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SOLVABILITY OF PERIODIC BOUNDARY-VALUE PROBLEMS FOR SECOND-ORDER NONLINEAR DIFFERENTIAL EQUATION INVOLVING FRACTIONAL DERIVATIVES

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Abstract. This article concerns the existence of solutions to periodic boundary-value problems for second-order nonlinear differential equation involving fractional derivatives. Under certain linear growth condition of the nonlinearity, we obtain solutions, by using coincidence degree theory. An example illustrates our results.

1. Introduction

Fractional calculus is a generalization of ordinary differentiation and integration on an arbitrary order that can be noninteger. This subject, as old as the problem of ordinary differential calculus, can go back to the times when Leibniz and Newton invented differential calculus. As is known to all, the problem for fractional derivative was originally raised by Leibniz in a letter, dated September 30, 1695.

A fractional derivative arises from many physical processes, such as a non-Markovian diffusion process with memory [22], charge transport in amorphous semiconductors [26], propagations of mechanical waves in viscoelastic media [19], etc. Moreover, phenomena in electromagnetics, acoustics, viscoelasticity, electrochemistry and material science are also described by differential equations of fractional order [9, 11, 12, 20, 23]. For instance, to describe the horizontal vibration of the rigid thin plate with massless spring immersing vertically in ideal fluid, Torvik and Bagley [28] introduced the well known fractional differential equation

$$Ax''(t) + BD_t^{3/2}x(t) + Cx(t) = f(t).$$

Recently, fractional differential equations have been of great interest due to the intensive development of the theory of fractional calculus itself and its applications. For example, for fractional initial value problems, the existence and multiplicity of solutions (or positive solutions) were discussed in [2, 8, 16, 17]. On the other hand, for fractional boundary value problems, Agarwal et al. [1] considered a two-point boundary value problem at nonresonance, and Bai [3] considered a m-point boundary value problem at resonance. Moreover, for fractional periodic boundary value problems, Belmekki et al [5] discussed the existence of periodic solutions, and

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Kaslik et al [14] discussed the no-existence of periodic solutions. For more articles on fractional boundary value problems, see [4, 6, 7, 10, 13, 18, 27] and the references therein.

In the present article, motivated by the works mentioned previously, we investigate the existence of solutions for the periodic boundary-value problem (PBVP for short)

$$\begin{gathered} x''(t)=f(t,x(t),D_{0+}^{\alpha}x(t)), \quad t\in[0,1],\\ x(0)=x(1), \quad D_{0+}^{\alpha}x(0)=D_{0+}^{\alpha}x(1), \end{gathered} \tag{1.1}$$

where $0<\alpha<2$ is a real number, D_{0+}^{α} is a Caputo fractional derivative, and $f:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ is continuous.

The rest of this article is organized as follows. Section 2 contains some necessary notation, definitions and lemmas. In Section 3, basing on the coincidence degree theory of Mawhin [21], we establish a theorem on existence of solutions for PBVP (1.1) under linear growth restriction of f. Finally, in Section 4, an example is given to illustrate the main result.

2. Preliminaries

For the convenience of the reader, we present here some necessary basic knowledge and definitions about fractional calculus theory, which can be found, for instance, in [24, 25].

Definition 2.1. The Riemann-Liouville fractional integral operator of order $\alpha>0$ of a function $u:(0,+\infty)\to\mathbb{R}$ is given by

$$I_{0+}^{\alpha}u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}u(s)\,ds,$$

provided that the right side integral is pointwise defined on $(0,+\infty)$.

Definition 2.2. The Caputo fractional derivative of order $\alpha>0$ of a continuous function $u:(0,+\infty)\to\mathbb{R}$ is given by

$$D_{0+}^{\alpha}u(t)=I_{0+}^{n-\alpha}\frac{d^nu(t)}{dt^n}=\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-s)^{n-\alpha-1}u^{(n)}(s)\,ds,$$

where n is the smallest integer greater than or equal to α, provided that the right side integral is pointwise defined on $(0,+\infty)$.

