Electronic Journal of Differential Equations, Vol. 2014 (2014), No. 26, pp. 1–16.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

RELAXATION IN CONTROL SYSTEMS OF FRACTIONAL SEMILINEAR EVOLUTION EQUATIONS

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Abstract. We consider a control system described by fractional semilinear evolution equations with a mixed multivalued control constraint whose values are nonconvex closed sets. Along with the original system, we consider the system in which the constraint on the control is the closed convex hull of the original constraint. We obtain existence results for the control systems and study relations between the solution sets of the two systems. An example is given to illustrate the abstract results.

1. Introduction

Let $J = [0, b]$ and $0 < \alpha < 1$. In this paper, we consider a control system described by fractional semilinear evolution equations of the form

$$\begin{gathered} {}^C D_t^\alpha x(t) = Ax(t) + h(t, x(t)) + g(t)u(t), \quad t \in J, \\ x(0) = x_0, \end{gathered} \tag{1.1}$$

with the mixed nonconvex constraint on the control

$$u(t) \in U(t, x(t)) \quad \text{a.e. on } J, \tag{1.2}$$

where ${}^C D_t^\alpha$ is the Caputo fractional derivative of order α, A is the infinitesimal generator of a strongly continuous semigroup $\{T(t), t \geq 0\}$ in a separable reflexive Banach space X, $g : J \to \mathcal{L}(Y, X)$ ($\mathcal{L}(Y, X)$ is the space of continuous linear operators from Y into X), $h : J \times X \to X$ is a nonlinear function and $U : J \times X \to 2^Y \backslash \{\emptyset\}$ is a multivalued map with closed values (not necessarily convex). The space Y is a separable, reflexive Banach space modeling the control space.

Along with the constraint (1.2) on the control, we also consider the constraint

$$u(t) \in \overline{\mathrm{co}}\, U(t, x(t)) \quad \text{a.e. on } J \tag{1.3}$$

on the control. Here $\overline{\mathrm{co}}$ stands for the closed convex hull of a set.

The solutions to the control systems considered in this paper are in the mild sense and the precise definition will be given in Definition 2.5 below.

We denote by $\mathcal{R}_U$, $\mathcal{T}r_U$ ($\mathcal{R}_{\overline{\mathrm{co}}U}$, $\mathcal{T}r_{\overline{\mathrm{co}}U}$) the sets of all solutions, all admissible trajectories of the control system (1.1), (1.2) (with the control system (1.1), (1.3), respectively).

2000 *Mathematics Subject Classification.* 34G20, 93C20, 34A08.
Key words and phrases. Fractional semilinear evolution equation; relaxation property; feedback control; nonconvex constraint; mild solution.

Submitted October 15, 2013. Published January 14, 2014.

The main results obtained in this paper are that: $\mathcal{T}r_{\overline{\text{co}}U}$ is a compact set in $C(J,X)$ and the relaxation property

$$\mathcal{T}r_{\overline{\text{co}}U} = \overline{\mathcal{T}r_U} \tag{1.4}$$

holds, where the bar stands for the closure in $C(J,X)$.

Recently, fractional calculus and differential equations have been proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. We can find its numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc., see [7, 8, 10] for example. There has been a great deal of interest in the existence of solutions of fractional differential equations. One can see the monographs of Kilbas et al [12], Miller et al [18], the survey of Agarwal et al [1, 2], Liu et al [14, 15] and the references therein.

Abstract fractional semilinear differential equations represent a class of fractional partial differential equations. For the study of their existence results, we can refer to Zhou and Jiao [31, 32], Wang and Zhou [29] and the references therein. For control systems governed by fractional semilinear differential equations, many literatures were devoted to give sufficient conditions for their (approximate) controllability and optimal control theory. For instance, Kumar and Sukavanam [13], Sakthivel et al [19, 20, 21, 22], Ganesh et al [6] (approximate controllability). Wang and Zhou [30] (optimal control theory).

Relaxation property, such as (1.4), if true, has important ramifications in control theory, since it implies that every trajectory of the convexified (full) system can be approximated in $C(J,X)$ norm, with arbitrary degree of accuracy, by trajectories of the original system. There are many papers dealing with the verification of the relaxation property for various classes of control systems, for instance, Tolstonogov [23] of control systems of subdifferential type, Migórski [16, 17], Tolstonogov[24], Tolstonogov et al [26], Denkowski et al [4] (c.f. Section 7.4) of nonlinear evolution inclusions or equations.

In this paper, we study the relaxation property for control systems described by a class of fractional semilinear evolution equations. Please note that the control systems studied here are closed-loop systems (feedback control systems) while the ones considered in papers related to this work cited above were concerned with open-loop systems.

The rest of the paper is organized as follows: In section 2, we introduce some useful preliminaries and give the assumptions on the data of our problems. Some auxiliary results needed in the proof of the main results are given in section 3. Section 4 deals with the existence of solutions for the control systems. The main results are presented in section 5. An example and some concluding remarks are given in sections 6.

2. Preliminaries and assumptions

Let $J=[0,b]$ be the closed interval of the real line with the Lebesgue measure μ and the σ-algebra Σ of μ measurable sets. The norm of the space X (or Y) will be denoted by $\|\cdot\|_X$ (or $\|\cdot\|_Y$). We denote by $C(J,X)$ the space of all continuous functions from J into X with the supnorm given by $\|x\|_C = \sup_{t\in J}\|x(t)\|_X$ for $x\in C(J,X)$. For any Banach space V, the symbol ω-V stands for V equipped with the weak $\sigma(V,V^*)$ topology. The same notation will be used for subsets of V.

In all other cases, we assume that V and its subsets are equipped with the strong (normed) topology.

We first recall the following known definitions from the theory of fractional calculus. For more details, please see [12, 18].

Definition 2.1. The fractional integral of order α with the lower limit zero for a function f is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > 0,\ \alpha > 0,$$

provided the right hand side is point-wise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2. The Riemann-Liouville derivative of order α with the lower limit zero for a function f is defined as

$$^L D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}} ds, \quad t > 0,\ n-1 < \alpha < n.$$

Definition 2.3. The Caputo derivative of order α with the lower limit zero for a function f is defined as

$$^C D^\alpha f(t) = {}^L D^\alpha \Big(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \Big), \quad t > 0,\ n-1 < \alpha < n.$$

If f is an abstract function with values in X, then integrals which appear in Definitions 2.1 and 2.2 are taken in Bochner's sense.

We now proceed to some basic definitions and results from multivalued analysis. For more details on multivalued analysis, see the books [3, 11].

We use the following symbols: $P_f(Y)$ is the set of all nonempty closed subsets of Y, $P_{bf}(Y)$ is the set of all nonempty, closed and bounded subsets of Y.

On $P_{bf}(Y)$, we have a metric known as the "Hausdorff metric" and defined by

$$h(A, B) = \max\big\{ \sup_{a\in A} d(a, B),\ \sup_{b\in B} d(b, A) \big\},$$

where $d(x, C)$ is the distance from a point x to a set C. We say a multivalued map is h-continuous if it is continuous in the Hausdorff metric $h(\cdot, \cdot)$.

