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GROUND STATES FOR SCHRÖDINGER-POISSON SYSTEMS WITH THREE GROWTH TERMS

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Abstract. In this article we study the existence and nonexistence of ground states of the Schrödinger-Poisson system

$$-\Delta u + V(x)u + K(x)\phi u = Q(x)u^3, \quad x \in \mathbb{R}^3,$$
$$-\Delta\phi = K(x)u^2, \quad x \in \mathbb{R}^3,$$

where V, K, and Q are asymptotically periodic in the variable x. The proof is based on the the method of Nehari manifold and concentration compactness principle. In particular, we develop the method of Nehari manifold for Schrödinger-Poisson systems with three times growth.

1. Introduction and statement of the main result

The Schrödinger-Poisson system

$$\begin{aligned} -\Delta u + V(x)u + K(x)\phi u &= f(x,u), \quad x \in \mathbb{R}^3, \\ -\Delta\phi &= K(x)u^2, \quad x \in \mathbb{R}^3, \end{aligned} \tag{1.1}$$

has great importance for describing the interaction of a charged particle with an electromagnetic field. For more information on the physical aspects about system (1.1) we refer the reader to [6].

There are many existence and nonexistence results about nontrivial solutions, radial and nonradial solutions, ground states, multiplicity of solutions and concentration of solutions for system (1.1) and similar problems. See the references in this article. Especially, the study of ground states has made great progress and attracted many authors attention for its great physical interests. Many results are focus on the case that (1.1) with more than three times growth. As we know, the first result on the existence of ground states of (1.1) was obtained by Azzollini and Pomponio [5], they treated (1.1) with $f(x,u) = |u|^{q-2}u$ and $f(x,u) = u^5 + |u|^{q-2}u$ ($4 < q < 6$) respectively and obtained ground states when V is constant or nonconstant, possibly unbounded below. Later, Cerami and Vaira [7] obtained positive ground states of (1.1) with $V(x) = 1$, $f(x,u) = a(x)|u|^{p-1}u$, $3 < p < 5$, and a, K have the limit at infinity. Alves et al [1] studied (1.1) with $K(x) = 1$ and $f(x,u) = f(u)$ continuous and discussed the existence of ground states when V is

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periodic and asymptotically periodic in the meaning that there exists a periodic function V_p such that $\lim_{|x|\to\infty}|V(x)-V_p(x)|=0$. More recently, in some weaker asymptotically periodic sense, assuming that V, K and f are all asymptotically periodic in x, we [26] showed that (1.1) possesses ground states.

Comparing with the above case, the case that f is with less than or equal to three times growth, there is no higher-order term in the nonlinearity. Then there is no Mountain-Pass structure and the standard variational methods can not be used. So there needs some techniques or new variational framework. In [15, 27], with the help of some parameters, the compactness of the PS sequence was recovered, and then the authors deduced that the existence of nontrivial solutions.

By the motivation of above works, without use of parameters, we try to show the existence of ground states for (1.1) with three times growth. Moreover, we want to know when there is no ground state.

In this article we study the system

$$\begin{gathered} -\Delta u+V(x)u+K(x)\phi u=Q(x)u^3,\ x\in\mathbb{R}^3,\\ -\Delta\phi=K(x)u^2,\quad x\in\mathbb{R}^3. \end{gathered} \tag{1.2}$$

Letting $\mathcal{F}$ be the class of functions $\tilde{h}\in L^\infty(\mathbb{R}^3)$ such that, for every $\epsilon>0$ the set $\{x\in\mathbb{R}^3:|\tilde{h}(x)|\geq\epsilon\}$ has finite Lebesgue measure, we assume that:

- (H1) $V,K,Q\in L^\infty(\mathbb{R}^3)$, and there exist functions $V_p,K_p,Q_p\in L^\infty(\mathbb{R}^3)$, are 1-periodic in $x_i, 1\leq i\leq 3$, such that $V-V_p, K-K_p, Q-Q_p\in\mathcal{F}$;
- (H2) there exist positive constants a_0, b_0 and q_0 such that $a_0<V\leq V_p$, $b_0<K\leq K_p$, $Q\geq Q_p>q_0$, where V_p,K_p,Q_p are given in (H1);
- (H2') there exist positive constants a_0, b_0 and q_0 such that $a_0<V_p\leq V$, $b_0<K_p\leq K$, $Q_p\geq Q>q_0$, where V_p,K_p,Q_p are given in (H1).

Our main results read as follows.

