\varphi_1|_2^2 < 0.$$

By a slight modification to the proof above, we can prove (ii) if $l = +\infty$. □

Lemma 2.2 ([20, 21, 22])**.** *Let $u \in H$. Then*

$$\sup_{u\in H, \|u\|_H \leq 1} \int_{\mathbb{R}^N} (\exp \alpha|u|^2 - 1)\, dx \leq C \quad \text{for } \alpha \leq 4\pi^2.$$

The inequality is sharp: for any $\alpha > 4\pi^2$ the corresponding supremum is $+\infty$.

Lemma 2.3. *Let $N = 2$ and $\varphi_1 > 0$ be a λ_1-eigenfunction with $\|\varphi_1\| = 1$ and assume* (H1)–(H3) *and* (SCE) *hold. If $f_0 < \lambda_1 < l \leq +\infty$, then*

(i) *There exist $\rho, \alpha > 0$ such that $I(u) \geq \alpha$ for all $u \in H$ with $\|u\| = \rho$,*
(ii) *$I(t\varphi_1) \to -\infty$ as $t \to +\infty$.*

Proof. By (SCE) and (H1)–(H3), for any $\varepsilon > 0$, there exist $A_1 = A_1(\varepsilon)$, $B_1 = B_1(\varepsilon)$, $\kappa > 0$ and $q > 2$ such that for all $(x, s) \in \mathbb{R}^N \times \mathbb{R}$,

$$F(x,s) \leq \frac{1}{2}(f_0 + \varepsilon)|s|^2 + A_1(\exp(\kappa|s|^2) - 1)|s|^q. \tag{2.2}$$

Choose $\varepsilon > 0$ such that $(f_0 + \varepsilon) < \lambda_1$. By (2.2), the Holder inequality and the Moser-Trudinger embedding, we obtain

$$\begin{aligned} I(u) &\geq \frac{1}{2}\|u\|^2 - \frac{f_0+\varepsilon}{2}|u|_2^2 - A_1 \int_{\mathbb{R}^N} (\exp(\kappa|u|^2) - 1)|u|^q\, dx \\ &\geq \frac{1}{2}(1 - \frac{f_0+\varepsilon}{\lambda_1})\|u\|^2 - A_1\Big(\int_{\mathbb{R}^N} (\exp(\kappa r\|u\|_H^2 (\frac{|u|}{\|u\|_H})^2 - 1))\, dx\Big)^{1/r} \end{aligned}$$

$$
\begin{aligned}
&\times \Big(\int_\Omega |u|^{r'q}\,dx\Big)^{1/r'} \\
&\geq \frac{1}{2}(1 - \frac{f_0+\varepsilon}{\lambda_1})\|u\|^2 - C\|u\|^q,
\end{aligned}
$$

where $r > 1$ sufficiently close to 1, $\|u\|_H \leq \sigma$ and $\kappa r\sigma^2 < 4\pi^2$. So, part (i) is proved if we choose $\|u\| = \rho > 0$ small enough.

On the other hand, by the definition of I and (H2) with $l > \lambda_1$, we have

$$
\lim_{t\to-\infty} \frac{I(t\varphi_1)}{t^2} \leq \frac{1}{2}(\lambda_1 - l)|\varphi_1|_2^2 = \frac{\lambda_1 - l}{2\lambda_1} < 0
$$

By a slight modification to the proof above, we can prove (ii) if $l = +\infty$. □

Lemma 2.4. *For the functional I defined by* (1.2)*, if $u_n(x) \leq 0$ a.e. $x \in \mathbb{R}^N$, $n \in \mathbb{N}$ and*

$$
\langle I'(u_n), u_n\rangle \to 0 \text{ quadas } n \to \infty,
$$

then there exists subsequence, still denoted by $\{u_n\}$, such that

$$
I(tu_n) \leq \frac{1+t^2}{2n} + I(u_n) \quad \text{for all } t \geq 0 \text{ and } n \in \mathbb{N}.
$$