We say that a multivalued map $F : J \to P_f(Y)$ is measurable if $F^{-1}(E) = \{t \in J : F(t) \cap E \neq \emptyset\} \in \Sigma$ for every closed set $E \subseteq Y$. If $F : J \times X \to P_f(Y)$, then the measurability of F means that $F^{-1}(E) \in \Sigma \otimes \mathcal{B}_X$, where $\Sigma \otimes \mathcal{B}_X$ is the σ-algebra of subsets in $J \times X$ generated by the sets $A \times B$, $A \in \Sigma$, $B \in \mathcal{B}_X$, and $\mathcal{B}_X$ is the σ-algebra of the Borel sets in X.

Suppose V, Z are two Hausdorff topological spaces and $F : V \to 2^Z \backslash \{\emptyset\}$. We say that F is lower semicontinuous in the sense of Vietoris (l.s.c. for short) at a point $x_0 \in V$, if for any open set $W \subseteq Z$, $F(x_0) \cap W \neq \emptyset$, there is a neighborhood $O(x_0)$ of x_0 such that $F(x) \cap W \neq \emptyset$ for all $x \in O(x_0)$. F is said to be upper semicontinuous in the sense of Vietoris (u.s.c. for short) at a point $x_0 \in V$, if for any open set $W \subseteq Z$, $F(x_0) \subseteq W$, there is a neighborhood $O(x_0)$ of x_0 such that $F(x) \subseteq W$ for all $x \in O(x_0)$. For the properties of l.s.c and u.s.c, please refer to the book [11].

Besides the standard norm on $L^q(J,Y)$ (here Y is a separable, reflexive Banach space), $1<q<\infty$, we also consider the so called weak norm

$$\|u(\cdot)\|_\omega = \sup_{0\le t_1\le t_2\le b} \Big\| \int_{t_1}^{t_2} u(s)ds \Big\|_Y, \text{ for } u\in L^q(J,Y). \tag{2.1}$$

The space $L^q(J,Y)$ furnished with this norm will be denoted by $L^q_\omega(J,Y)$. The following result establishes a relation between convergence in ω-$L^q(J,Y)$ and convergence in $L^q_\omega(J,Y)$.

Lemma 2.4 ([24]). *If a sequence $\{u_n\}_{n\ge1}\subseteq L^q(J,Y)$ is bounded and converges to u in $L^q_\omega(J,Y)$, then it converges to u in ω-$L^q(J,Y)$.*

We assume the following assumptions on the data of our problems in the whole paper.

(H1) : The operator A generates a strongly continuous semigroup $T(t)$, $t\ge0$ in X, and there exists a constant $M_A\ge1$ such that $\sup_{t\in[0,\infty)}\|T(t)\|\le M_A$. For any $t>0$, $T(t)$ is compact.

(H2) The operator $g:J\to\mathcal{L}(Y,X)$ is such that:
  (1) the map $t\to g(t)u$ is measurable for any $u\in Y$;
  (2) for a.e. $t\in J$,
$$\|g(t)\|_{\mathcal{L}(Y,X)}\le d, \text{ with } d>0. \tag{2.2}$$

(H3) The function $h:J\times X\to X$ satisfies the following:
  (1) $t\to h(t,x)$ is measurable for all $x\in X$;
  (2) there exists a function $l\in L^\infty(J,\mathbb{R}^+)$ such that for a.e. $t\in J$ and all x, $y\in X$,
$$\|h(t,x)-h(t,y)\|_X\le l(t)\|x-y\|_X; \tag{2.3}$$
  (3) there exists a constant $0<\beta<\alpha$ such that for a.e. $t\in J$, and all $x\in X$, $\|h(t,x)\|_X\le a_1(t)+c_1\|x\|_X$, where $a_1\in L^{1/\beta}(J,\mathbb{R}^+)$ and $c_1>0$.

(H4) The multivalued map $U:J\times X\to P_f(Y)$ is such that:
  (1) for all $x\in X$, $t\to U(t,x)$ is measurable;
  (2) $h(U(t,x),U(t,y))\le k_1(t)\|x-y\|_X$ a.e. on J, with $k_1\in L^\infty(J,\mathbb{R}^+)$;
  (3) for a.e. $t\in J$, and all $x\in X$, $\|U(t,x)\|_Y=\sup\{\|v\|_Y : v\in U(t,x)\}\le a_2(t)+c_2\|x\|_X$, where $a_2\in L^{1/\beta}(J,\mathbb{R}^+)$ and $c_2>0$.

Definition 2.5 ([31, 32]). A pair of functions (x,u) is a solution (mild solution) of the control system (1.1), (1.2), if $x(0)=x_0$, $x\in C(J,X)$ and there exists $u\in L^1(J,Y)$ such that $u(t)\in U(t,x(t))$ a.e. $t\in J$ and

$$x(t)=P_\alpha(t)x_0+\int_0^t (t-s)^{\alpha-1}Q_\alpha(t-s)\big(g(s)u(s)+h(s,x(s))\big)ds. \tag{2.4}$$

A similar definition can be introduced for the system (1.1), (1.3). Here

$$P_\alpha(t)=\int_0^\infty \xi_\alpha(\theta)T(t^\alpha\theta)d\theta, \quad Q_\alpha(t)=\alpha\int_0^\infty \theta\xi_\alpha(\theta)T(t^\alpha\theta)d\theta,$$
$$\xi_\alpha(\theta)=\frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}}\varpi_\alpha\big(\theta^{-\frac{1}{\alpha}}\big)\ge0,$$
$$\varpi_\alpha(\theta)=\frac{1}{\pi}\sum_{n=1}^\infty(-1)^{n-1}\theta^{-n\alpha-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha), \quad \theta\in(0,\infty),$$

and ξ_α is a probability density function defined on $(0,\infty)$; that is,

$$\xi_\alpha(\theta) \geq 0, \quad \theta \in (0,\infty), \quad \int_0^\infty \xi_\alpha(\theta)d\theta = 1.$$

It is not difficult to verify that

$$\int_0^\infty \theta\xi_\alpha(\theta)d\theta = \frac{1}{\Gamma(1+\alpha)}. \tag{2.5}$$

Lemma 2.6 ([31, 32])**.** *Let* (H1) *hold. Then the operators P_α and Q_α have the following properties:*

1. *For any fixed $t \geq 0$, $P_\alpha(t)$ and $Q_\alpha(t)$ are linear and bounded operators, i.e., for any $x \in X$,*
$$\|P_\alpha(t)x\|_X \leq M_A\|x\|_X, \quad \|Q_\alpha(t)x\|_X \leq \frac{\alpha M_A}{\Gamma(1+\alpha)}\|x\|_X;$$
2. *$\{P_\alpha(t), t \geq 0\}$ and $\{Q_\alpha(t), t \geq 0\}$ are strongly continuous;*
3. *For every $t > 0$, $P_\alpha(t)$ and $Q_\alpha(t)$ are compact operators.*

The proof of the above lemma can be found in [31].