Lemma 2.3 ([15])**.** *Let $\alpha>0$. Assume that $u,D_{0+}^{\alpha}u\in L(0,1)$. Then the following equality holds*

$$I_{0+}^{\alpha}D_{0+}^{\alpha}u(t)=u(t)+c_0+c_1t+\dots+c_{n-1}t^{n-1},$$

where $c_i\in\mathbb{R}$, $i=0,1,\dots,n-1$, here n is the smallest integer greater than or equal to α.

Now, we briefly recall some notation and an abstract existence result, which can be found in [21].

Let X,Y be real Banach spaces, $L:\operatorname{dom}L\subset X\to Y$ be a Fredholm operator with index zero, and $P:X\to X$, $Q:Y\to Y$ be projectors such that

$$\operatorname{Im}P=\ker L,\quad \ker Q=\operatorname{Im}L,\quad X=\ker L\oplus\ker P,\quad Y=\operatorname{Im}L\oplus\operatorname{Im}Q.$$

It follows that

$$L|_{\operatorname{dom}L\cap\ker P}:\operatorname{dom}L\cap\ker P\to\operatorname{Im}L$$

is invertible. We denote the inverse by K_P.

If Ω is an open bounded subset of X such that $\operatorname{dom} L \cap \overline{\Omega} \neq \emptyset$, then the map $N : X \to Y$ will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N : \overline{\Omega} \to X$ is compact.

Lemma 2.4 ([21]). *Let $L : \operatorname{dom} L \subset X \to Y$ be a Fredholm operator of index zero and $N : X \to Y$ be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied*

1. $Lx \neq \lambda Nx$ *for every* $(x, \lambda) \in [(\operatorname{dom} L \setminus \ker L) \cap \partial\Omega] \times (0, 1)$*;*
2. $Nx \notin \operatorname{Im} L$ *for every* $x \in \ker L \cap \partial\Omega$*;*
3. $\deg(QN|_{\ker L}, \Omega \cap \ker L, 0) \neq 0$*, where* $Q : Y \to Y$ *is a projection such that* $\operatorname{Im} L = \ker Q$*.*

Then the equation $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$*.*

In this article, we take $Y = C[0, 1]$ with the norm $\|y\|_\infty = \max_{t\in[0,1]} |y(t)|$, and $X = \{x : x, D^\alpha_{0+}x \in Y\}$ with the norm $\|x\|_X = \max\{\|x\|_\infty, \|D^\alpha_{0+}x\|_\infty\}$. By means of the linear functional analysis theory, we can prove that X is a Banach space.

Define the operator $L : \operatorname{dom} L \subset X \to Y$ by

$$Lx = x'', \tag{2.1}$$

where

$$\operatorname{dom} L = \{x \in X : x'' \in Y,\ x(0) = x(1), D^\alpha_{0+}x(0) = D^\alpha_{0+}x(1)\}.$$

Let $N : X \to Y$ be the Nemytskii operator

$$Nx(t) = f(t, x(t), D^\alpha_{0+}x(t)), \quad \forall t \in [0, 1]. \tag{2.2}$$

Then PBVP (1.1) is equivalent to the operator equation

$$Lx = Nx, \quad x \in \operatorname{dom} L.$$

From the definition of L, we can obtain that

$$\ker L = \{x \in X : x(t) = c,\ \forall t \in [0, 1],\ c \in \mathbb{R}\}, \tag{2.3}$$

$$\operatorname{Im} L = \Big\{y \in Y : \int_0^1 (1-s)^{1-\alpha} y(s)\,ds = 0\Big\}. \tag{2.4}$$

Let us define the linear continuous projector operators $P : X \to X$ and $Q : Y \to Y$ by

$$Px(t) = x(0), \quad \forall t \in [0, 1],$$

$$Qy(t) = (2-\alpha) \int_0^1 (1-s)^{1-\alpha} y(s)\,ds, \quad \forall t \in [0, 1].$$

Obviously

$$\operatorname{Im} P = \ker L, \quad \ker Q = \operatorname{Im} L, \quad X = \ker L \oplus \ker P,$$

and the operator $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$ can be written as

$$K_P y(t) = \int_0^t (t-s)y(s)\,ds - \int_0^1 (1-s)y(s)\,ds \cdot t, \quad \forall t \in [0, 1].$$

3. Existence result

In this section, a theorem on existence of solutions for PBVP (1.1) is given.