Proof. This lemma is essentially due to [7]. For the sake of completeness, we prove it here. By $\langle I'(u_n), u_n\rangle \to 0$ as $n \to \infty$, for a suitable subsequence, we may assume that

$$
-\frac{1}{n} < \langle I'(u_n), u_n\rangle = \|u_n\|^2 - \int_{\mathbb{R}^N} f(x, u_n(x))u_n\,dx < \frac{1}{n} \quad \text{for all } n. \tag{2.3}
$$

We claim that for any $t \geq 0$ and $n \in \mathbb{N}$,

$$
I(tu_n) \leq \frac{t^2}{2n} + \int_{\mathbb{R}^N} \{\frac{1}{2}f(x, u_n(x))u_n - F(x, u_n(x))\}\,dx. \tag{2.4}
$$

Indeed, for any $t \geq 0$, at fixed $x \in \mathbb{R}^N$ and $n \in \mathbb{N}$ if we set

$$
h(t) = \frac{1}{2}t^2 f(x, u_n)u_n(x) - F(x, tu_n(x)),
$$

then

$$
\begin{aligned}
h'(t) &= tf(x, u_n)u_n(x) - f(x, tu_n)u_n(x) \\
&= tu_n(x)\{f(x, u_n) - f(x, tu_n(x))/t\} \\
&\begin{cases} \geq 0 & \text{for } 0 < t \leq 1 \\ \leq 0 & \text{for } t \geq 1 \end{cases}
\end{aligned}
$$

by (H4); hence $h(t) \leq h(1)$ for all $t \geq 0$. Therefore,

$$
\begin{aligned}
I(tu_n) &= \frac{1}{2}t^2\|u_n\|^2 - \int_{\mathbb{R}^N} F(x, tu_n(x))\,dx \\
&< \frac{1}{2}t^2\{\frac{1}{n} + \int_{\mathbb{R}^N} f(x, u_n(x))u_n(x)\,dx\} - \int_{\mathbb{R}^N} F(x, tu_n(x))\,dx \\
&\leq \frac{t^2}{2n} + \int_{\mathbb{R}^N} \{\frac{1}{2}t^2 f(x, u_n(x))u_n(x) - F(x, tu_n(x))\}\,dx \\
&\leq \frac{t^2}{2n} + \int_{\mathbb{R}^N} \{\frac{1}{2}f(x, u_n(x))u_n(x) - F(x, u_n(x))\}\,dx
\end{aligned}
$$

and our claim (2.4) is proved.

On the other hand,

$$\begin{aligned} I(u_n) &= \frac{1}{2}\|u_n\|^2 - \int_{\mathbb{R}^N} F(x,u_n(x))\,dx \\ &\geq \frac{1}{2}\{-\frac{1}{n} + \int_{\mathbb{R}^N} f(x,u_n(x))u_n(x)\,dx\} - \int_{\mathbb{R}^N} F(x,u_n(x))\,dx; \end{aligned}$$

that is,

$$\int_{\mathbb{R}^N} \{\frac{1}{2} f(x,u_n(x))u_n(x) - F(x,u_n(x))\}\,dx \leq \frac{1}{2n} + I(u_n). \tag{2.5}$$

Combining (2.4) and (2.5), we find that

$$I(tu_n) \leq \frac{1+t^2}{2n} + I(u_n) \text{ for all } t \geq 0 \text{ and } n \in \mathbb{N}. \tag{2.6}$$

□

3. Proofs of the main results

We prove only Theorems 1.3, 1.5, 1.7 and 1.9. Others followed from these results.

Proof of Theorem 1.3. By Lemma 2.1, the geometry conditions of Mountain Pass Theorem hold. So, we only need to verify condition (PS). Let $\{u_n\} \subset E$ be a (PS) sequence such that for every $n \in \mathbb{N}$,

$$\big|\frac{1}{2}\|u_n\|^2 - \int_{\mathbb{R}^N} F(x,u_n)\,dx\big| \leq c, \tag{3.1}$$

$$\big|\int_{\mathbb{R}^N} \nabla u_n \nabla v\,dx + \int_{\mathbb{R}^N} V(x)u_n v\,dx - \int_{\mathbb{R}^N} f(x,u_n)v\,dx\big| \leq \varepsilon_n\|v\|, \quad v \in E, \tag{3.2}$$

where $c > 0$ is a positive constant and $\{\varepsilon_n\} \subset \mathbb{R}^+$ is a sequence which converges to zero.