Theorem 1.1. *Let* (H1) *and* (H2) *hold. Then* (1.2) *has a positive ground state.*

Theorem 1.2. *Let* (H1) *and* (H2') *hold. In addition, if one of the three conditions*

$$V\not\equiv V_p\quad K\not\equiv K_p,\quad Q\not\equiv Q_p,$$

is satisfied, then (1.2) *has no ground state.*

The outline for the proof: For system (1.2), we do not use the parameters as in [15, 27, 19], but improve the method of Nehari manifold [22] to prove Theorems 1.1 and 1.2. In the process of using the method of Nehari manifold, since our problem is lack of higher-order term of nonlinearity, the standard method of Nehari manifold [22] needs to be re-established. We find that, although the Nehari manifold is not homeomorphic to the unit sphere, it is homeomorphic to an open set of the unit sphere. So we can still reduce the problem of looking for a ground state into that of finding a minimizer of the functional on Nehari manifold. Then we use concentration compactness principle to deal with the minimizing problem. Since (1.2) is non-periodic, we cannot use the invariance of the functional under translation to look for a minimizer. By the periodicity of the limit system and the relation of the functionals and derivatives of (1.2) and its limit system, we find the minimizer. In addition, we take advantage of the ground states of the limit system and the relation of the functionals and derivatives of (1.2) and its limit system to obtain sufficient conditions for the nonexistence of ground states.

The article is organized as follows. In Section 2 we give some preliminaries. In Section 3 we introduce the variational setting. In Section 4 we prove Theorems 1.1 and 1.2.

2. Notation and preliminaries

In this article we use the following notation:

$\int_{\mathbb{R}^3} h(x)\,dx$ will be represented by $\int h\,dx$. By (H1), (H2) or (H2'), we can define the scalar product and norm in $H^1(\mathbb{R}^3)$ by

$$\langle u,v\rangle = \int (\nabla u\cdot\nabla v + V(x)uv)\,dx, \quad \|u\|^2 = \langle u,u\rangle.$$

Moreover,

$$\|u\|_p^2 = \int (|\nabla u|^2 + V_p(x)u^2)\,dx,$$

is an equivalent norm in $H^1(\mathbb{R}^3)$. $D^{1,2}(\mathbb{R}^3)$ is the Sobolev space endowed with the scalar product and norm

$$(u,v)_{D^{1,2}} = \int \nabla u\cdot\nabla v\,dx, \quad \|u\|_{D^{1,2}}^2 = \int |\nabla u|^2\,dx.$$

$S = \{u\in H^1(\mathbb{R}^3) : \|u\|^2 = 1\}$. The norm in $L^r(\mathbb{R}^3)$ $(1\le r\le\infty)$ is denoted by $|\cdot|_r$. For $\varrho>0$ and $z\in\mathbb{R}^3$, $B_\varrho(z)$ denotes the ball of radius ϱ centered at z.

The system (1.2) can be easily transformed into a Schrödinger equation with a nonlocal term. Actually, for all $u\in H^1(\mathbb{R}^3)$, considering the linear functional L_u defined in $D^{1,2}(\mathbb{R}^3)$ by

$$L_u(v) = \int K(x)u^2v\,dx.$$

By the Hölder inequality and the Sobolev inequality, we have

$$|L_u(v)| \le |K|_\infty |u|_{12/5}^2 |v|_6 \le C|u|_{12/5}^2\|v\|_{D^{1,2}}, \tag{2.1}$$

here and below C may indicate different constants. Hence the Lax-Milgram theorem implies that there exists a unique $\phi_u\in D^{1,2}(\mathbb{R}^3)$ such that

$$\begin{aligned}\int \nabla\phi_u\nabla v\,dx &= (\phi_u,v)_{D^{1,2}} = L_u(v)\\ &= \int K(x)u^2v\,dx, \quad \forall\, v\in D^{1,2}(\mathbb{R}^3).\end{aligned} \tag{2.2}$$

Namely, ϕ_u is the unique solution of $-\Delta\phi = K(x)u^2$. Moreover, ϕ_u can be expressed as

$$\phi_u(x) = \int \frac{K(y)}{|x-y|}u^2(y)\,dy.$$

Substituting ϕ_u into the first equation of (1.2), we obtain

$$-\Delta u + V(x)u + K(x)\phi_u u = Q(x)u^3. \tag{2.3}$$

By (2.1) and (2.2) we obtain

$$\|\phi_u\|_{D^{1,2}} = \|L_u\|_{\mathcal{L}(D^{1,2}(\mathbb{R}^3),\mathbb{R})} \le C|u|_{12/5}^2.$$

Then we obtain

$$\begin{aligned}\Big|\int K(x)\phi_u u^2\,dx\Big| &\le |K|_\infty|\phi_u|_6|u|_{12/5}^2\\ &\le C|K|_\infty\|\phi_u\|_{D^{1,2}}|u|_{12/5}^2\\ &\le C_0|u|_{12/5}^4 \le C_1\|u\|^4.\end{aligned}\tag{2.4}$$

