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CRITICAL EXPONENT AND BLOW-UP RATE FOR THE ω-DIFFUSION EQUATIONS ON GRAPHS WITH DIRICHLET BOUNDARY CONDITIONS

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Abstract. In this article, we study the ω-diffusion equation on a graph with Dirichlet boundary conditions

$$\begin{gathered} u_t(x,t) = \Delta_\omega u(x,t) + e^{\beta t} u^p(x,t), \quad (x,t) \in S \times (0,\infty), \\ u(x,t) = 0, \quad (x,t) \in \partial S \times [0,\infty), \\ u(x,0) = u_0(x) \geq 0, \quad x \in V, \end{gathered}$$

where Δ_ω is the discrete weighted Laplacian operator. First, we prove the existence and uniqueness of the local solution via Banach fixed point theorem. Then, by the method of supersolutions and subsolutions we prove that the ω-diffusion problem has a critical exponent p_β: when $p > p_\beta$, the solution becomes global; while when $1 < p < p_\beta$, the solution blows up in finite time. Under appropriate hypotheses, we estimate the blow-up rate in the L^∞-norm. Some numerical experiments illustrate our results.

1. Introduction

In this article we are concerned with the blow-up properties of the problem

$$\begin{gathered} u_t(x,t) = \Delta_\omega u(x,t) + e^{\beta t} u^p(x,t), \quad (x,t) \in S \times (0,\infty), \\ u(x,t) = 0, \quad (x,t) \in \partial S \times [0,\infty), \\ u(x,0) = u_0(x) \geq 0, \quad x \in V, \end{gathered} \tag{1.1}$$

where $p > 1$, $\beta > 0$, $u_0(x)$ is nonnegative and nontrivial, and V is the set of vertices of a graph $G(V, E, \omega)$. In general, we can split the set of vertexes V into two disjoint subjects S and ∂S such that $V = S \cup \partial S$, which are called the interior and the boundary of V, respectively. We write $(x, y) \in E$ when two vertices x, y are adjacent and connected by an edge. Throughout this paper, all the graphs in our concern are assumed to be simple, finite, connected, undirected and weighted. Besides, $\Delta_\omega u(x,t)$ is defined as

$$\sum_{y \in V} \omega(x,y)\big(u(y,t) - u(x,t)\big), \quad \forall x \in V,$$

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where $\omega : V \times V \to \mathbb{R}$ denotes the weighted function, which has the following properties:

(1) $\omega(x,x) = 0$, for any $x \in V$,
(2) $\omega(x,y) = \omega(y,x)$, for any $x, y \in V$,
(3) $\omega(x,y) = 0$, if and only if $(x,y) \notin E$.

A function on a graph is understood as a function defined just on the set of vertices of the graph. The integration of a function $f : V \to \mathbb{R}$ on a graph G is defined by

$$\int_V f = \sum_{x \in V} f(x).$$

As usual, the set $C\big(V \times (0,\infty)\big)$ consists of all function u defined on $V \times (0,\infty)$ which satisfies $u(x,t) \in C(0,\infty)$ for each $x \in V$.

The ω-diffusion equation of the form $u_t(x,t) = \Delta_\omega u(x,t)$ and its variations can be used to model diffusion process. Recently, more diffusion equations are taken into account to graphs [2, 11, 14]. In [3, 8, 18], the extinction and positivity of the solution of the ω-diffusion equation with absorption and its variations was considered. In [4], the decay rates for evolution equations of the averaging operator $\partial t - \Delta_F$ were studied. In [5], the quenching phenomena for a non-local diffusion equation with a singular absorption was discussed. Meanwhile, Xin et al [19] considered the blow-up for the ω-heat equation with Dirichlet boundary conditions and a reaction term $u^p(x,t)$ on graphs

$$\begin{gathered}
u_t(x,t) = \Delta_\omega u(x,t) + u^p(x,t), \quad (x,t) \in S \times (0,\infty), \\
u(x,t) = 0, \quad (x,t) \in \partial S \times (0,\infty), \\
u(x,0) = u_0(x), \quad x \in V.
\end{gathered} \tag{1.2}$$

They proved that when $p > 1$, the solution of problem (1.2) blows up in finite time under some suitable conditions and when $p \le 1$, every solution is global.