Step 1. To prove that $\{u_n\}$ has a convergence subsequence, we first show that it is a bounded sequence. To do this, we argue by contradiction assuming that for a subsequence, which we follow denoting by $\{u_n\}$, we have

$$\|u_n\| \to +\infty \quad \text{as } n \to \infty.$$

Without loss of generality, we can assume $\|u_n\| > 1$ for all $n \in \mathbb{N}$ and define $z_n = \frac{u_n}{\|u_n\|}$. Obviously, $\|z_n\| = 1$ for all $n \in \mathbb{N}$ and then, it is possible to extract a subsequence (denoted also by $\{z_n\}$) such that

$$z_n \rightharpoonup z_0 \quad \text{in } E, \tag{3.3}$$

$$z_n \to z_0 \quad \text{in } L^2(\mathbb{R}^N), \tag{3.4}$$

$$z_n(x) \to z_0(x) \quad \text{a.e. } x \in \mathbb{R}^N, \tag{3.5}$$

$$|z_n(x)| \leq q(x) \quad \text{a.e. } x \in \mathbb{R}^N, \tag{3.6}$$

where $z_0 \in E$ and $q \in L^2(\mathbb{R}^N)$. Dividing both sides of (3.2) by $\|u_n\|$, we obtain

$$\big|\int_{\mathbb{R}^N} \nabla z_n \nabla v\,dx + \int_{\mathbb{R}^N} V(x)z_n v\,dx - \int_{\mathbb{R}^N} \frac{f(x,u_n)}{\|u_n\|} v\,dx\big| \leq \frac{\varepsilon_n}{\|u_n\|}\|v\| \quad \text{for all } v \in E.$$

Passing to the limit we deduce from (3.3) that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \frac{f(x,u_n)}{\|u_n\|} v\,dx = \int_{\mathbb{R}^N} \nabla z_0 \nabla v\,dx + \int_{\mathbb{R}^N} V(x)z_0 v\,dx \tag{3.7}$$

for all $v \in E$.

Now we claim that $z_0(x) \le 0$ for a.e. $x \in \mathbb{R}^N$. To verify this, let us observe that by choosing $v = z_0^+ = \max\{z_0, 0\}$ in (3.7) we have

$$\lim_{n\to\infty} \int_{\Theta} \frac{f(x,u_n)}{\|u_n\|} z_0\,dx = \int_{\Theta} |\nabla z_0|^2\,dx + \int_{\Theta} V(x)|z_0|^2\,dx < +\infty, \tag{3.8}$$

where $\Theta = \{x \in \mathbb{R}^N | z_0(x) > 0\}$. On the other hand, from conditions (SCPI), (H1), (H2), (H3), (3.5) and (3.6), we have

$$\frac{f(x,u_n(x))}{\|u_n\|} z_0(x) \ge (-K_1) q(x) z_0(x), \quad \text{a.e. } x \in \Theta$$

for some positive constant $K_1 > 0$, and

$$\lim_{n\to\infty} \frac{f(x,u_n(x))}{\|u_n\|} z_0(x) = \lim_{n\to\infty} \frac{f(x,u_n(x))}{u_n} z_n(x) z_0(x) = +\infty, \quad \text{a.e. } x \in \Theta.$$

Therefore, if $|\Theta| > 0$, by the Fatou's Lemma, we obtain

$$\lim_{n\to\infty} \int_{\Theta} \frac{f(x,u_n(x))}{\|u_n\|} z_0(x)\,dx = +\infty,$$

which contradicts (3.8). Thus $|\Theta| = 0$ and the claim is proved.