In addition, one easily show that the functional

$$I(u)=\frac{1}{2}\|u\|^2+\frac{1}{4}\int K(x)\phi_u u^2\,dx-\frac{1}{4}\int Q(x)u^4\,dx$$

is of class C^1 and its critical points are solutions of (2.3). By the above argument, looking for ground states in $H^1(\mathbb{R}^3)\times D^{1,2}(\mathbb{R}^3)$ for (1.2) is equivalent to seeking for ground states in $H^1(\mathbb{R}^3)$ for (2.3). A solution $\tilde{u}\in H^1(\mathbb{R}^3)$ of (2.3) is called a ground state if

$$I(\tilde{u})=\min\{I(u):u\in H^1(\mathbb{R}^3)\backslash\{0\},\ I'(u)=0\}.$$

In the process of finding ground states for (1.2), the corresponding periodic system of (1.2) is very important. The corresponding periodic system is defined by

$$\begin{gathered}-\Delta u+V_p(x)u+K_p(x)\phi u=Q_p(x)u^3,\quad x\in\mathbb{R}^3,\\ -\Delta\phi=K_p(x)u^2,\quad x\in\mathbb{R}^3.\end{gathered}\tag{2.5}$$

As before this system can be transformed into the equation

$$-\Delta u+V_p(x)u+K_p(x)\tilde{\phi}_u u=Q_p(x)u^3.\tag{2.6}$$

Here $\tilde{\phi}_u\in D^{1,2}(\mathbb{R}^3)$ is the unique solution of the equation

$$-\Delta\phi=K_p(x)u^2.$$

Moreover, the functional for (2.6) is

$$I_p(u)=\frac{1}{2}\|u\|_p^2+\frac{1}{4}\int K_p(x)\tilde{\phi}_u u^2\,dx-\frac{1}{4}\int Q_p(x)u^4\,dx.$$

By [26, Lemmas 2.2, 2.3 and Remark 2.1], we have the following two results.

Lemma 2.1. *Let $K_p\in L^\infty(\mathbb{R}^3)$ be 1-periodic in x_i, $1\le i\le 3$, and $\inf_{\mathbb{R}^3}K_p>0$. If $z\in\mathbb{Z}^3$ and $\breve{w}(x)=w(x+z)$ for any $w\in H^1(\mathbb{R}^3)$, then*

$$\int K_p(x)\tilde{\phi}_{\breve{u}}\breve{u}\breve{v}\,dx=\int K_p(x)\tilde{\phi}_u uv\,dx,\quad u,v\in H^1(\mathbb{R}^3).$$

Lemma 2.2. *Let* (H1) *hold. If* (H2) *or* (H2') *is satisfied, then I' and I_p' are weakly sequentially continuous.*

3. Variational setting

In this section we describe the variational framework for our problem. In order to find ground states, we shall use the method of Nehari manifold [22]. A very important condition using the method of Nehari manifold is that the functional has a unique maximum point along the direction u, for any $u\in H^1(\mathbb{R}^3)\setminus\{0\}$. However, since our functional I is lack of the higher-order term of the nonlinearity, $I(tu)$ ($t\ge 0$) may not have the maximum, and therefore the standard method of Nehari manifold cannot be used. Partially inspired by [12], where the authors considered the existence of infinitely many nontrivial solutions of quasilinear Schrödinger equations, we find that if we restrict the functional $I(tu)$ in a set, then the functional has a unique maximum. In [22], the authors showed that when the functional has a unique maximum, the Nehari manifold is homeomorphic to the unit sphere. So it is natural to think that the Nehari manifold is homeomorphic to the intersection of the above set and the unit sphere. Then we can use the one-to-one correspondence of the functionals on the manifold and the intersection to improve the method of Nehari manifold in [22], and therefore find ground states.

First we give the Nehari manifold M corresponding to I,

$$M = \{u \in H^1(\mathbb{R}^3)\backslash\{0\} : \langle I'(u), u\rangle = 0\},$$

where

$$\langle I'(u), u\rangle = \|u\|^2 + \int K(x)\phi_u u^2\,dx - \int Q(x)u^4\,dx,$$

and the least energy on M is defined by $c := \inf_M I$.

Lemma 3.1. *Let $V, K, Q \in L^\infty(\mathbb{R}^3)$ be such that $\inf_{\mathbb{R}^3} V > 0$, $\inf_{\mathbb{R}^3} K > 0$, and $\inf_{\mathbb{R}^3} Q > 0$. Then I is coercive on M.*

Proof. For all $u \in M$, we have

$$I(u) = I(u) - \frac{1}{4}\langle I'(u), u\rangle = \frac{1}{4}\|u\|^2. \tag{3.1}$$

Then $I|_M$ is coercive. □

By the above statement, we need to use a new set to construct the new variational framework. We define

$$\Theta := \{u \in H^1(\mathbb{R}^3) : \int K(x)\phi_u u^2\,dx < \int Q(x)u^4\,dx\}.$$