Theorem 3.1. *Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous. Assume that*

- (H1) *there exist nonnegative functions $a, b, c \in Y$ such that*
$$|f(t,u,v)| \leq a(t) + b(t)|u| + c(t)|v|, \quad \forall t \in [0,1], (u,v) \in \mathbb{R}^2;$$
- (H2) *there exists a constant $B > 0$ such that either*
$$uf(t,u,v) > 0, \quad \forall t \in [0,1],\ v \in \mathbb{R},\ |u| > B$$
or
$$uf(t,u,v) < 0, \quad \forall t \in [0,1],\ v \in \mathbb{R},\ |u| > B.$$

Then PBVP (1.1) *has at least one solution, provided that*
$$\frac{1}{\Gamma(3-\alpha)}\Big(\frac{2\|b\|_\infty}{\Gamma(\alpha+1)} + \|c\|_\infty\Big) < 1. \tag{3.1}$$

Next, we introduce some lemmas that are useful in what follows.

Lemma 3.2. *Let L be defined by* (2.1)*, then L is a Fredholm operator of index zero.*

Proof. For any $y \in Y$, we have
$$Q^2y(t) = Qy(t)(2-\alpha)\int_0^1 (1-s)^{1-\alpha}\,ds = Qy(t). \tag{3.2}$$
Let $y_1 = y - Qy$, then from (3.2) we obtain
$$\begin{aligned}\int_0^1 (1-s)^{1-\alpha}y_1(s)\,ds &= \int_0^1 (1-s)^{1-\alpha}y(s)\,ds - \int_0^1 (1-s)^{1-\alpha}Qy(s)\,ds\\ &= \frac{1}{2-\alpha}Qy(t) - \frac{1}{2-\alpha}Q^2y(t) = 0,\end{aligned}$$
which implies $y_1 \in \operatorname{Im} L$. Hence $Y = \operatorname{Im} L + \operatorname{Im} Q$. Since $\operatorname{Im} L \cap \operatorname{Im} Q = \{0\}$, we have
$$Y = \operatorname{Im} L \oplus \operatorname{Im} Q.$$
Thus,
$$\dim\ker L = \dim \operatorname{Im} Q = \operatorname{codim} \operatorname{Im} L = 1.$$
This means that L is a Fredholm operator of index zero. The proof is complete. $\square$

Lemma 3.3. *Let L be defined by* (2.1) *and N be defined by* (2.2)*. Assume $\Omega \subset X$ is an open bounded subset such that* $\operatorname{dom} L \cap \overline{\Omega} \neq \emptyset$*, then N is L-compact on $\overline{\Omega}$.*

Proof. By the continuity of f, we can show that $QN(\overline{\Omega})$ and $K_P(I-Q)N(\overline{\Omega})$ are bounded. Moreover, there exists a constant $T > 0$ such that $|(I-Q)Nx| \leq T$ for all $x \in \overline{\Omega}$, $t \in [0,1]$. Thus, in view of the Arzelà-Ascoli theorem, we need only prove that $K_P(I-Q)N(\overline{\Omega}) \subset X$ is equicontinuous.

For $0 \leq t_1 < t_2 \leq 1, x \in \overline{\Omega}$, we have
$$\begin{aligned}&|K_P(I-Q)Nx(t_2) - K_P(I-Q)Nx(t_1)|\\ &= \Big|\int_0^{t_2}(t_2-s)(I-Q)Nx(s)\,ds - \int_0^{t_1}(t_1-s)(I-Q)Nx(s)\,ds\end{aligned}$$

$$\begin{aligned}
&- \int_0^1 (1-s)(I-Q)Nx(s)\,ds\cdot(t_2-t_1)\Big| \\
&\le T\Big[\int_0^{t_1}(t_2-t_1)\,ds + \int_{t_1}^{t_2}(t_2-s)\,ds + \int_0^1 (1-s)\,ds\cdot(t_2-t_1)\Big] \\
&= \frac{T}{2}(t_2^2 - t_1^2 + t_2 - t_1).
\end{aligned}$$