Clearly, $z_0(x) \not\equiv 0$. By (H2), there exists $c > 0$ such that $\frac{|f(x,u_n)|}{|u_n|} \le c$ for a.e. $x \in \mathbb{R}^N$. By using Lebesgue dominated convergence theorem in (3.7), we have

$$\int_{\mathbb{R}^N} \nabla z_0 \nabla v\,dx + \int_{\mathbb{R}^N} V(x) z_0 v\,dx - \int_{\mathbb{R}^N} l z_0 v\,dx = 0 \tag{3.9}$$

for all $v \in E$. This contradicts our assumption, i.e., $l > \lambda_1$.

Step 2. Now, we prove that $\{u_n\}$ has a convergence subsequence. In fact, we can suppose that

$$\begin{gathered} u_n \rightharpoonup u \quad \text{in } E, \\ u_n \to u \quad \text{in } L^q(\mathbb{R}^N),\ \forall 1 \le q < 2^*, \\ u_n(x) \to u(x) \quad \text{a.e. } x \in \mathbb{R}^N. \end{gathered}$$

Since f has the subcritical growth on $\mathbb{R}^N$, for every $\epsilon > 0$, we can find a constant $C(\epsilon) > 0$ such that

$$f(x,s) \le C(\epsilon) + \epsilon |s|^{2^*-1}, \quad \forall (x,s) \in \mathbb{R}^N \times \mathbb{R},$$

then we obtain

$$\begin{aligned} &|\int_{\mathbb{R}^N} f(x,u_n)(u_n - u)\,dx| \\ &\le C(\epsilon) \int_{\mathbb{R}^N} |u_n - u|\,dx + \epsilon \int_{\mathbb{R}^N} |u_n - u||u_n|^{2^*-1}\,dx \\ &\le C(\epsilon) \int_{\mathbb{R}^N} |u_n - u|\,dx + \epsilon \Big(\int_{\mathbb{R}^N} (|u_n|^{2^*-1})^{\frac{2^*}{2^*-1}}\,dx \Big)^{\frac{2^*-1}{2^*}} \Big(\int_{\mathbb{R}^N} |u_n - u|^{2^*} \Big)^{1/2^*} \\ &\le C(\epsilon) \int_{\mathbb{R}^N} |u_n - u|\,dx + \epsilon C. \end{aligned}$$

Similarly, since $u_n \rightharpoonup u$ in E, $\int_{\mathbb{R}^N} |u_n - u|\,dx \to 0$. Since $\epsilon > 0$ is arbitrary, we can conclude that

$$\int_{\mathbb{R}^N} (f(x,u_n) - f(x,u))(u_n - u)\,dx \to 0 \quad \text{as } n \to \infty. \tag{3.10}$$

By (3.2), we have

$$\langle I'(u_n) - I'(u), (u_n - u)\rangle \to 0 \quad \text{as } n \to \infty. \tag{3.11}$$

From (3.10) and (3.11), we obtain

$$\int_{\mathbb{R}^N} (\nabla u_n - \nabla u)(\nabla u_n - \nabla u) + \int_{\mathbb{R}^N} V(x)|u_n - u|^2\,dx \to 0 \quad \text{as } n \to \infty. \tag{3.12}$$

We have $u_n \to u$ in E which means that I satisfies (PS). □

Proof of Theorem 1.5. Since $l = \lambda_1$, obviously, Lemma 2.1 (i) holds. We only need to show that Lemma 2.1 (ii) holds. Let $u = -t\varphi_1$, then

$$\begin{aligned} I(-t\varphi_1) &= \frac{1}{2}t^2 \int_{\mathbb{R}^N} (|\nabla\varphi_1|^2 + V(x)|\varphi_1|^2)\,dx - \int_{\mathbb{R}^N} F(x,-t\varphi_1)\,dx \\ &= \frac{1}{2}t^2 \int_{\mathbb{R}^N} (|\nabla\varphi_1|^2 + V(x)|\varphi_1|^2)\,dx - \frac{1}{2}\int_{\mathbb{R}^N} f(x,-t\varphi_1)(-t\varphi_1)\,dx \\ &\quad - \int_{\mathbb{R}^N} \{F(x,-t\varphi_1) + \frac{f(x,-t\varphi_1)t\varphi_1}{2}\}\,dx. \end{aligned}$$