Since $K, Q \in L^\infty(\mathbb{R}^3)$ by (H1) and $\inf_{\mathbb{R}^3} K > 0$ and $\inf_{\mathbb{R}^3} Q > 0$ by (H2) or (H2'), we claim that $\Theta \neq \emptyset$. In fact, let $u_0 \in C_0^\infty(\mathbb{R}^3, [0,1])$ be such that $u_0 \equiv 1$ in $B_r(0)$, $u_0 \equiv 0$ in $\mathbb{R}^3 \setminus B_{2r}(0)$, where r is to be determined. Then

$$\begin{aligned}\int_{\mathbb{R}^3} K(x)\phi_{u_0} u_0^2\,dx &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{K(x)K(y)}{|x-y|} u_0^2(x)u_0^2(y)\,dy\,dx \\ &\le |K|_\infty^2 \int_{|x|\le 2r}\int_{|y|\le 2r} \frac{1}{|x-y|}\,dy\,dx \\ &= |K|_\infty^2 \int_{|x|\le 2r}\int_{|x-z|\le 2r} \frac{1}{|z|}dz\,dx \\ &\le |K|_\infty^2 \int_{|x|\le 2r}\int_{|z|\le 4r} \frac{1}{|z|}dz\,dx.\end{aligned} \tag{3.2}$$

Using the sphere coordinate transformation method, we have

$$\int_{|z|\le 4r} \frac{1}{|z|}dz = \int_0^{4r}\int_0^{2\pi}\int_0^{\pi} \frac{\rho^2\sin\varphi}{\rho}\,d\varphi\,d\theta d\rho \le c_1 r^2,$$

where $z = (z_1, z_2, z_3) = (\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)$. Inserting the above inequality in (3.2), we have

$$\int_{\mathbb{R}^3} K(x)\phi_{u_0} u_0^2\,dx \le |K|_\infty^2 \int_{|x|\le 2r} c_1 r^2\,dx \le c_2 r^5.$$

Moreover,

$$\int Q(x)u_0^4\,dx \geq \inf_{\mathbb{R}^3} Q \int_{|x|\leq r} dx = c_3 r^3.$$

Choose small r be such that $c_2 r^5 < c_3 r^3$, then

$$\int K(x)\phi_{u_0} u_0^2\,dx < \int Q(x)u_0^4\,dx.$$

Hence, $\Theta \neq \emptyset$.

Now we consider the function

$$h(t) := I(tu) = \frac{t^2}{2}\|u\|^2 + \frac{t^4}{4}\Big[\int K(x)\phi_u u^2\,dx - \int Q(x)u^4\,dx\Big].$$

Lemma 3.2. *Under the assumptions of Lemma 3.1, we have:*

- (i) *For all $u \in \Theta$, there exists an unique $t_u > 0$ such that $h'(t) > 0$ for $0 < t < t_u$, and $h'(t) < 0$ for $t > t_u$. Moreover, $t_u u \in M$ and $I(t_u u) = \max_{t>0} I(tu)$.*
- (ii) *If $u \notin \Theta$, then $tu \notin M$ for any $t > 0$.*
- (iii) *For each compact subset W of $\Theta \cap S$, there exists $C_W > 0$ such that $t_w \leq C_W$ for all $w \in W$.*

Proof. (i) For each $u \in \Theta$, one easily has that $h(t) > 0$ when t is sufficiently small, and $h(t) < 0$ when t is large enough. Then h has a positive maximum point in $(0, \infty)$. Moreover, the maximum point t satisfies that

$$\|u\|^2 = t^2\Big[\int Q(x)u^4\,dx - \int K(x)\phi_u u^2\,dx\Big]. \tag{3.3}$$

Then the maximum point is unique, and denoted by t_u. Therefore conclusion (i) holds.

(ii) We argue by contradiction. Assume that there exists $t > 0$ such that $tu \in M$. Then $\langle I'(tu), tu\rangle = 0$. So (3.3) holds. Then $\int K(x)\phi_u u^2\,dx < \int Q(x)u^4\,dx$. Namely, $u \in \Theta$. This contradicts with $u \notin \Theta$. So the conclusion (ii) holds.

(iii) Suppose that there exist a compact subset $W \subset \Theta \cap S$ and a sequence $w_n \in W$ such that $t_{w_n} \to \infty$. Assume $w \in W$ satisfies $w_n \to w$ in $H^1(\mathbb{R}^3)$. Then one easily has that

$$\int K(x)\phi_{w_n} w_n^2\,dx - \int Q(x)w_n^4\,dx \to \int K(x)\phi_w w^2\,dx - \int Q(x)w^4\,dx < 0.$$

So

$$\frac{I(t_{w_n} w_n)}{t_{w_n}^2} = \frac{1}{2} + \frac{t_{w_n}^2}{4}\Big[\int K(x)\phi_{w_n} w_n^2\,dx - \int Q(x)w_n^4\,dx\Big] \to -\infty. \tag{3.4}$$

However, by (3.1), we know that $I(t_{w_n} w_n) \geq 0$. This is a contradiction. This completes the proof. $\square$

Lemma 3.3. *Under the assumptions of Lemma 3.1, we have:*

1. *there exists $\rho > 0$ such that $\inf_{S_\rho} I > 0$ and then $c = \inf_M I \geq \inf_{S_\rho} I > 0$, where $S_\rho = \{u \in H^1(\mathbb{R}^3) : \|u\|^2 = \rho\}$;*
2. *$\|u\|^2 \geq 4c$ for all $u \in M$.*