On the other hand, from the definition of $D_{0^+}^\alpha$, one has

$$D_{0^+}^\alpha K_P y(t) = I_{0^+}^{2-\alpha}\frac{d^2}{dt^2}K_P y(t) = \frac{1}{\Gamma(2-\alpha)}\int_0^t (t-s)^{1-\alpha}y(s)\,ds.$$

Thus, when $0<\alpha\le 1$, we have

$$\begin{aligned}
&|D_{0^+}^\alpha K_P(I-Q)Nx(t_2) - D_{0^+}^\alpha K_P(I-Q)Nx(t_1)| \\
&= \frac{1}{\Gamma(2-\alpha)}\Big|\int_0^{t_2}(t_2-s)^{1-\alpha}(I-Q)Nx(s)\,ds \\
&\quad - \int_0^{t_1}(t_1-s)^{1-\alpha}(I-Q)Nx(s)\,ds\Big| \\
&\le \frac{T}{\Gamma(2-\alpha)}\Big\{\int_0^{t_1}[(t_2-s)^{1-\alpha}-(t_1-s)^{1-\alpha}]\,ds + \int_{t_1}^{t_2}(t_2-s)^{1-\alpha}\,ds\Big\} \\
&= \frac{T}{\Gamma(3-\alpha)}(t_2^{2-\alpha}-t_1^{2-\alpha}).
\end{aligned}$$

When $1\le\alpha<2$, we have

$$\begin{aligned}
&|D_{0^+}^\alpha K_P(I-Q)Nx(t_2) - D_{0^+}^\alpha K_P(I-Q)Nx(t_1)| \\
&\le \frac{T}{\Gamma(2-\alpha)}\Big\{\int_0^{t_1}[(t_1-s)^{1-\alpha}-(t_2-s)^{1-\alpha}]\,ds + \int_{t_1}^{t_2}(t_2-s)^{1-\alpha}\,ds\Big\} \\
&= \frac{T}{\Gamma(3-\alpha)}[t_1^{2-\alpha}-t_2^{2-\alpha}+2(t_2-t_1)^{2-\alpha}] \\
&\le \frac{T}{\Gamma(3-\alpha)}[t_2^{2-\alpha}-t_1^{2-\alpha}+2(t_2-t_1)^{2-\alpha}].
\end{aligned}$$

Since t^2 and $t^{2-\alpha}$ are uniformly continuous on $[0,1]$, we obtain that $K_P(I-Q)N(\overline{\Omega})\subset Y$ and $D_{0^+}^\alpha K_P(I-Q)N(\overline{\Omega})\subset Y$ are equicontinuous. Hence, $K_P(I-Q)N:\overline{\Omega}\to X$ is compact. The proof is complete. □

Lemma 3.4. *Suppose* (H1), (H2) *hold, then the set*

$$\Omega_1 = \{x\in \operatorname{dom} L\setminus\ker L : Lx=\lambda Nx,\ \lambda\in(0,1)\}$$

is bounded.

Proof. Take $x\in\Omega_1$, then $Lx=\lambda Nx$ and $Nx\in\operatorname{Im}L$. By (2.4), we have

$$\int_0^1 (1-s)^{1-\alpha}f(s,x(s),D_{0^+}^\alpha x(s))\,ds = 0.$$

Then, by the mean value theorem for integrals, there exists a constant $\xi\in(0,1)$ such that $f(\xi,x(\xi),D_{0^+}^\alpha x(\xi))=0$. So, from (H2), we get $|x(\xi)|\le B$. By Lemma 2.3, one has

$$x(t) = x(\xi) - I_{0^+}^\alpha D_{0^+}^\alpha x(t)|_{t=\xi} + I_{0^+}^\alpha D_{0^+}^\alpha x(t)$$

$$
\begin{aligned}
&= x(\xi) - \frac{1}{\Gamma(\alpha)}\int_0^\xi (\xi - s)^{\alpha-1} D_{0^+}^\alpha x(s)\,ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t - s)^{\alpha-1} D_{0^+}^\alpha x(s)\,ds.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
|x(t)| &\le |x(\xi)| + \frac{2}{\Gamma(\alpha)}\int_0^1 (1 - s)^{\alpha-1} |D_{0^+}^\alpha x(s)|\,ds \\
&\le B + \frac{2}{\Gamma(\alpha)}\|D_{0^+}^\alpha x\|_\infty \frac{1}{\alpha} \\
&= B + \frac{2}{\Gamma(\alpha+1)}\|D_{0^+}^\alpha x\|_\infty, \quad \forall t \in [0,1].
\end{aligned}
$$