Blow-up phenomenon has been widely studied in various evolution equations [1, 6, 7, 9, 12, 13, 15, 16, 17, 21, 22]. Meier [10] considered the problem

$$\begin{gathered}
u_t = \Delta u(x,t) + e^{\beta t} u^p(x,t), \quad (x,t) \in \Omega \times (0,\infty), \\
u(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,\infty), \\
u(x,0) = u_0(x), \quad x \in \Omega,
\end{gathered} \tag{1.3}$$

and proved that the critical exponent is $p_\beta = 1 + \frac{\beta}{\lambda_1}$, where λ_1 is the first Dirichlet eigenvalue of the Laplacian in Ω. Zhang and Wang [20] studied the nonlocal diffusion equations with Dirichlet boundary condition

$$\begin{gathered}
u_t = \int_{\mathbb{R}^N} J(x-y)\,(u(y,t) - u(x,t))\,\mathrm{d}y + e^{\beta t} u^p(x,t), \quad (x,t) \in \Omega \times (0,\infty), \\
u(x,t) = 0, \quad (x,t) \notin \Omega \times [0,\infty), \\
u(x,0) = u_0(x) \ge 0, \quad x \in \Omega,
\end{gathered} \tag{1.4}$$

where $p > 1, \beta > 0$ and the kernel $J \in C^1(\mathbb{R}^N)$ satisfies $J \ge 0$ in B_1 (the unit ball); $J = 0$ in $\mathbb{R}^N \setminus B_1$ with $\int_{B_1} J(z)\,\mathrm{d}z = 1$. They showed that the critical exponent is coincident with that of [10].

Motivated by above research, we consider the critical exponent for problem (1.1). We first study the local existence of the solution via Banach fixed point theorem.

Then, we deal with the existence of the critical exponent of problem (1.1) by the method of supersolutions and subsolutions. Meanwhile, we prove that the nonnegative and nontrivial solution blows up in finite time and give the blow-up rate on L^∞-norm. Finally, we take two examples to support our results.

This paper is organized as follows. In Section 2, we consider the local existence of the solution. In Section 3, we share many important properties of problem (1.1), such as critical exponent or blow-up rate. In the end, we take two examples to check our results theoretically in Section 4.

2. Local existence of solution

In this section we prove the existence of local a solution for problem (1.1) via Banach fixed point theorem. We first define the following Banach space:

$$X_{t_0} = \{u(x,t) : u(x,t) \in C\left(V \times [0,t_0]\right); u(x,t) \equiv 0, \forall x \in \partial S\},$$

with the norm

$$\|u\|_{X_{t_0}} = \max_{t\in[0,t_0]} \max_{x\in S} |u(x,t)|,$$

where $t_0 > 0$ is a fixed constant. And then, we consider the operator $D : X_{t_0} \to X_{t_0}$ for the fixed $v_0(x)$ which is defined on V

$$D_{v_0}[v](x,t) = \begin{cases} v_0(x) + \int_0^t \Delta_\omega v(x,\tau)\,\mathrm{d}\tau + \int_0^t e^{\beta t} v^p(x,\tau)\,\mathrm{d}\tau, & x \in S, \\ 0, & x \in \partial S. \end{cases}$$

In the following lemmas, we prove that the operator $D_{v_0} : X_{t_0} \to X_{t_0}$ is well defined and strictly contractive under some suitable conditions.

Lemma 2.1. *The operator D_{v_0} is well defined, mapping X_{t_0} to X_{t_0}. Moreover, let u_0, v_0 be defined on V and $u, v \in X_{t_0}$, then, there exists a positive constant $C = C(p, \omega(x,y), \|u\|_{X_{t_0}}, \|v\|_{X_{t_0}}, \beta, |V|)$ such that*

$$\|D_{u_0}[u] - D_{v_0}[v]\|_{X_{t_0}} \le \max_{x\in V} |u_0(x) - v_0(x)| + Ct\|u - v\|_{X_{t_0}}, \tag{2.1}$$

where $|V|$ denotes the number of the nodes of the graph G.