Proof. By (2.4), one easily has that there exists $\rho > 0$ such that $\inf_{S_\rho} I > 0$. As a consequence of Lemma 3.2(i), for any $u \in M$, there is $t > 0$ such that $tu \in S_\rho$. Note that $I(u) \geq I(tu)$, then $\inf_{S_\rho} I \leq \inf_M I = c$. Hence $c > 0$. Then the conclusion (1) holds. By (3.1), the conclusion (ii) easily follows. □

From Lemma 3.1(1), we define the mapping $\hat{m} : \Theta \to M$ by $\hat{m}(u) = t_u u$. In addition, for all $v \in \mathbb{R}^+ u$ we have $\hat{m}(v) = \hat{m}(u)$. Let $U := \Theta \cap S$, we easily infer that U is an open subset of S. Define $m := \hat{m}|_U$. Then m is a bijection from U to M. Moreover, by Lemmas 3.2 and 3.9, as in the proof of [22, Proposition 3.1], we have:

Lemma 3.4. *Under the assumptions of Lemma 3.1, the mapping m is a homeomorphism between U and M, and the inverse of m is given by $m^{-1}(u) = \frac{u}{\|u\|}$.*

We consider the functional $\Psi : U \to \mathbb{R}$ given by $\Psi(w) := I(m(w))$, and we easily deduce that:

Lemma 3.5. *Under the assumptions of Lemma 3.1, the following results hold:*

(1) *If $\{w_n\}$ is a PS sequence for Ψ, then $\{m(w_n)\}$ is a PS sequence for I. If $\{u_n\} \subset M$ is a bounded PS sequence for I, then $\{m^{-1}(u_n)\}$ is a PS sequence for Ψ.*
(2) *w is a critical point of Ψ if and only if $m(w)$ is a nontrivial critical point of I. Moreover, $\inf_M I = \inf_U \Psi$;*
(3) *A minimizer of I on M is a ground state of* (2.3).

From Lemma 3.5 (3), we know that the problem of seeking for a ground state for (2.3) can be reduced into that of finding a minimizer of $I|_M$. In the process of finding the minimizer, since (2.3) is non-periodic, we cannot use the invariance of the functional under translation to look for a minimizer. However, the approached equation of (2.3) as $|x| \to \infty$ (i.e. the equation $(1.2)'_p$) is periodic, we shall take advantage of the periodicity of the equation $(1.2)'_p$ and the relation of the functionals and derivatives of (2.3) and $(1.2)'_p$ to find the minimizer.

Below we give some lemmas for studying the relation of the functionals and derivatives of (2.3) and $(1.2)'_p$. By (H2), one easily has the following lemma.

Lemma 3.6. *Let* (H2) *hold. Then $I(u) \leq I_p(u)$, for all $u \in H^1(\mathbb{R}^3)$.*

The following lemma is obtained in [26], we give it for reader's convenience.

Lemma 3.7. *Let* (H1) *hold. Assume that $u_n \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$ and $\{\varphi_n\}$ is a bounded sequence in $H^1(\mathbb{R}^3)$. Then*

$$\int [V(x) - V_p(x)] u_n \varphi_n \, dx \to 0,$$

$$\int [K(x)\phi_{u_n} u_n \varphi_n - K_p(x)\tilde{\phi}_{u_n} u_n \varphi_n] \, dx \to 0,$$

$$\int (Q(x) - Q_p(x)) u_n^3 \varphi_n \, dx \to 0.$$

To show the nonexistence of ground states, we still need the following results.

Lemma 3.8. *Let* (H2') *hold. If one of the three conditions $V \not\equiv V_p$, $K \not\equiv K_p$, $Q \not\equiv Q_p$ is satisfied, then $I_p(u) < I(u)$, for all $u > 0$ in $H^1(\mathbb{R}^3)$.*

Lemma 3.9. *Let (H1) hold. Assume* $u_n(x) = u_0(x - y_n)$*, where* $u_0 \in H^1(\mathbb{R}^3)$ *and* $y_n \in \mathbb{Z}^3$*. If* $|y_n| \to \infty$*, then*

$$\int [V(x)u_n^2 - V_p(x)u_0^2]\, dx \to 0, \tag{3.5}$$

$$\int [K(x)\phi_{u_n} u_n^2 - K_p(x)\tilde{\phi}_{u_0} u_0^2]\, dx \to 0, \tag{3.6}$$

$$\int [Q(x)u_n^4 - Q_p(x)u_0^4]\, dx \to 0\,. \tag{3.7}$$

Proof. Noting that $u_n(x) = u_0(x - y_n)$ with $|y_n| \to \infty$, then it is easy to show that $u_n \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$. Replacing φ_n by u_n, from Lemma 3.4 it follows that

$$\int [V(x) - V_p(x)]u_n^2\, dx \to 0,$$

$$\int [K(x)\phi_{u_n} u_n^2 - K_p(x)\tilde{\phi}_{u_n} u_n^2]\, dx \to 0,$$

$$\int (Q(x) - Q_p(x))u_n^4\, dx \to 0.$$

By $y_n \in \mathbb{Z}^3$ and the periodicity of V_p, K_p, Q_p, (3.5), (3.6) and (3.7) yield. The proof is complete. □

4. Proof of main results

Proof of Theorem 1.1. By the statement in Section 3, it suffices to show that c is attained.