That is,

$$
\|x\|_\infty \le B + \frac{2}{\Gamma(\alpha+1)}\|D_{0^+}^\alpha x\|_\infty. \tag{3.3}
$$

By $Lx = \lambda Nx$, we obtain

$$
\begin{aligned}
D_{0^+}^\alpha x(t) &= I_{0^+}^{2-\alpha}\frac{d^2x(t)}{dt^2} = \lambda I_{0^+}^{2-\alpha} Nx(t) \\
&= \frac{\lambda}{\Gamma(2-\alpha)}\int_0^t (t - s)^{1-\alpha} f(s, x(s), D_{0^+}^\alpha x(s))\,ds.
\end{aligned}
$$

So, from (H1), we have

$$
\begin{aligned}
&|D_{0^+}^\alpha x(t)| \\
&\le \frac{1}{\Gamma(2-\alpha)}\int_0^t (t - s)^{1-\alpha} |f(s, x(s), D_{0^+}^\alpha x(s))|\,ds \\
&\le \frac{1}{\Gamma(2-\alpha)}\int_0^t (t - s)^{1-\alpha} (a(s) + b(s)|x(s)| + c(s)|D_{0^+}^\alpha x(s)|)\,ds \\
&\le \frac{1}{\Gamma(2-\alpha)}(\|a\|_\infty + \|b\|_\infty\|x\|_\infty + \|c\|_\infty\|D_{0^+}^\alpha x\|_\infty)\frac{1}{2-\alpha}t^{2-\alpha} \\
&\le \frac{1}{\Gamma(3-\alpha)}(\|a\|_\infty + \|b\|_\infty\|x\|_\infty + \|c\|_\infty\|D_{0^+}^\alpha x\|_\infty), \quad \forall t \in [0,1],
\end{aligned}
$$

which, together with (3.3), yields

$$
\|D_{0^+}^\alpha x\|_\infty \le \frac{1}{\Gamma(3-\alpha)}\Big[\|a\|_\infty + B\|b\|_\infty + \Big(\frac{2\|b\|_\infty}{\Gamma(\alpha+1)} + \|c\|_\infty\Big)\|D_{0^+}^\alpha x\|_\infty\Big]. \tag{3.4}
$$

In view of (3.1), from (3.4), we can see that there exists a constant $M_1 > 0$ such that

$$
\|D_{0^+}^\alpha x\|_\infty \le M_1. \tag{3.5}
$$

Thus, from (3.3), we get

$$
\|x\|_\infty \le B + \frac{2M_1}{\Gamma(\alpha+1)} := M_2. \tag{3.6}
$$

Combining (3.5) with (3.6), we have

$$
\|x\|_X = \max\{\|x\|_\infty, \|D_{0^+}^\alpha x\|_\infty\} \le \max\{M_1, M_2\} := M.
$$

Therefore, Ω_1 is bounded. The proof is complete. $\square$

Lemma 3.5. *Suppose* (H2) *holds, then the set* $\Omega_2 = \{x \in \ker L : Nx \in \operatorname{Im} L\}$ *is bounded.*

Proof. For $x \in \Omega_2$, we have $x(t) = c$, $c \in \mathbb{R}$ and $Nx \in \operatorname{Im} L$. Then

$$\int_0^1 (1-s)^{1-\alpha} f(s, c, 0)\, ds = 0,$$

which together with (H2) implies $|c| \le B$. Thus, we have

$$\|x\|_X \le \max\{B, 0\} = B.$$

Hence, Ω_2 is bounded. The proof is complete. □

Lemma 3.6. *Suppose the first part of* (H2) *holds, then the set*

$$\Omega_3 = \{x \in \ker L : \lambda x + (1-\lambda)QNx = 0,\ \lambda \in [0,1]\}$$

is bounded.