Proof. We first show that the operator D_{v_0} maps X_{t_0} to X_{t_0}. On the one hand, for any $(x,t) \in S \times [0,t_0]$, we have

$$\begin{aligned} |D_{v_0}[v](x,t) - D_{v_0}[v](x,0)| &= \Big| \int_0^t \Delta_\omega v(x,\tau)\,\mathrm{d}\tau + \int_0^t e^{\beta\tau} v^p(x,\tau)\,\mathrm{d}\tau \Big| \\ &\le \Big(2 \max_{(x,y)\in E} |\omega(x,t)||V|\|v\|_{X_{t_0}} + e^{\beta t_0}\|v\|^p_{X_{t_0}}\Big) t. \end{aligned} \tag{2.2}$$

From this inequality, we know that the operator D_{v_0} is continuous at $t = 0$. Similarly, for any (x,t_1), $(x,t_2) \in S \times [0,t_0]$, we have

$$\begin{aligned} &|D_{v_0}[v](x,t_1) - D_{v_0}[v](x,t_2)| \\ &= \Big| \int_{t_1}^{t_2} \Delta_\omega v(x,\tau)\,\mathrm{d}\tau + \int_{t_1}^{t_2} e^{\beta\tau} v^p(x,\tau)\,\mathrm{d}\tau \Big| \\ &\le \Big(2 \max_{(x,y)\in E} |\omega(x,t)||V|\|v\|_{X_{t_0}} + e^{\beta t_0}\|v\|^p_{X_{t_0}}\Big)|t_2 - t_1|, \end{aligned} \tag{2.3}$$

which shows that D_{u_0} is continuous in time for any $t \in [0,t_0]$. On the other hand, the convolution $\sum_{y\in V} \omega(x,y)\left(u(y,t) - u(x,t)\right)$ is uniformly continuous. So it is easy to see that the operator D_{v_0} is continuous as the function of x. Hence, the operator D_{v_0} maps X_{t_0} to X_{t_0}.

Next, we prove (2.1). For any $(x,t) \in S \times [0,t_0]$, we have

$$
\begin{aligned}
&|D_{u_0}[u](x,t) - D_{v_0}[v](x,t)| \\
&\le \max_{x\in S} |u_0(x) - v_0(x)| + \int_0^t |\Delta_\omega (u(x,\tau) - v(x,\tau))| \,\mathrm{d}\tau \\
&\quad + \int_0^t e^{\beta\tau} |u^p(x,\tau) - v^p(x,\tau)| \,\mathrm{d}\tau \\
&\le \max_{x\in S} |u_0(x) - v_0(x)| + 2 \max_{(x,y)\in E} |\omega(x,t)|\, |V| \|u - v\|_{X_{t_0}} t \\
&\quad + e^{\beta t_0} p \xi^{p-1} \|u - v\|_{X_{t_0}} t \\
&= \max_{x\in S} |u_0(x) - v_0(x)| + C \|u - v\|_{X_{t_0}} t,
\end{aligned} \tag{2.4}
$$

where $\xi = \max\{\|u\|_{X_{t_0}}, \|v\|_{X_{t_0}}\}$ and

$$
C = 2 \max_{(x,y)\in E} |\omega(x,t)||V| \|u - v\|_{X_{t_0}} + e^{\beta t_0} p \xi^{p-1}.
$$

The arbitrariness of $(x,t) \in S \times [0,t_0]$ gives the desired estimate (2.1). $\square$

Lemma 2.2. *Suppose t_0 to be small enough, then D_{v_0} is strictly contractive in the ball $B(u_0,\, 2\|u_0\|_{L^\infty(V)})$.*

Proof. Let $u_0 = v_0$, (2.1) ensures that D_{v_0} is a strict contraction in the ball $B\left(u_0, 2\|u_0\|_{L^\infty(V)}\right)$ provided that t_0 is small enough. In fact, for any u, $v \in B\left(u_0, 2\|u_0\|_{L^\infty(V)}\right)$, we have

$$
\|u\|_{X_{t_0}} \le 3\|u_0\|_{L^\infty(V)}, \quad \|v\|_{X_{t_0}} \le 3\|u_0\|_{L^\infty(V)},
$$

and thus, we obtain

$$
\|D_{u_0}[u](x,t) - D_{u_0}[v](x,t)\|_{X_{t_0}} \le C_1 t_0 \|u - v\|_{X_{t_0}},
$$

where

$$
C_1 = e^{\beta t_0} p \left(1 + 3\|u_0\|_{L^\infty(V)}^{p-1}\right) + 2 \max_{(x,y)\in E} |\omega(x,t)||V|.
$$