3. Auxiliary results

In this section, we shall give some auxiliary results needed in the proof of the main results. We begin with the a prior estimation of the trajectory of the control systems.

Lemma 3.1. *For any admissible trajectory x of the control system* (1.1), (1.3)*; i.e., $x \in \mathcal{T}r_{\overline{\mathrm{co}}U}$, there is a constant L such that*

$$\|x\|_C \leq L. \tag{3.1}$$

Proof. Let any $x \in \mathcal{T}r_{\overline{\mathrm{co}}U}$. From Definition 2.5, we have that there exists a $u(t) \in \overline{\mathrm{co}}\,U(t,x(t))$ a.e. $t \in J$ and

$$x(t) = P_\alpha(t)x_0 + \int_0^t (t-s)^{\alpha-1}Q_\alpha(t-s)\big(g(s)u(s) + h(s,x(s))\big)ds.$$

Then by Lemma 2.6, we obtain

$$\begin{aligned}\|x(t)\|_X &\leq M_A\|x_0\|_X + \frac{\alpha M_A}{\Gamma(1+\alpha)}\int_0^t (t-s)^{\alpha-1}\|h(s,x(s))\|_X ds \\ &\quad + \frac{\alpha M_A}{\Gamma(1+\alpha)}\int_0^t (t-s)^{\alpha-1}\|g(s)u(s)\|_X ds.\end{aligned} \tag{3.2}$$

From (H3)(2), (H3)(3) and the Hölder inequality, we have

$$\begin{aligned}&\int_0^t (t-s)^{\alpha-1}\|h(s,x(s))\|_X ds \\ &\leq \int_0^t (t-s)^{\alpha-1}\|h(s,x(s)) - h(s,0)\|_X ds + \int_0^t (t-s)^{\alpha-1}\|h(s,0)\|_X ds \\ &\leq \int_0^t (t-s)^{\alpha-1}l(s)\|x(s)\|_X ds + \int_0^t (t-s)^{\alpha-1}a_1(s)ds \\ &\leq \Big[\frac{1-\beta}{\alpha-\beta}b^{\frac{\alpha-\beta}{1-\beta}}\Big]^{1-\beta}\|a_1\|_{L^{1/\beta}(J)} + \|l\|_{L^\infty(J)}\int_0^t (t-s)^{\alpha-1}\|x(s)\|_X ds.\end{aligned} \tag{3.3}$$

Similarly, by (H2)(2) and (H4)(3), we obtain

$$\begin{aligned} &\int_0^t (t-s)^{\alpha-1}\|g(s)u(s)\|_X ds \\ &\le d\int_0^t (t-s)^{\alpha-1}\big(a_2(s)+c_2\|x(s)\|_X\big)ds \\ &\le d\Big[\frac{1-\beta}{\alpha-\beta}b^{\frac{\alpha-\beta}{1-\beta}}\Big]^{1-\beta}\|a_2\|_{L^{1/\beta}(J)} + dc_2\int_0^t (t-s)^{\alpha-1}\|x(s)\|_X ds. \end{aligned} \tag{3.4}$$

Combining (3.3), (3.4) with (3.2), we obtain

$$\begin{aligned} \|x(t)\|_X \le M_A\|x_0\|_X + \frac{\alpha M_A}{\Gamma(1+\alpha)}\big(dc_2+\|l\|_{L^\infty(J)}\big)\int_0^t (t-s)^{\alpha-1}\|x(s)\|_X ds \\ + \frac{\alpha M_A}{\Gamma(1+\alpha)}\Big[\frac{1-\beta}{\alpha-\beta}b^{\frac{\alpha-\beta}{1-\beta}}\Big]^{1-\beta}\Big(\|a_1\|_{L^{1/\beta}(J)}+d\|a_2\|_{L^{1/\beta}(J)}\Big). \end{aligned}$$

From the above inequality, using the well-known singular-version Gronwall inequality (see [5, Theorem 3.1]), we can deduce that there exists a constant $L>0$ such that $\|x\|_C \le L$. $\square$

Let $\mathrm{pr}_L : X \to X$ be the L-radial retraction; i.e.,

$$\mathrm{pr}_L(x) = \begin{cases} x, & \|x\|_X \le L, \\ \frac{Lx}{\|x\|_X}, & \|x\|_X > L. \end{cases}$$

This map is Lipschitz continuous. We define $U_1(t,x) = U(t, \mathrm{pr}_L\, x)$. Evidently, U_1 satisfies (H4)(1) and (H4)(2). Moreover, by the properties of pr_L, we have, for a.e. $t \in J$, all $x \in X$ and all $u \in U_1(t,x)$ such that

$$\|u\|_Y \le a_2(t) + c_2 L \text{ and } \|u\|_Y \le a_2(t) + c_2\|x\|_X.$$

Hence, Lemma 3.1 is still valid with $U(t,x)$ substituted by $U_1(t,x)$. Consequently, henceforth we assume without any loss of generality that, for a.e. $t \in J$ and all $x \in X$,

$$\sup\{\|v\|_Y : v \in U(t,x)\} \le \varphi(t) = a_2(t) + c_2 L, \quad \text{with } \varphi \in L^{1/\beta}(J, \mathbb{R}^+). \tag{3.5}$$

Let φ be defined by (3.5), we put

$$Y_\varphi = \{u \in L^{1/\beta}(J,Y) : \|u(t)\|_Y \le \varphi(t) \text{ a.e. } t \in J\}, \tag{3.6}$$

$$X_\varphi = \{f \in L^{1/\beta}(J,X) : \|f(t)\|_X \le d\varphi(t) + a_1(t) + c_1 L \text{ a.e. } t \in J\}. \tag{3.7}$$

In accordance with (H2) and (H3), for any $x \in C(J,X)$ and $u \in L^{1/\beta}(J,Y)$, the function $t \to g(t)u(t) + h(t,x(t))$ is an element of the space $L^{1/\beta}(J,X)$. Hence, we can consider an operator $\mathcal{A} : C(J,X) \times L^{1/\beta}(J,Y) \to L^{1/\beta}(J,X)$ defined by

$$\mathcal{A}(x,u)(t) = g(t)u(t) + h(t,x(t)). \tag{3.8}$$

Lemma 3.2. *The map $(x,u) \to \mathcal{A}(x,u)$ is sequentially continuous from $C(J,X) \times \omega\text{-}L^{1/\beta}(J,Y)$ into $\omega\text{-}L^{1/\beta}(J,X)$.*