Assume that $w_n \in U$ satisfies that $\Psi(w_n) \to \inf_U \Psi$. By the Ekeland variational principle, we may suppose that $\Psi'(w_n) \to 0$. Then from Lemma 3.5 (1) it follows that $I'(u_n) \to 0$, where $u_n = m(w_n) \in M$. By Lemma 3.5 (2), we have $I(u_n) = \Psi(w_n) \to c$. By Lemma 3.1, we obtain that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Up to a subsequence, we assume that $u_n \rightharpoonup \tilde{u}$ in $H^1(\mathbb{R}^3)$, $u_n \to \tilde{u}$ in $L^2_{\rm loc}(\mathbb{R}^3)$ and $u_n \to \tilde{u}$ a.e. on $\mathbb{R}^3$. Using Lemma 2.2, we have $I'(\tilde{u}) = 0$. We discuss for two cases that $\tilde{u} \neq 0$ and $\tilde{u} = 0$.

Case 1: $\tilde{u} \neq 0$. Then $\tilde{u} \in M$. By (3.1) we obtain

$$\begin{aligned} c + o_n(1) &= I(u_n) - \frac{1}{4}\langle I'(u_n), u_n\rangle = \frac{1}{4}\|u_n\|^2 \\ &\geq \frac{1}{4}\|\tilde{u}\|^2 + o_n(1) \\ &= I(\tilde{u}) - \frac{1}{4}\langle I'(\tilde{u}), \tilde{u}\rangle + o_n(1) = I(\tilde{u}) + o_n(1), \end{aligned} \tag{4.1}$$

where the inequality of (4.1) follows from Fatou Lemma. Then $I(\tilde{u}) \leq c$. Since $\tilde{u} \in M$, we have $I(\tilde{u}) \geq c$. Hence $I(\tilde{u}) = c$.

Case 2: $\tilde{u} = 0$. This case is more complicated than the previous case. We study when $\{u_n\}$ is vanishing or non-vanishing. It is easy to see that the case of vanishing does not happen since the energy $c > 0$ by Lemma 3.3 (1). In the case of non-vanishing, we can follow the similar idea in [20] to construct a minimizer. However, since our equation (2.3) has a Poisson term, the process is somewhat different from [20].

Suppose $\{u_n\}$ is vanishing. Namely

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^3}\int_{B_1(y)} u_n^2(x)\,dx=0.$$

Then Lions Compactness Lemma implies that $u_n\to 0$ in $L^4(\mathbb{R}^3)$ and $u_n\to 0$ in $L^{\frac{12}{5}}(\mathbb{R}^3)$. Then by (2.4) we obtain

$$\int K(x)\phi_{u_n}u_n^2\,dx\to 0,\int Q(x)|u_n|^4\,dx\to 0. \tag{4.2}$$

Note that

$$I(u_n)\to c,\quad \langle I'(u_n),u_n\rangle\to 0.$$

Namely

$$\begin{gathered}c=\frac12\|u_n\|^2+\frac14\int K(x)\phi_{u_n}u_n^2\,dx-\frac14\int Q(x)|u_n|^4\,dx+o_n(1),\\ \|u_n\|^2+\int K(x)\phi_{u_n}u_n^2\,dx=\int Q(x)|u_n|^4\,dx+o_n(1).\end{gathered}$$

Combining with (4.2) we easily have $c=0$. However, from Lemma 3.3 (1) we obtain $c>0$. This is a contradiction.

Hence $\{u_n\}$ is non-vanishing. Then there exists $x_n\in\mathbb{R}^3$ and $\delta_0>0$ such that

$$\int_{B_1(x_n)} u_n^2(x)\,dx>\delta_0. \tag{4.3}$$

Without loss of generality, we assume that $x_n\in\mathbb{Z}^3$. Since $u_n\to\tilde u$ in $L^2_{\rm loc}(\mathbb{R}^3)$ and $\tilde u=0$, we may suppose that $|x_n|\to\infty$ up to a subsequence. Denote $\bar u_n$ by $\bar u_n(\cdot)=u_n(\cdot+x_n)$. Similarly, passing to a subsequence, we assume that $\bar u_n\rightharpoonup\bar u$ in $H^1(\mathbb{R}^3)$, $\bar u_n\to\bar u$ in $L^2_{\rm loc}(\mathbb{R}^3)$, and $\bar u_n\to\bar u$ a.e. on $\mathbb{R}^3$. By (4.3) we have

$$\int_{B_1(0)}\bar u_n^2(x)\,dx>\delta_0.$$

So $\bar u\neq 0$.