Therefore, if t_0 is small enough such that $C_1 t_0 < \frac{1}{2}$, we obtain that D_{v_0} is a strict contraction in the ball $B\left(u_0, 2\|u_0\|_{L^\infty(V)}\right)$. We complete the proof. $\square$

Theorem 2.3. *If $p > 1$, then problem (1.1) has a unique solution in $[0,T)$ for some $T > 0$ to be sufficiently small.*

Proof. By the Banach fixed point theorem, Lemma 2.1 and Lemma 2.2, we can easily obtain the existence and the uniqueness of solution to (1.1) in the time interval $[0,t_0]$. Thus, if $\|u\|_{X_{t_0}} < \infty$ and initial data is taken as $u(x,t_0)$, then, the solution can be extended to some interval $[0,t_1)$, where $t_1 > t_0$. Therefore, there exists a $T > 0$, which is sufficiently small, such that problem (1.1) has a unique solution in $[0,T)$. $\square$

3. Global existence and blow-up phenomenon

To study the global existence and blow-up properties, we have the following statements.

Definition 3.1. A nonnegative function $\overline{u}(x,t) \in C^1(V \times [0,T))$ is a supersolution of problem (1.1) if it satisfies

$$\begin{gathered}\overline{u}(x,t) \geq \sum_{y\in V} \omega(x,y)\left(\overline{u}(y,t)-\overline{u}(x,t)\right) + e^{\beta t}\overline{u}^p(x,t), \quad (x,t) \in S \times [0,T), \\ \overline{u}(x,t) \geq 0, \quad (x,t) \in \partial S \times [0,T), \\ \overline{u}(x,0) \geq u_0(x), \quad x \in V.\end{gathered} \tag{3.1}$$

Similarly, we can define the subsolution $\underline{u}(x,t)$ by reversing the inequalities.

Now, we introduce the comparison principle for the nonnegative solutions of (1.1) which plays an important role in the proof of the existence of the critical exponent.

Theorem 3.2. *Let $\overline{u}(x,t), \underline{u}(x,t)$, be supersolution and subsolution of (1.1), respectively. Meanwhile, there exists a point $y \in S$ such that $\omega(x,y) \neq 0$ for any $x \in S$. Then, for any $(x,t) \in V \times [0,T)$, we have $\overline{u}(x,t) \geq \underline{u}(x,t)$.*

Proof. Denote $v = \underline{u} - \overline{u}$, and choose $T_1 < T$. In $V \times [0,T_1]$, we have

$$\begin{aligned}\frac{\partial v(x,t)}{\partial t} &\leq \Delta_\omega v(x,t) + e^{\beta t}\left(\underline{u}^p(x,t) - \overline{u}^p(x,t)\right) \\ &= \Delta_\omega v(x,t) + pe^{\beta t}\xi^{p-1}(x,t)v(x,t),\end{aligned} \tag{3.2}$$

where $\xi(x,t)$ is bounded in $V \times [0,T_1]$.

Let $v_+ = \max\{v,0\}$. Multiplying both sides of (3.2) by v_+ and integrating on V, we have

$$\frac{1}{2}\Big(\int_V v_+^2(x,t)\Big)_t \leq \int_V \Delta_\omega v(x,t)v_+(x,t) + \int_V pe^{\beta t}\xi^{p-1}v_+^2(x,t), \tag{3.3}$$

for all $(x,t) \in V \times [0,T_1]$, due to the fact that $v_+ = 0$ on ∂S. Now, set $I(t) = \{x \in V : \underline{u} > \overline{u}\}$. A direct use of definition of $\Delta_w u(x,t)$ yields

$$\begin{aligned}\int_V \Delta_\omega v(x,t)v_+(x,t) &= \int_{I(t)} \Delta_\omega v(x,t)v_+(x,t) \\ &= \sum_{x\in I(t)}\sum_{y\in V}\omega(x,y)\left(v(y,t)-v(x,t)\right)v_+(x,t) \\ &= \sum_{x\in I(t)}\sum_{y\in I(t)}\omega(x,y)\left(v(y,t)-v(x,t)\right)v_+(x,t) \\ &\quad + \sum_{x\in I(t)}\sum_{y\in V\setminus I(t)}\omega(x,y)\left(v(y,t)-v(x,t)\right)v_+(x,t).\end{aligned} \tag{3.4}$$