Proof. Suppose that $x_n \to x$ in $C(J,X)$ and $u_n \to u$ in $\omega\text{-}L^{1/\beta}(J,Y)$. Let any $h \in L^{1/(1-\beta)}(J,X^*)$ be fixed. Now we may assume that $\|x_n\|_C \le M$ for some constant $M>0$ and $n \ge 1$. Then from (H2) and (H3), we can have the following facts

$$h(t,x_n(t)) \to h(t,x(t)) \quad \text{in } X \text{ a.e. } t \in J, \tag{3.9}$$

$$\|h(t, x_n(t))\|_X \leq a_1(t) + c_1 M, \tag{3.10}$$

$$\int_J \langle g^*(t)h(t), u_n(t)\rangle dt \to \int_J \langle g^*(t)h(t), u(t)\rangle dt, \tag{3.11}$$

where $g^*(t)$ is the operator adjoint to $g(t)$. From (3.9) and (3.10), using Lebesgue's dominated convergence theorem, we obtain

$$h(t, x_n(t)) \to h(t, x(t)) \quad \text{in } L^{1/\beta}(J, X). \tag{3.12}$$

Since $\langle h(t), g(t)u(t)\rangle = \langle g^*(t)h(t), u(t)\rangle$ and $h \in L^{1/(1-\beta)}(J, X^*)$ is arbitrary, by (3.11), we deduce that

$$g(t)u_n(t) \to g(t)u(t) \quad \text{in } \omega\text{-}L^{1/\beta}(J, X).$$

This together and (3.12) imply

$$\mathcal{A}(x_n, u_n) \to \mathcal{A}(x, u) \quad \text{in } \omega\text{-}L^{1/\beta}(J, X).$$

The lemma is proved. □

Now we consider the auxiliary problem:

$$\begin{gathered} {}^C D_t^\alpha x(t) = Ax(t) + f(t),\ t \in J = [0, b], \\ x(0) = x_0. \end{gathered} \tag{3.13}$$

It is clear that, for every $f \in L^{1/\beta}(J, X)$, equation (3.13) has a unique mild solution $S(f) \in C(J, X)$ which is given by

$$S(f)(t) = P_\alpha(t)x_0 + \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s) f(s) ds.$$

The following lemma concerns with the property of the solution map S which is crucial in our investigation.

Lemma 3.3. *The solution map $S : X_\varphi \to C(J, X)$ is continuous from ω-X_φ into $C(J, X)$.*

Proof. Consider the operator $H : L^{1/\beta}(J, X) \to C(J, X)$ defined by

$$H(f)(t) = \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s) f(s) ds.$$

We know H is linear. From simple calculation, one has

$$\|H(f)\|_C \leq \frac{\alpha M_A}{\Gamma(1+\alpha)} \Big[\frac{1-\beta}{\alpha-\beta} b^{\frac{\alpha-\beta}{1-\beta}}\Big]^{1-\beta} \|f\|_{L^{1/\beta}(J,X)}; \tag{3.14}$$

i.e., the operator H is continuous from $L^{1/\beta}(J, X)$ to $C(J, X)$, hence H is also continuous from ω-$L^{1/\beta}(J, X)$ to ω-$C(J, X)$.

Let $C \in P_b(L^{1/\beta}(J, X))$ and suppose that for any $f \in C$, $\|f\|_{L^{1/\beta}(J,X)} \leq K$ ($K > 0$ is a constant). Next we will show that H is completely continuous.

(a) From (3.14), we know that $\|H(f)(t)\|_X$ is uniformly bounded for any $t \in J$ and $f \in C$.

(b) H is equicontinuous on C. Let $0 \leq t_1 < t_2 \leq b$. For any $f \in C$, we obtain

$$\begin{aligned} &\|H(f)(t_2) - H(f)(t_1)\|_X \\ &= \Big\| \int_0^{t_2} (t_2-s)^{\alpha-1} Q_\alpha(t_2-s) f(s) ds - \int_0^{t_1} (t_1-s)^{\alpha-1} Q_\alpha(t_1-s) f(s) ds \Big\|_X \end{aligned}$$

$$
\begin{aligned}
&\leq \Big\| \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} Q_\alpha(t_2-s) f(s)ds \Big\|_X \\
&\quad + \Big\| \int_0^{t_1} \Big((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\Big) Q_\alpha(t_2-s) f(s)ds \Big\|_X \\
&\quad + \Big\| \int_0^{t_1} (t_1-s)^{\alpha-1} \Big(Q_\alpha(t_2-s) - Q_\alpha(t_1-s)\Big) f(s)ds \Big\|_X \\
&=: I_1 + I_2 + I_3.
\end{aligned}
$$

By using analogous arguments as in Lemma 3.1, we have

$$
I_1 \leq \frac{\alpha M_A}{\Gamma(1+\alpha)} \Big[\frac{1-\beta}{\alpha-\beta}\Big]^{1-\beta} K (t_2-t_1)^{\alpha-\beta},
$$

$$
\begin{aligned}
I_2 &\leq \frac{\alpha M_A}{\Gamma(1+\alpha)} \Big(\int_0^{t_1} \big((t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}\big)^{1/(1-\beta)} ds \Big)^{1-\beta} K \\
&\leq \frac{\alpha M_A}{\Gamma(1+\alpha)} \Big(\int_0^{t_1} \big((t_1-s)^{\frac{\alpha-1}{1-\beta}} - (t_2-s)^{\frac{\alpha-1}{1-\beta}}\big) ds \Big)^{1-\beta} K \\
&= \frac{\alpha M_A}{\Gamma(1+\alpha)} \Big[\frac{1-\beta}{\alpha-\beta}\Big]^{1-\beta} \Big(t_1^{\frac{\alpha-\beta}{1-\beta}} - t_2^{\frac{\alpha-\beta}{1-\beta}} + (t_2-t_1)^{\frac{\alpha-\beta}{1-\beta}} \Big)^{1-\beta} K \\
&\leq \frac{2\alpha M_A}{\Gamma(1+\alpha)} \Big[\frac{1-\beta}{\alpha-\beta}\Big]^{1-\beta} \big(t_2-t_1\big)^{\alpha-\beta} K.
\end{aligned}
$$

For $t_1 = 0$, $0 < t_2 \leq b$, it is easy to see that $I_3 = 0$. For $t_1 > 0$ and $\epsilon > 0$ be small enough, we have

$$
\begin{aligned}
I_3 &\leq \Big\| \int_0^{t_1-\epsilon} (t_1-s)^{\alpha-1} \Big(Q_\alpha(t_2-s) - Q_\alpha(t_1-s)\Big) f(s)ds \Big\|_X \\
&\quad + \Big\| \int_{t_1-\epsilon}^{t_1} (t_1-s)^{\alpha-1} \Big(Q_\alpha(t_2-s) - Q_\alpha(t_1-s)\Big) f(s)ds \Big\|_X \\
&\leq \sup_{s\in[0,t_1-\epsilon]} \|Q_\alpha(t_2-s) - Q_\alpha(t_1-s)\| \Big[\frac{1-\beta}{\alpha-\beta}\Big]^{1-\beta} \big(t_1^{\frac{\alpha-\beta}{1-\beta}} - \epsilon^{\frac{\alpha-\beta}{1-\beta}} \big)^{1-\beta} K \\
&\quad + \frac{2\alpha M_A}{\Gamma(1+\alpha)} \Big[\frac{1-\beta}{\alpha-\beta}\Big]^{1-\beta} \epsilon^{\alpha-\beta} K.
\end{aligned}
$$

Combining the estimations for I_1, I_2, I_3, and letting $t_2 \to t_1$ and $\epsilon \to 0$ in I_3, we know that H is equicontinuous. For more details, please see [32].