Proof. For $x \in \Omega_3$, we have $x(t) = c$, $c \in \mathbb{R}$ and

$$\lambda c + (1-\lambda)(2-\alpha) \int_0^1 (1-s)^{1-\alpha} f(s, c, 0)\, ds = 0. \tag{3.7}$$

If $\lambda = 0$, then $|c| \le B$ because of the first part of (H2). If $\lambda \in (0,1]$, we can also obtain $|c| \le B$. Otherwise, if $|c| > B$, in view of the first part of (H2), one has

$$\lambda c^2 + (1-\lambda)(2-\alpha) \int_0^1 (1-s)^{1-\alpha} c f(s, c, 0)\, ds > 0,$$

which contradicts (3.7). Therefore, Ω_3 is bounded. The proof is complete. □

Remark 3.7. If the second part of (H2) holds, then the set

$$\Omega_3' = \{x \in \ker L : -\lambda x + (1-\lambda)QNx = 0,\ \lambda \in [0,1]\}$$

is bounded.

Proof of Theorem 3.1. Set

$$\Omega = \{x \in X : \|x\|_X < \max\{M, B\} + 1\}.$$

Obviously, $\Omega_1 \cup \Omega_2 \cup \Omega_3 \subset \Omega$ (or $\Omega_1 \cup \Omega_2 \cup \Omega_3' \subset \Omega$). It follows from Lemma 3.2 and Lemma 3.3 that L (defined by (2.1)) is a Fredholm operator of index zero and N (defined by (2.2)) is L-compact on $\overline{\Omega}$. By Lemma 3.4 and Lemma 3.5, the following two conditions are satisfied

(1) $Lx \ne \lambda Nx$ for every $(x, \lambda) \in [(\operatorname{dom} L \setminus \ker L) \cap \partial\Omega] \times (0,1)$;
(2) $Nx \notin \operatorname{Im} L$ for every $x \in \ker L \cap \partial\Omega$.

It remains verifying condition (3) of Lemma 2.4. To do that, let

$$H(x, \lambda) = \pm\lambda x + (1-\lambda)QNx.$$

Based on Lemma 3.6 (or Remark 3.7), we have

$$H(x, \lambda) \ne 0, \quad \forall x \in \partial\Omega \cap \ker L.$$

Thus, by the homotopy property of degree, we have

$$\begin{aligned}
\deg(QN|_{\ker L}, \Omega \cap \ker L, 0) &= \deg(H(\cdot, 0), \Omega \cap \ker L, 0) \\
&= \deg(H(\cdot, 1), \Omega \cap \ker L, 0) \\
&= \deg(\pm I, \Omega \cap \ker L, 0) \ne 0.
\end{aligned}$$

So that condition (3) of Lemma 2.4 is satisfied.

Consequently, by using Lemma 2.4, the operator equation $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$. Namely, PBVP (1.1) has at least one solution in X. The proof is complete. $\square$

4. An example

To illustrate our main result, we consider the periodic boundary-value problem

$$\begin{gathered} x''(t) = -2 + \frac{1}{2}x(t) + te^{-(D_{0^+}^{3/2}x(t))^2}, \quad t \in [0,1], \\ x(0) = x(1), \quad D_{0^+}^{3/2}x(0) = D_{0^+}^{3/2}x(1). \end{gathered} \tag{4.1}$$

Corresponding to PBVP (1.1), we have $\alpha = 3/2$ and

$$f(t,u,v) = -2 + \frac{1}{2}u + te^{-v^2}.$$

Choose $a(t) = 3$, $b(t) = 1/2$, $c(t) = 0$, $B = 4$. By a simple calculation, we obtain that $\|b\|_\infty = 1/2$, $\|c\|_\infty = 0$ and

$$\begin{gathered} uf(t,u,v) = u\Big[\frac{1}{2}(u-4) + te^{-v^2}\Big] > 0, \quad \forall t \in [0,1],\ v \in \mathbb{R},\ |u| > 4, \\ \frac{1}{\Gamma(3-\frac{3}{2})}\Big(\frac{2 \times \frac{1}{2}}{\Gamma(\frac{3}{2}+1)} + 0\Big) < 1. \end{gathered}$$

Obviously, (4.1) satisfies all the assumptions of Theorem 3.1. Hence, it has at least one solution.

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