We first note that

$$\begin{aligned}
&\sum_{x\in I(t)}\sum_{y\in I(t)}\omega(x,y)\left(v(y,t)-v(x,t)\right)v_+(x,t)\\
&=\frac{1}{2}\sum_{x\in I(t)}\sum_{y\in I(t)}\omega(x,y)\left(v(y,t)-v(x,t)\right)v(x,t)\\
&\quad+\frac{1}{2}\sum_{x\in I(t)}\sum_{y\in I(t)}\omega(x,y)\left(v(x,t)-v(y,t)\right)v(y,t)\\
&=-\frac{1}{2}\sum_{x\in I(t)}\sum_{y\in I(t)}\omega(x,y)\left(v(y,t)-v(x,t)\right)^2\le 0.
\end{aligned}\tag{3.5}$$

Then, notice that if $x\in I(t)=\{x\in V:\underline{u}>\overline{u}\}$ and $y\in V\setminus I(t)=\{y\in V:\overline{u}\ge\underline{u}\}$, then

$$\underline{u}(x,t)>\overline{u}(x,t),\quad \overline{u}(y,t)\ge\underline{u}(y,t),$$

so we have

$$\underline{u}(y,t)-\underline{u}(x,t)\le\overline{u}(y,t)-\overline{u}(x,t).$$

Then

$$\sum_{x\in I(t)}\sum_{y\in V\setminus I(t)}\omega(x,y)\left(v(y,t)-v(x,t)\right)v_+(x,t)\le 0\,. \tag{3.6}$$

Therefore,

$$\int_V \Delta_\omega v(x,t)v_+(x,t)\le 0 \tag{3.7}$$

and

$$\begin{aligned}
\int_V pe^{\beta t}\xi^{p-1}v_+^2(x,t)&=\sum_{x\in V}pe^{\beta t}\xi^{p-1}v_+^2(x,t)\\
&\le pe^{\beta T_1}\xi^{p-1}\sum_{x\in V}v_+^2(x,t)=C\sum_{x\in V}v_+^2(x,t),
\end{aligned}\tag{3.8}$$

where $C=pe^{\beta T_1}\xi^{p-1}$. From (3.3), (3.7) and (3.8), we have

$$\Big(\int_V v_+^2(x,t)\Big)_t\le 2C\int_V v_+^2(x,t).$$

Since $v_+(\cdot,0)=0$, we arrive at $v_+(x,t)=0$ in $V\times[0,T_1]$. Due to the arbitrariness of T_1, we obtain

$$\overline{u}(x,t)\ge\underline{u}(x,t),\quad \forall (x,t)\in V\times[0,T).$$

The proof is complete. □

Then, we study the existence of the critical exponent for problem (1.1).

Definition 3.3. p_β is called the critical exponent of problem (1.1), if it satisfies:

(1) when $p>p_\beta$, there is a nonnegative and nontrivial global solution u of equation (1.1);
(2) when $1<p<p_\beta$, the nontrivial solution u of(1.1) blows up in finite time.

The following theorems imply that if $p>1$, problem (1.1) admits a critical exponent $p_\beta=1+\frac{\beta}{\lambda_1}$, where λ_1 is the principle eigenvalue of the eigenvalue problem

$$\begin{gathered}
-\Delta_\omega\varphi(x)=\lambda_1\varphi(x),\quad x\in S,\\
\varphi(x)=0,\quad x\in\partial S.
\end{gathered}\tag{3.9}$$

Theorem 3.4. *Suppose $p > p_\beta = 1 + \frac{\beta}{\lambda_1}$ and the initial value $u_0 \le z_0\phi_1(x)$, then the solution of (1.1) is global and positive. Here $\phi_1(x)$ which corresponds to λ_1 is the positive eigenfunction with $\|\phi_1(x)\|_{L^\infty} = 1$ and z_0 is a constant satisfying $0 < z_0 < \big(\lambda_1 - \frac{\beta}{p-1}\big)^{\frac{1}{p-1}}$.*

Proof. Let $v(x,t) = \phi_1(x)e^{-\lambda_1 t}$, then it is the solution of the problem

$$\begin{gathered}\frac{\partial v(x,t)}{\partial t} = \Delta_\omega v(x,t), \quad (x,t) \in S \times (0,\infty),\\ v(x,t) = 0, \quad (x,t) \in \partial S \times [0,\infty),\\ v(x,0) = \phi_1(x), \quad x \in V.\end{gathered} \tag{3.10}$$