Since $f(x,s) = \lambda_1 s + \circ(s)$ as $s \to -\infty$, we have

$$I(-t\varphi_1) \to -\infty \quad \text{as } t \to +\infty$$

and the claim is proved. By Proposition 1.2, there exists a sequence $\{u_n\} \subset E$ such that

$$I(u_n) = \frac{1}{2}\|u_n\|^2 - \int_{\mathbb{R}^N} F(x,u_n)\,dx = c + \circ(1), \tag{3.13}$$

$$(1 + \|u_n\|)\|I'(u_n)\|_{E^*} \to 0 \quad \text{as } n \to \infty. \tag{3.14}$$

Clearly, (3.14) implies

$$\langle I'(u_n), u_n\rangle = \|u_n\|^2 - \int_{\mathbb{R}^N} f(x,u_n(x))u_n\,dx = \circ(1). \tag{3.15}$$

To complete our proof, we need to verify that $\{u_n\}$ is bounded in E. Similar to the proof of Theorem 1.3, we have $z_0(x) \le 0$, $x \in \Omega$, $z_0(x) \not\equiv 0$ and

$$\int_{\mathbb{R}^N} (\nabla z_0 \nabla v + V(x) z_0 v)\,dx - \int_{\mathbb{R}^N} l z_0 v\,dx = 0$$

for all $v \in E$. By the maximum principle, $z_0 < 0$ is an eigenfunction of λ_1 then $|u_n(x)| \to \infty$ for a.e. $x \in \mathbb{R}^N$. By our assumptions, we have

$$\lim_{n\to\infty} (f(x,u_n(x))u_n(x) - 2F(x,u_n(x))) = -\infty$$

uniformly in $x \in \mathbb{R}^N$, which implies that

$$\int_{\mathbb{R}^N} (f(x,u_n(x))u_n(x) - 2F(x,u_n(x)))\,dx \to -\infty \quad \text{as } n \to \infty. \tag{3.16}$$

On the other hand, (3.15) implies that

$$2I(u_n) - \langle I'(u_n), u_n\rangle \to 2c \quad \text{as } n \to \infty.$$

Thus

$$\int_{\mathbb{R}^N} (f(x,u_n)u_n - 2F(x,u_n))\,dx \to 2c \quad \text{as } n \to \infty,$$

which contradicts (3.16). Hence $\{u_n\}$ is bounded. According to the Step 2 proof of Theorem 1.3, we have $u_n \to u$ in E which means that I satisfies $(\mathrm{C})_c$. □

Proof. Proof of Theorem 1.7] By Lemma 2.1 and Proposition 1.2 (3.13)-(3.15) hold. We still can prove that $\{u_n\}$ is bounded in E. Assume $\|u_n\| \to +\infty$ as $n \to \infty$. Similar to the proof of Theorem 1.3, we have $z_0(x) \le 0$ and when $z_0(x) < 0$, $u_n = z_n\|u_n\| \to -\infty$ as $n \to \infty$. Let

$$s_n = \frac{2\sqrt{c}}{\|u_n\|}, \quad w_n = s_n u_n = \frac{2\sqrt{c}u_n}{\|u_n\|}. \tag{3.17}$$

Since $\{w_n\}$ is bounded in E, it is possible to extract a subsequence (denoted also by $\{w_n\}$) such that

$$w_n \rightharpoonup w_0 \quad \text{in } E, \tag{3.18}$$

$$w_n \to w_0 \quad \text{in } L^2(\mathbb{R}^N), \tag{3.19}$$

$$w_n(x) \to w_0(x) \quad \text{a.e. } x \in \mathbb{R}^N, \tag{3.20}$$

$$|w_n(x)| \le h(x) \quad \text{a.e. } x \in \mathbb{R}^N, \tag{3.21}$$

where $w_0 \in E$ and $h \in L^2(\mathbb{R}^N)$.