(c) The set $\Pi(t) = \{H(f)(t) : f \in C\}$ is relatively compact in X. Clearly, $\Pi(0) = \{0\}$ is compact, and hence, it is only necessary to consider $t > 0$. For each $h \in (0,t)$, $t \in (0,b]$, $f \in C$, and $\delta > 0$ being arbitrary, we define

$$
\Pi_{h,\delta}(t) = \{H_{h,\delta}(f)(t) : f \in C\},
$$

where

$$
\begin{aligned}
H_{h,\delta}(f)(t) &= \alpha \int_0^{t-h} \int_\delta^\infty \theta (t-s)^{\alpha-1} \xi_\alpha(\theta) T((t-s)^\alpha \theta) f(s) d\theta ds \\
&= \alpha \int_0^{t-h} \int_\delta^\infty \theta (t-s)^{\alpha-1} \xi_\alpha(\theta) T(h^\alpha \delta) T\big((t-s)^\alpha \theta - h^\alpha \delta\big) f(s) d\theta ds \\
&= \alpha T(h^\alpha \delta) \int_0^{t-h} \int_\delta^\infty \theta (t-s)^{\alpha-1} \xi_\alpha(\theta) T\big((t-s)^\alpha \theta - h^\alpha \delta\big) f(s) d\theta ds.
\end{aligned}
$$

From the compactness of $T(h^\alpha\delta)$ $(h^\alpha\delta > 0)$, we obtain that the set $\Pi_{h,\delta}(t)$ is relatively compact in X for any $h \in (0,t)$ and $\delta > 0$. Moreover, we have

$$\begin{aligned}
&\|H(f)(t) - H_{h,\delta}(f)(t)\|_X \\
&= \alpha\Big\| \int_0^t \int_0^\delta \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta)f(s)d\theta ds \\
&\quad + \int_0^t \int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta)f(s)d\theta ds \\
&\quad - \int_0^{t-h} \int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta)f(s)d\theta ds\Big\|_X \\
&\le \alpha\Big\| \int_0^t \int_0^\delta \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta)f(s)d\theta ds\Big\|_X \\
&\quad + \alpha\Big\| \int_{t-h}^t \int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta)f(s)d\theta ds\Big\|_X \\
&\le M_A\alpha\Big(\int_0^t (t-s)^{\frac{\alpha-1}{1-\beta}} ds\Big)^{1-\beta}\|f\|_{L^{1/\beta}(J,X)}\int_0^\delta \theta\xi_\alpha(\theta)d\theta \\
&\quad + M_A\alpha\Big(\int_{t-h}^t (t-s)^{\frac{\alpha-1}{1-\beta}} ds\Big)^{1-\beta}\|f\|_{L^{1/\beta}(J,X)}\int_\delta^\infty \theta\xi_\alpha(\theta)d\theta \\
&\le M_A K\alpha\Big[\frac{1-\beta}{\alpha-\beta}\Big]^{1-\beta}\Big(b^{\alpha-\beta}\int_0^\delta \theta\xi_\alpha(\theta)d\theta + \frac{1}{\Gamma(1+\alpha)}h^{\alpha-\beta}\Big).
\end{aligned}$$

By (2.5), the last term of the preceding inequality tends to zero as $h \to 0$ and $\delta \to 0$. Therefore, there are relatively compact sets arbitrarily close to the set $\Pi(t)$, $t > 0$. Hence the set $\Pi(t)$, $t > 0$ is also relatively compact in X.

Since X_φ is a convex compact metrizable subset of ω-$L^{1/\beta}(J,X)$, it suffices to prove the sequential continuity of the map S. Now let $\{f_n\}_{n\ge1} \subseteq X_\varphi$ such that

$$f_n \to f \quad \text{in } \omega\text{-}L^{1/\beta}(J,X),\ f \in X_\varphi. \tag{3.15}$$

By the property of the operator H, we have $H(f_n) \to H(f)$ in ω-$C(J,X)$. Since $\{f_n\}_{n\ge1}$ is bounded, there is a subsequence $\{f_{n_k}\}_{k\ge1}$ of the sequence $\{f_n\}_{n\ge1}$ such that $H(f_{n_k}) \to z$ in $C(J,X)$ for some $z \in C(J,X)$. From the facts that

$$H(f_n) \to H(f) \quad \text{in } \omega\text{-}C(J,X), \quad \text{and} \quad H(f_{n_k}) \to z \text{ in } C(J,X),$$

we obtain that $z = H(f)$ and $H(f_n) \to H(f)$ in $C(J,X)$.

By the definitions of the operators S and H, we have that $S(f)(t) = P_\alpha(t)x_0 + H(f)(t)$. Then due to the arguments above, we have $S(f_n) \to S(f)$ in $C(J,X)$. This completes the proof of the lemma. $\square$

4. Existence results for control systems

In this section, we shall prove the existence of solutions for the control systems (1.1), (1.2) and (1.1), (1.3).

Let $\Lambda = S(X_\varphi)$. From Lemma 3.3, we have Λ is a compact subset of $C(J,X)$. It follows from (3.5) and (3.7) that $\mathcal{T}r_U \subseteq \mathcal{T}r_{\overline{\text{co}}U} \subseteq \Lambda$. Let $\overline{U} : C(J,X) \to 2^{L^{1/\beta}(J,Y)}$ be defined by

$$\overline{U}(x) = \{h : J \to Y \text{ measurable} : h(t) \in U(t,x(t)) \text{ a.e.}\},\ x \in C(J,X). \tag{4.1}$$

Theorem 4.1. *The set $\mathcal{R}_U$ is nonempty and the set $\mathcal{R}_{\overline{\text{co}}U}$ is a compact subset of the space $C(J,X) \times \omega\text{-}L^{1/\beta}(J,Y)$.*

Proof. By the hypotheses (H4)(1) and (H4)(2), we have that for any measurable function $x : J \to X$, the map $t \to U(t, x(t))$ is measurable and has closed values [11, Proposition 2.7.9]. Therefore it has measurable selectors [9]. So the operator $\overline{U}$ is well defined and its values are closed decomposable subsets of $L^{1/\beta}(J,Y)$. We claim that $x \to \overline{U}(x)$ is l.s.c. Let $x_* \in C(J,X)$, $h_* \in \overline{U}(x_*)$ and let $\{x_n\}_{n\geq 1} \subseteq C(J,X)$ be a sequence converging to x_*. It follows from [33, Lemma 3.2] that there is a sequence $h_n \in \overline{U}(x_n)$ such that

$$\|h_*(t) - h_n(t)\|_Y \leq d_Y(h_*(t), U(t, x_n(t))) + \frac{1}{n}, \quad \text{a.e. } t \in J. \tag{4.2}$$