We first claim that

$$I_p'(\bar u)=0. \tag{4.4}$$

Indeed, for all $\psi\in H^1(\mathbb{R}^3)$, set $\psi_n(\cdot):=\psi(\cdot-x_n)$. From Lemma 3.4, replacing φ_n by ψ_n it follows that

$$\begin{gathered}\int[V(x)-V_p(x)]u_n\psi_n\,dx\to 0,\\ \int[K(x)\phi_{u_n}u_n\psi_n-K_p(x)\tilde\phi_{u_n}u_n\psi_n]\,dx\to 0,\\ \int(Q(x)-Q_p(x))u_n^3\psi_n\,dx\to 0.\end{gathered}$$

Consequently,

$$\langle I'(u_n),\psi_n\rangle-\langle I_p'(u_n),\psi_n\rangle\to 0.$$

Since $I'(u_n)\to 0$ and $\|\psi_n\|=\|\psi\|$, we have $\langle I'(u_n),\psi_n\rangle\to 0$. So $\langle I_p'(u_n),\psi_n\rangle\to 0$. Moreover, by the fact that $x_n\in\mathbb{Z}^3$, (H1) and Lemma 2.1, we obtain

$$\langle I_p'(\bar u_n),\psi\rangle=\langle I_p'(u_n),\psi_n\rangle.$$

Then $\langle I_p'(\bar u_n),\psi\rangle \to 0$. By the arbitrary of ψ, $I_p'(\bar u_n) \rightharpoonup 0$ in $H^{-1}(\mathbb{R}^3)$. Since I_p' is weakly sequentially continuous by Lemma 2.2, (4.4) holds.

Now we turn to prove that

$$I_p(\bar u) \le c. \tag{4.5}$$

Replacing φ_n by u_n, Lemma 3.4 yields

$$\int (V(x) - V_p(x)) u_n^2\, dx \to 0. \tag{4.6}$$

Then we infer that

$$\begin{aligned}
c + o_n(1) &= I(u_n) - \frac{1}{4}\langle I'(u_n) u_n\rangle = \frac{1}{4}\int |\nabla u_n|^2\, dx + \frac{1}{4}\int V(x) u_n^2\, dx \\
&= \frac{1}{4}\int |\nabla u_n|^2\, dx + \frac{1}{4}\int V_p(x) u_n^2\, dx + o_n(1) \\
&= \frac{1}{4}\int |\nabla \bar u_n|^2\, dx + \frac{1}{4}\int V_p(x) \bar u_n^2\, dx + o_n(1) \\
&\ge \frac{1}{4}\int |\nabla \bar u|^2\, dx + \frac{1}{4}\int V_p(x) \bar u^2\, dx + o_n(1) \\
&= I_p(\bar u) - \frac{1}{4}\langle I_p'(\bar u), \bar u\rangle + o_n(1) = I_p(\bar u) + o_n(1),
\end{aligned}$$

where we have used Fatou's Lemma and (4.4). So we have $I_p(\bar u) \le c$.

We shall verify that $\max_{t>0} I_p(t\bar u) = I_p(\bar u)$. Indeed, let $\chi(t) = I_p(t\bar u)$, $t > 0$. Then

$$\chi'(t) = t^3\big(\frac{1}{t^2}\|\bar u\|_p^2 + \int K_p(x)\tilde\phi_{\bar u}\bar u^2\, dx - \int Q_p(x)\bar u^4\, dx\big) := t^3 \tilde A(t).$$

Since $I_p'(\bar u) = 0$ by (4.4), $\tilde A(1) = 0$. Noting that $\tilde A$ is decreasing in $(0,\infty)$, then $\tilde A(t) > 0$ when $0 < t < 1$ and $\tilde A(t) < 0$ when $t > 1$. Hence $\chi'(t) > 0$ when $0 < t < 1$ and $\chi'(t) < 0$ when $t > 1$. Therefore, $\max_{t>0} I_p(t\bar u) = I_p(\bar u)$.

Note that $I_p'(\bar u) = 0$, then

$$\int K_p(x)\tilde\phi_{\bar u}\bar u^2\, dx < \int Q_p(x)\bar u^4\, dx.$$

By the condition that $K \le K_p$ and $Q \ge Q_p$, we obtain

$$\int K(x)\phi_{\bar u}\bar u^2\, dx < \int Q(x)\bar u^4\, dx.$$

Then $\bar u \in \Theta$. Using Lemma 3.2 (i), there exists $t_{\bar u} > 0$ such that $t_{\bar u}\bar u \in M$. Then by Lemma 3.6 we infer

$$I(t_{\bar u}\bar u) \le I_p(t_{\bar u}\bar u) \le \max_{t>0} I_p(t\bar u) = I_p(\bar u).$$

With the use of (4.5), we have $I(t_{\bar u}\bar u) \le c$. Noting that $t_{\bar u}\bar u \in M$, we obtain $I(t_{\bar u}\bar u) \ge c$. Then $I(t_{\bar u}\bar u) = c$.