If $\|u_n\| \to +\infty$ as $n \to \infty$, then $w_0(x) \equiv 0$. In fact, letting $\Theta^- = \{x \in \Omega : w_0(x) < 0\}$ and noticing $l = +\infty$, from (H3) it follows that

$$\frac{f(x,u_n)}{u_n} \ge M \quad \text{uniformly for all } x \in \Theta^-,$$

where M is a constant, large enough . Therefore, by (3.15) and (3.17), we have

$$\begin{aligned}
4c &= \lim_{n\to\infty} \|w_n\|^2 \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^N} \frac{f(x,u_n)}{u_n}|w_n|^2\,dx \\
&\ge \lim_{n\to\infty} \int_{\Theta^-} \frac{f(x,u_n)}{u_n}|w_n|^2\,dx \\
&\ge M\int_{\Theta^-} |w_0|^2\,dx.
\end{aligned}$$

So $w_0 \equiv 0$ for a.e. $x \in \mathbb{R}^N$. But, if $w_0 \equiv 0$, then $\int_{\mathbb{R}^N} F(x,w_n)\,dx \to 0$. Hence

$$I(w_n) = \frac{1}{2}\|w_n\|^2 + \circ(1) = 2c + \circ(1). \tag{3.22}$$

On the other hand, since $\|u_n\| \to \infty$ as $n \to \infty$, we have $s_n \to 0$ as $n \to \infty$. From Lemma 2.4 and (3.13), we obtain

$$I(w_n) = I(s_n u_n) \le \frac{1+(s_n)^2}{2n} + I(u_n) \le c, \quad \text{as } n \to \infty.$$

Obviously, it contradicts (3.22). So $\{u_n\}$ is bounded in E. According to the Step 2 proof of Theorem 1.3, we have $u_n \to u$ in E which means that I satisfies $(\mathrm{C})_c$. □

Proof of Theorem 1.9. By Lemma 2.3, the geometry conditions of Mountain Pass Theorem hold. So, we only need to verify condition (PS). Similar to the Step 1 proof of Theorem 1.3, we easily know that (PS) sequence $\{u_n\}$ is bounded in E. Next, we prove that $\{u_n\}$ has a convergence subsequence. Without loss of generality, suppose that

$$\begin{gathered}\|u_n\| \le \beta,\\ u_n \rightharpoonup u \quad \text{in } E,\\ u_n \to u \quad \text{in } L^q(\mathbb{R}^N),\ \forall q \ge 1,\\ u_n(x) \to u(x) \quad \text{a.e. } x \in \mathbb{R}^N.\end{gathered}$$

Now, since f has the subcritical exponential growth (SCE) on $\mathbb{R}^N$, we can find a constant $C_{\beta_0} > 0$ such that

$$|f(x,t)| \le C_{\beta_0}(\exp(\frac{\alpha_N}{2\beta_0^2}|t|^2) - 1), \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R},$$

where $\beta_0 = \gamma\beta$and γ is defined

$$\|u\|_H \le \gamma\|u\|, \quad u \in E.$$

Thus, by the Moser-Trudinger inequality (see Lemma 2.2),

$$\begin{aligned} &|\int_{\mathbb{R}^N} f(x,u_n)(u_n-u)\,dx| \\ &\le C\Big(\int_{\mathbb{R}^N} (\exp(\frac{\alpha_N}{\beta_0^2}|u_n|^2) - 1)\,dx\Big)^{1/2}|u_n-u|_2 \\ &\le C\Big(\int_{\mathbb{R}^N} (\exp(\frac{\alpha_N}{\beta_0^2}\|u_n\|_H^2|\frac{u_n}{\|u_n\|_H}|^2) - 1)\,dx\Big)^{1/2}|u_n-u|_2 \\ &\le C|u_n-u|_2 \to 0. \end{aligned}$$

Similar to the proof of Theorem 1.3, we have $u_n \to u$ in E which means that I satisfies (PS). □

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