Since the map $y \to U(t,y)$ is h-continuous a.e. $t \in J$ ((H4)(2)), then for a.e. $t \in J$, the map $y \to U(t,y)$ is l.s.c. [11, Proposition 1.2.66]. Hence by Proposition 1.2.26 in [11], the function $y \to d_Y(h_*(t), U(t,y))$ is u.s.c. for a.e. $t \in J$. It follows from (4.2) that, for a.e. $t \in J$,

$$\begin{aligned}\lim_{n\to\infty} \|h_*(t) - h_n(t)\|_Y &\leq \limsup_{n\to\infty} d_Y(h_*(t), U(t, x_n(t))) \\ &\leq d_Y(h_*(t), U(t, x_*(t))) = 0.\end{aligned}$$

This together with (3.5) implies that $h_n \to h_*$ in $L^{1/\beta}(J,Y)$. Therefore the map $x \to \overline{U}(x)$ is l.s.c. By [27, Proposition 2.2] (also see [11, Theorem 2.8.7]), there exists a continuous function $m : \Lambda \to L^{1/\beta}(J,Y)$ such that

$$m(x) \in \overline{U}(x), \quad \text{for all } x \in \Lambda. \tag{4.3}$$

Consider the map $\mathcal{P} : L^{1/\beta}(J,X) \to L^{1/\beta}(J,Y)$ defined by $\mathcal{P}(f) = m(S(f))$. Thanks to Lemma 3.3 and the continuity of m, the map $\mathcal{P}$ is continuous from $\omega\text{-}X_\varphi$ into $L^{1/\beta}(J,Y)$. Then by Lemma 3.2, we deduce that the map $f \to \mathcal{A}(S(f), \mathcal{P}(f))$ is continuous from $\omega\text{-}X_\varphi$ into $\omega\text{-}L^{1/\beta}(J,X)$. It follows from (3.5), (3.7) and (3.8) that $\mathcal{A}(S(f), \mathcal{P}(f)) \in X_\varphi$ for every $f \in X_\varphi$. Therefore, the map $f \to \mathcal{A}(S(f), \mathcal{P}(f))$ is continuous from $\omega\text{-}X_\varphi$ into $\omega\text{-}X_\varphi$. Since $\omega\text{-}X_\varphi$ is a convex metrizable compact set in $\omega\text{-}L^{1/\beta}(J,X)$, Schauder's fixed point theorem implies that this map has a fixed point $f_* \in X_\varphi$; i.e., $f_* = \mathcal{A}(S(f_*), \mathcal{P}(f_*))$. Let $u_* = \mathcal{P}(f_*)$ and $x_* = S(f_*)$, then we have $u_* = m(x_*)$ and $f_* = \mathcal{A}(x_*, u_*)$. That is to say we have

$$x_*(t) = S(f_*)(t) = P_\alpha(t)x_0 + \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s)\big(g(s)u_*(s) + h(s, x_*(s))\big)ds,$$
$$u_*(t) \in U(t, x_*(t)) \text{ a.e. } t \in J.$$

These imply that $(x_*(\cdot), u_*(\cdot))$ is a solution of the control system (1.1), (1.2). Hence $\mathcal{R}_U$ is nonempty.

It is easy to see that $\mathcal{R}_{\overline{\text{co}}U} \subseteq \Lambda \times Y_\varphi$. Since Λ is compact in $C(J,X)$ and Y_φ is metrizable convex compact in $\omega\text{-}L^{1/\beta}(J,Y)$, we have that $\mathcal{R}_{\overline{\text{co}}U}$ is relatively compact in $C(J,X) \times \omega\text{-}L^{1/\beta}(J,Y)$. Hence to complete the proof of this theorem, it is sufficient to prove that $\mathcal{R}_{\overline{\text{co}}U}$ is sequentially closed in $C(J,X) \times \omega\text{-}L^{1/\beta}(J,Y)$.

Let $\{(x_n(\cdot), u_n(\cdot))\}_{n\geq 1} \subseteq \mathcal{R}_{\overline{\text{co}}U}$ be a sequence converging to $(x(\cdot), u(\cdot))$ in the space $C(J,X) \times \omega\text{-}L^{1/\beta}(J,Y)$. Denote

$$\begin{aligned} f_n(t) &= g(t)u_n(t) + h(t, x_n(t)), \\ f(t) &= g(t)u(t) + h(t, x(t)). \end{aligned}$$

According to Lemma 3.2, $f_n \to f$ in ω-$L^{1/\beta}(J,X)$. Since $f_n \in X_\varphi$ and $x_n = S(f_n)$, $n \geq 1$, Lemma 3.3 implies that
$$x = S(f).$$
Hence, to prove that $(x(\cdot), u(\cdot)) \in \mathcal{R}_{\overline{\mathrm{co}}U}$, we only need to verify that $u(t) \in \overline{\mathrm{co}}\, U(t, x(t))$ a.e. $t \in J$.

Since $u_n \to u$ in ω-$L^{1/\beta}(J,Y)$, by Mazur's theorem, we have
$$u(t) \in \cap_{n=1}^{\infty} \overline{\mathrm{co}} \left(\cup_{k=n}^{\infty} u_k(t) \right), \quad \text{for a.e. } t \in J. \tag{4.4}$$
By (H4)(2) and the fact that $h(\overline{\mathrm{co}}\, A, \overline{\mathrm{co}}\, B) \leq h(A,B)$ for sets A, B, the map $x \to \overline{\mathrm{co}}\, U(t,x)$ is h-continuous. Then from Proposition 1.2.86 in [11], the map $x \to \overline{\mathrm{co}}\, U(t,x)$ has property Q. Therefore we have
$$\cap_{n=1}^{\infty} \overline{\mathrm{co}} \Big(\cup_{k=n}^{\infty} \overline{\mathrm{co}}\, U(t, x_k(t)) \Big) \subseteq \overline{\mathrm{co}}\, U(t, x(t)), \quad \text{for a.e. } t \in J. \tag{4.5}$$
By (4.4) and (4.5), we obtain that $u(t) \in \overline{\mathrm{co}}\, U(t, x(t))$ a.e. $t \in J$. This means that $\mathcal{R}_{\overline{\mathrm{co}}U}$ is compact in $C(J,X) \times \omega$-$L^{1/\beta}(J,Y)$. The proof is complete. □

5. Main results

Now we are in a position to obtain our main results.

Theorem 5.1. *For any $(x_*(\cdot), u_*(\cdot)) \in \mathcal{R}_{\overline{\mathrm{co}}U}$, we have that there exists a sequence $(x_n(\cdot), u_n(\cdot)) \in \mathcal{R}_U$, $n \geq 1$, such that*
$$x_n \to x_* \quad \text{in } C(J,X), \tag{5.1}$$
$$u_n \to u_* \quad \text{in } L^{1/\beta}_{\omega}(J,Y) \quad \text{and} \quad \omega\text{-}L^{1/\beta}(J,Y). \tag{5.2}$$
Moreover, we have
$$\overline{\mathcal{T}r_U} = \mathcal{T}r_{\overline{\mathrm{co}}U}, \tag{5.3}$$
where the bar stands for the closure in the space $C(J,X)$.