In addition, let $z(t)$ be the solution of the initial-value problem

$$\begin{gathered}\frac{\mathrm{d}z}{\mathrm{d}t} = e^{\beta t}\|v(\cdot,t)\|_{L^\infty}^{p-1} z^p(t),\\ z(0) = z_0 > 0,\end{gathered} \tag{3.11}$$

where z_0 is a constant satisfying $z_0 < \Big(\lambda_1 - \frac{\beta}{p-1}\Big)^{\frac{1}{p-1}}$. Solve the ODE problem of (3.11) with $\|v(\cdot,t)\|_{L^\infty}^{p-1} = e^{-\lambda_1(p-1)t}$, then we have

$$z(t) = \Big(z_0^{1-p} - \frac{p-1}{(p-1)\lambda_1 - \beta}\big(1 - e^{(\beta-(p-1)\lambda_1)t}\big)\Big)^{\frac{1}{1-p}},$$

which is bounded uniformly for $t \in [0,\infty)$.

Now, let $\overline{u}(x,t) = z(t)v(x,t)$ for $(x,t) \in S \times (0,\infty)$, then

$$\begin{aligned}\frac{\partial \overline{u}(x,t)}{\partial t} &= \frac{\mathrm{d}z(t)}{\mathrm{d}t}v(x,t) + z(t)\frac{\partial v}{\partial t}\\ &= e^{\beta t}\|v(\cdot,t)\|_{L^\infty}^{p-1} z^p(t)v(x,t) + z(t)\Delta_\omega v(x,t)\\ &\ge \Delta_\omega \overline{u}(x,t) + e^{\beta t}v^p z^p(t)\\ &= \Delta_\omega \overline{u}(x,t) + e^{\beta t}\overline{u}^p(x,t),\end{aligned} \tag{3.12}$$

and

$$\overline{u}(x,t) = 0, \quad \forall (x,t) \in \partial S \times [0,\infty), \tag{3.13}$$

furthermore,

$$\overline{u}(x,0) = z_0\phi_1(x) \ge u_0(x). \tag{3.14}$$

From (3.12), (3.13) and (3.14), we obtain that $\overline{u}(x,t)$ is a supersolution of problem (1.1). By Theorem 3.2, we conclude that (1.1) admits a global positive solution provided that the initial value u_0 is small. □

Theorem 3.5. *Suppose that $u_0(x)$ is nonnegative and nontrivial. If $1 < p < p_\beta = 1 + \frac{\beta}{\lambda_1}$, then the corresponding solution to (1.1) on graph G blows up in the sense of $\lim_{t\to T^*-} \sum_{x\in S} u(x,t)\varphi(x) = +\infty$ and the blow-up time T^* satisfies*

$$T^* = \frac{\ln\big(1 - \frac{\beta-(p-1)\lambda_1}{1-p}G_0^{1-p}\big)}{\beta - (p-1)\lambda_1}.$$

Proof. Consider the eigenvalue problem

$$\begin{gathered} -\Delta_\omega\varphi(x) = \lambda_1\varphi(x), \quad x \in S, \\ \varphi(x) = 0, \quad x \in \partial S, \\ \|\varphi(x)\|_{L^\infty} = 1, \end{gathered} \tag{3.15}$$

where λ_1 is the principle eigenvalue of the eigenvalue problem. Multiplying $\varphi(x)$ on the both sides of (1.1) and summing on S, we have

$$\sum_{x\in S} u_t(x,t)\varphi(x) - \sum_{x\in S}\Delta_\omega u(x,t)\varphi(x) = \sum_{x\in S} e^{\beta t}u^p(x,t)\varphi(x). \tag{3.16}$$

Let $G(t) = \sum_{x\in S} u(x,t)\varphi(x)$, then $G'(t) = \sum_{x\in S} u_t(x,t)\varphi(x)$, and