In a word, we deduce that c is attained and the corresponding minimizer is a ground state of (2.3). Below we shall look for a positive ground state for (2.3). Assume that the ground state we found is u_0. Then $u_0 \in M$ and $I(u_0) = c$. By Lemma 3.2 (ii), we have that $u_0 \in \Theta$. Then $|u_0| \in \Theta$. By Lemma 3.2 (i) there exists $t_0 > 0$ such that $t_0|u_0| \in M$. Then $I(t_0|u_0|) \ge c$. Noting that $I(t_0|u_0|) \le I(t_0 u_0)$ and $I(t_0 u_0) \le I(u_0)$, we obtain $I(t_0|u_0|) \le c$. So $I(t_0|u_0|) = c$. Then $t_0|u_0|$ is also a ground state of (2.3). Applying the maximum principle to (2.3), we easily infer that $t_0|u_0| > 0$. Namely, we find a positive ground state for (2.3). This completes the proof. □

Proof of Theorem 1.2. We argue by contradiction. Suppose that $\hat{u}$ is a ground state of (2.3). Then $\hat{u} \in M$ and $I(\hat{u}) = c$. As the last paragraph in the proof of Theorem 1.1, we may assume that $\hat{u} > 0$. Define

$$M_p = \{u \in H^1(\mathbb{R}^3)\backslash\{0\} : \langle I_p'(u), u\rangle = 0\}, \quad c_p = \inf_{M_p} I_p,$$

$$\Theta_p = \{u \in H^1(\mathbb{R}^3) : \int K_p(x)\tilde{\phi}_u u^2\,dx < \int Q_p(x)u^4\,dx\}.$$

Below we claim that

$$c \le c_p. \tag{4.7}$$

Indeed, by Theorem 1.1, we know that (2.6) has a ground state and denoted by u_0. Then $u_0 \in M_p$ and $I_p(u_0) = c_p$. Letting $u_n(\cdot) = u_0(\cdot - y_n)$, $y_n \in \mathbb{Z}^3$ and $|y_n| \to \infty$. Since $u_0 \in M_p$, u_0 satisfies that

$$\|u_0\|_p^2 + \int K_p(x)\tilde{\phi}_{u_0}u_0^2\,dx = \int Q_p(x)u_0^4\,dx. \tag{4.8}$$

By Lemma 3.9 we infer that

$$\|u_0\|_p^2 + \int K(x)\phi_{u_n}u_n^2\,dx = \int Q(x)u_n^4\,dx + o_n(1).$$

Therefore, $u_n \in \Theta$ since $u_0 \neq 0$. By Lemma 3.2 (i), we have that there exists $t_n > 0$ such that $t_n u_n \in M$. Then t_n satisfies

$$\frac{1}{t_n^2}\|u_n\|^2 + \int K(x)\phi_{u_n}u_n^2\,dx = \int Q(x)u_n^4\,dx.$$

Using Lemma 3.9 again, we have

$$\frac{1}{t_n^2}\|u_0\|_p^2 + \int K_p(x)\tilde{\phi}_{u_0}u_0^2\,dx = \int Q_p(x)u_0^4\,dx + o_n(1).$$

Combining this with (4.2) we obtain that $t_n \to 1$. Then Lemma 3.9 implies that

$$\begin{aligned} c \le I(t_n u_n) &= \frac{t_n^2}{2}\|u_n\|^2 + \frac{t_n^4}{4}\Big[\int K(x)\phi_{u_n}u_n^2\,dx - \int Q(x)u_n^4\,dx\Big] \\ &= \frac{1}{2}\|u_0\|_p^2 + \frac{1}{4}[\int K_p(x)\tilde{\phi}_{u_0}u_0^2\,dx - \int Q_p(x)u_0^4\,dx] + o_n(1) \\ &= I_p(u_0) + o_n(1) = c_p + o_n(1). \end{aligned}$$

Then (4.7) holds.

Note that $\hat{u} \in M$, then one easily has that $\hat{u} \in \Theta$ and then $\hat{u} \in \Theta_p$ since $K \ge K_p$ and $Q \le Q_p$ by $(H_2)'$. Similar to Lemma 3.2 (i), we infer that there exists $t_0 > 0$ such that $t_0\hat{u} \in M_p$ and $I_p(t_0\hat{u}) = \max_{t\ge 0} I_p(t\hat{u})$. By Lemma 3.6, we have $I_p(t_0\hat{u}) < I(t_0\hat{u})$. Then

$$c_p \le I_p(t_0\hat{u}) < I(t_0\hat{u}) \le I(\hat{u}) = c.$$

This contradicts with (4.7). So the equation (2.3) has no ground state. The proof is complete. □

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