Proof. Let any $(x_*(\cdot), u_*(\cdot)) \in \mathcal{R}_{\overline{\mathrm{co}}U}$, then we have $u_*(t) \in \overline{\mathrm{co}}\, U(t, x_*(t))$ a.e. $t \in J$. It follows from (H4)(1), (H4)(2) and (3.5) that the map $t \to U(t, x_*(t))$ is measurable and integrally bounded. Hence by using [28, Theorem 2.2], we have that, for any $n \geq 1$, there exists a measurable selection $v_n(t)$ of the multivalued map $t \to U(t, x_*(t))$ such that
$$\sup_{0 \leq t_1 \leq t_2 \leq b} \Big\| \int_{t_1}^{t_2} (u_*(s) - v_n(s)) ds \Big\|_Y \leq \frac{1}{n}. \tag{5.4}$$
For each fixed $n \geq 1$, by (H4)(2), we have that, for any $x \in X$ and a.e. $t \in J$, there exists a $v \in U(t,x)$ such that
$$\|v_n(t) - v\|_Y < k_1(t) \|x_*(t) - x\|_X + \frac{1}{n}. \tag{5.5}$$
Let a map $H_n : J \times X \to 2^Y$ be defined by
$$H_n(t,x) = \{v \in Y : v \text{ satisfies inequality (5.5)}\}. \tag{5.6}$$
It follows from (5.5) that $H_n(t,x)$ is well defined for a.e. on J and all $x \in X$, and its values are open sets. Using [25, Corollary 2.1] (since we can assume without loss of generality that $U(t,x)$ is $\Sigma \otimes \mathcal{B}_X$ measurable, see [11, Proposition 2.7.9]), we obtain that, for any $\epsilon > 0$, there is a compact set $J_\epsilon \subseteq J$ with $\mu(J \backslash J_\epsilon) \leq \epsilon$, such that the restriction of $U(t,x)$ to $J_\epsilon \times X$ is l.s.c and the restrictions of $v_n(t)$ and $k_1(t)$ to J_ϵ are continuous. So (5.5) and (5.6) imply that the graph of the restriction of

$H_n(t,x)$ to $J_\epsilon \times X$ is an open set in $J_\epsilon \times X \times Y$. Let a map $H : J \times X \to 2^Y$ be defined by

$$H(t,x) = H_n(t,x) \cap U(t,x). \tag{5.7}$$

It is obvious that, for a.e. $t \in J$ and all $x \in X$, $H(t,x) \neq \emptyset$. Due to the arguments above and Proposition 1.2.47 in [11], we know that the restriction of $H(t,x)$ to $J_\epsilon \times X$ is l.s.c. and so does $\overline{H}(t,x) = \overline{H(t,x)}$, here the bar stands for the closure of a set in Y.

Now we consider the system (1.1) with the constraint on the control

$$u(t) \in \overline{H}(t,x(t)) \quad \text{a.e. on } J. \tag{5.8}$$

Since $\overline{H}(t,x) \subseteq U(t,x)$, the a priori estimate Lemma 3.1 also holds in this situation. Repeating the proof of Theorem 4.1, we obtain that there is a solution $(x_n(\cdot), u_n(\cdot))$ of the control system (1.1), (5.8). The definition of $\overline{H}$ implies that $(x_n(\cdot), u_n(\cdot)) \in \mathcal{R}_U$ and

$$\|v_n(t) - u_n(t)\|_Y \le k_1(t)\|x_*(t) - x_n(t)\|_X + \frac{1}{n}. \tag{5.9}$$

Since $(x_n(\cdot), u_n(\cdot)) \in \mathcal{R}_U$, $n \ge 1$, and $(x_*(\cdot), u_*(\cdot)) \in \mathcal{R}_{\overline{\text{co}}U}$, we have

$$x_*(t) = P_\alpha(t)x_0 + \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s)\big(g(s)u_*(s) + h(s, x_*(s))\big)ds \tag{5.10}$$

and

$$x_n(t) = P_\alpha(t)x_0 + \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s)\big(g(s)u_n(s) + h(s, x_n(s))\big)ds. \tag{5.11}$$

Theorem 4.1 and $\{(x_n(\cdot), u_n(\cdot))\}_{n\ge1} \subseteq \mathcal{R}_U \subseteq \mathcal{R}_{\overline{\text{co}}U}$ imply that we can assume, possibly up to a subsequence, that the sequence $(x_n(\cdot), u_n(\cdot)) \to (\overline{x}(\cdot), \overline{u}(\cdot)) \in \mathcal{R}_{\overline{\text{co}}U}$ in $C(J,X) \times \omega\text{-}L^{1/\beta}(J,Y)$. Subtracting (5.11) from (5.10), and using (H3)(2), (H2)(2) and (5.9), we have

$$\begin{aligned}
&\|x_*(t) - x_n(t)\|_X \\
&= \Big\| \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s)\big(g(s)u_*(s) - g(s)u_n(s)\big)ds \\
&\quad + \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s)\big(h(s,x_*(s)) - h(s,x_n(s))\big)ds \Big\|_X \\
&\le \Big\| \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s) g(s)\big(u_*(s) - v_n(s)\big)ds \Big\|_X \\
&\quad + \Big\| \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s) g(s)\big(v_n(s) - u_n(s)\big)ds \Big\|_X \\
&\quad + \Big\| \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s)\big(h(s,x_*(s)) - h(s,x_n(s))\big)ds \Big\|_X \\
&\le \Big\| \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s) g(s)\big(u_*(s) - v_n(s)\big)ds \Big\|_X \\
&\quad + \frac{\alpha M_A d}{\Gamma(1+\alpha)} \int_0^t (t-s)^{\alpha-1} \Big(\frac{1}{n} + k_1(s)\|x_*(s) - x_n(s)\|_X\Big) ds \\
&\quad + \frac{\alpha M_A \|l\|_{L^\infty}}{\Gamma(1+\alpha)} \int_0^t (t-s)^{\alpha-1} \|x_*(s) - x_n(s)\|_X ds
\end{aligned}$$

$$
\begin{aligned}
&\le \Big\| \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s) g(s)\big(u_*(s)-v_n(s)\big)ds\Big\|_X + \frac{\alpha M_A d b^\alpha}{n\alpha\Gamma(1+\alpha)} \\
&\quad + \frac{\alpha M_A(d\|k_1\|_{L^\infty}+\|l\|_{L^\infty})}{\Gamma(1+\alpha)} \int_0^t (t-s)^{\alpha-1}\|x_*(s)-x_n(s)\|_X ds. \qquad (5.12)
\end{aligned}
$$

Due to (5.4), one has $v_n \to u_*$ in ω-$L^{1/\beta}(J,Y)$ by Lemma 2.4. Then it is easy to show that $g(t)v_n(t) \to g(t)u_*(t)$ in ω-$L^{1/\beta}(J,X)$. By the property of the operator H defined in the proof of Lemma 3.3, we have that, for any $t \in J$,

$$
\Big\| \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s) g(s)\big(u_*(s)-v_n(s)\big)ds\Big\|_X \to 0, \quad \text{as } n \to \infty.
$$

Since $\|x_*(t)\|_X \le L$, $\|x_n(t)\|_X \le L$ for any n, $t \in J$ and $x_n \to \overline{x}$ in $C(J,X)$, letting $n \to \infty$ in (5.12), we obtain

$$
\|x_*(t)-\overline{x}(t)\|_X \le \frac{\alpha M_A(d\|k_1\|_{L^\infty}+\|l\|_{L^\infty})}{\Gamma(1+\alpha)} \int_0^t (t-s)^{\alpha-1}\|x_*(s)-\overline{x}(s)\|_X ds.
$$

Then by [5, Theorem 3.1], we obtain $x_* = \overline{x}$; i.e., we have $x_n \to x_*$ in $C(J,X)$. Hence from (5.9), we have $(v_n - u_n) \to 0$ in $L^{1/\beta}(J,Y)$. Therefore, $u_n = u_n - v_n + v_n \to u_*$ in ω-$L^{1/\beta}(J,Y)$ and $L^{1/\beta}_\omega(J,Y)$, i.e., (5.1) and (5.2) hold.

Since it is clear that $\mathcal{T}r_U \subseteq \mathcal{T}r_{\overline{\text{co}}U}$ and $\mathcal{T}r_{\overline{\text{co}}U}$ is compact in $C(J,X)$ by Theorem 4.1, then from the proof of the first part of this theorem, we have

$$
\overline{\mathcal{T}r_U} = \mathcal{T}r_{\overline{\text{co}}U},
$$

where the bar stands for the closure in $C(J,X)$. This completes the proof. $\square$

6. An example

In this section, we present an example of control systems governed by fractional partial differential equations. In particular, to illustrate the abstract results of this paper, we provide the following example which do not aim at generality but indicate how our theorems can be applied to concrete problems. Since the hypotheses on the operator g and the function h are very common, we mainly pay attention to the operator A and the multivalued map U here.

Let $J = [0,1]$ and $\Omega = [0,\pi]$. Put $X = Y = L^2(\Omega)$. We consider the fractional control system

$$
\begin{gathered}
{}^C D_t^\alpha x(t,z) = \partial_z^2 x(t,z) + \bar{h}(t,z,x(t,z)) + \bar{b}(t)\bar{u}(t,z), \quad t \in J,\ z \in \Omega, \\
x(t,0) = x(t,\pi) = 0, \\
x(0,z) = x_0(z), \\
\bar{u}(t,z) \in \bar{U}(t,z,x(t,z)), \quad \text{a.e. in } J \times \Omega,
\end{gathered} \qquad (6.1)
$$

where ${}^C D_t^\alpha$ is the Caputo fractional derivative of order $0 < \alpha < 1$, $\bar{h}$, $\bar{b}$ are suitable functions, $\bar{U}$ is a multivalued function which will be given below.

Define the operator A by $A\omega = \omega''$ with $D(A)$ consisting of all $\omega \in X$ with ω, ω' are absolutely continuous, $\omega'' \in X$ and $\omega(0) = \omega(\pi) = 0$. Then

$$
A\omega = -\sum_{n=1}^\infty n^2 \langle \omega, e_n\rangle e_n, \quad \omega \in D(A),
$$

where $e_n(z) = (2/\pi)^{\frac{1}{2}} \sin(nz)$, $z \in \Omega$, $n = 1,2,3,\cdots$, is the orthogonal set of eigenfunctions of A and $\langle \cdot,\cdot \rangle$ denotes the L^2 inner product. It is clear that A is the infinitesimal generator of a strongly continuous semigroup $\{T(t),\, t \geq 0\}$ in X and $T(t)$, $t > 0$ is also compact, which is given by

$$T(t)\omega = \sum_{n=1}^{\infty} e^{-n^2 t}\langle \omega, e_n\rangle e_n, \quad \omega \in X.$$

Hence the assumption (H1) is satisfied.

(H5) $\bar{U} : J \times \Omega \times \mathbb{R} \to \mathbb{R}$ is a multivalued function with closed values satisfying the following conditions:
 (1) the map $(t, z) \to \bar{U}(t, z, x)$ is measurable;
 (2) $h(\bar{U}(t, z, x_1), \bar{U}(t, z, x_2)) \leq \bar{k}_1(t)|x_1 - x_2|$ a.e. in $t \in J \times \Omega$ with $\bar{k}_1$ in $L_+^\infty(J)$;
 (3) $|\bar{U}(t, z, x)| \leq \bar{a}_2(t, z)+\bar{c}_2(t, z)|x|$ a.e. in $J\times\Omega$ with $\bar{a}_2 \in L^{1/\beta}(J, L_+^2(\Omega))$, $0 < \beta < \alpha$ and $\bar{c}_2 \in L_+^\infty(J \times \Omega)$.

Put $x(t) = x(t, \cdot)$; that is $x(t)(z) = x(t, z)$, $t \in J$, $z \in \Omega$. Define a multivalued map $U : J \times X \to 2^Y$ by

$$U(t, x) = \{u \in Y : u(z) \in \bar{U}(t, z, x(z)) \text{ a.e. in } \Omega\}, \quad x \in X.$$

Suppose assumption (H5) holds, then it is easy to verify that (H4)(1) and (H4)(2) are satisfied. Moveover, we have

$$\sup\{\|u\|_Y : u \in U(t, x)\} \leq |\bar{a}_2(t)|_2 + \|\bar{c}_2\|_{L^\infty}\|x\|_X,$$

where $|\bar{a}_2(\cdot)|_2 \in L^{1/\beta}(J, \mathbb{R}^+)$, $|\bar{a}_2(t)|_2 = (\int_\Omega a^2(t, z)dz)^{\frac{1}{2}}$. This means that (H4)(3) holds.

Let $h(t, x)(z) = \bar{h}(t, z, x(t)(z))$ and $g(t) = \bar{b}(t)$. With A and U defined above, the fractional control system (6.1) can be rewritten to our abstract form (1.1), (1.2). Hence the abstract results obtained in the previous sections can be applied to the control system (6.1).

Conclusions. Existence results and relaxation property of a class of fractional feedback control systems in Banach spaces have been investigated. With some auxiliary results provided in section 3, we obtained the existence results of the control systems by Schauder's fixed point theorem. To get the relaxation property, we used some tools from multivalued analysis.

Our future work will be devoted to study the following fractional control problems: systems with Riemann-Liouville fractional derivative, time optimal control, optimal control of Lagrange type and relaxed control systems by using other convexification techniques.

Acknowledgments. The authors would like to thank the anonymous reviewers and the editor for their valuable comments and suggestions. This work was supported in part by the NNSFC (grants 10671211, 11126170) and Hunan Provincial Natural Science Foundation (grant 11JJ4006).

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