$$\begin{aligned} &\sum_{x\in S}\Delta_\omega u(x,t)\varphi(x) \\ &= \sum_{x\in S}\sum_{y\in V}\omega(x,y)\left(u(y,t)-u(x,t)\right)\varphi(x) \\ &= \sum_{x\in S}\sum_{y\in S}\omega(x,y)\left(u(y,t)-u(x,t)\right)\varphi(x) - \sum_{x\in S}\sum_{y\in\partial S}\omega(x,y)u(x,t)\varphi(x) \\ &= -\frac{1}{2}\sum_{x\in S}\sum_{y\in S}\omega(x,y)\left(u(y,t)-u(x,t)\right)\left(\varphi(y)-\varphi(x)\right) \\ &\quad - \sum_{x\in S}\sum_{y\in\partial S}\omega(x,y)u(x,t)\varphi(x) \\ &= \sum_{x\in S}\sum_{y\in S}\omega(x,y)\left(\varphi(y)-\varphi(x)\right)u(x,t) - \sum_{x\in S}\sum_{y\in\partial S}\omega(x,y)u(x,t)\varphi(x) \\ &= \sum_{x\in S}\sum_{y\in V}\omega(x,y)\left(\varphi(y)-\varphi(x)\right)u(x,t) \\ &= \sum_{x\in S}\Delta_\omega\varphi(x)u(x,t) = -\lambda_1 G(t). \end{aligned} \tag{3.17}$$

Moreover, applying Jensen's inequality, we obtain

$$\sum_{x\in S} e^{\beta t}u^p(x,t)\varphi(x) \ge e^{\beta t}\Big(\sum_{x\in S} u(x,t)\varphi(x)\Big)^p = e^{\beta t}G^p(t). \tag{3.18}$$

Substituting (3.17) and (3.18) into (3.16), we have

$$G'(t) \ge -\lambda_1 G(t) + e^{\beta t}G^p(t). \tag{3.19}$$

Thus, using the comparison for the linear ODE, we have

$$G^{1-p}(t) \le \Big(G_0^{1-p} - \frac{1-p}{\beta-\lambda_1(p-1)} + \frac{1-p}{\beta-\lambda_1(p-1)}e^{\left(\beta-(p-1)\lambda_1\right)t}\Big)e^{(p-1)\lambda_1 t}, \tag{3.20}$$

and then

$$G^{p-1}(t) \ge \frac{1}{\big(G_0^{1-p} - \frac{1-p}{\beta-\lambda_1(p-1)} + \frac{1-p}{\beta-\lambda_1(p-1)}e^{(\beta-(p-1)\lambda_1)t}\big)e^{(p-1)\lambda_1 t}}. \tag{3.21}$$

Since $1 < p < p_\beta = 1 + \frac{\beta}{\lambda_1}$, we have $\frac{1-p}{\beta-(p-1)\lambda_1} < 0$ and $G^{1-p}(0) - \frac{1-p}{\beta-(p-1)\lambda_1} > 0$. Thus, $G(t) = \sum_{x\in S} u(x,t)\varphi(x)$ cannot be global.

From the right side of (3.21), we have the blow-up time is

$$T^* = \frac{\ln\left(1 - \frac{\beta-(p-1)\lambda_1}{1-p} G_0^{1-p}\right)}{\beta - (p-1)\lambda_1}.$$

□

Next we have a blow-up rate in the L^∞-norm.

Theorem 3.6. *Let $1 < p < p_\beta$ and $u(x,t)$ be a solution of problem (1.1) which blows up at time T, then*

$$\lim_{t\to T^-} (T-t)^{\frac{1}{p-1}} \max_{x\in V} u(x,t) = \Big(\frac{e^{-\beta T}}{p-1}\Big)^{\frac{1}{p-1}}. \tag{3.22}$$

Proof. Let $U(t) = u\big(x(t),t\big) = \max_{x\in V} u(x,t)$. On the one hand,

$$U'(t) = \sum_{y\in V} \omega(x,y)\left(U(y,t) - U(x,t)\right) + e^{\beta t} U^p(x,t). \tag{3.23}$$

Integrating over (t,T), we obtain

$$\frac{1}{1-p}\left(U^{1-p}(T) - U^{1-p}(t)\right) \le \frac{1}{\beta}(e^{\beta T} - e^{\beta t}),$$

then

$$\max_{x\in V} u(x,t) = U(t) \ge \big(\frac{\beta}{p-1}\big)^{\frac{1}{p-1}} (e^{\beta T} - e^{\beta t})^{-\frac{1}{p-1}},$$

so

$$\lim_{t\to T^-} (T-t)^{\frac{1}{p-1}} \max_{x\in V} u(x,t) \ge \lim_{t\to T^-} \Big(\frac{(T-t)\frac{\beta}{p-1}}{e^{\beta T} - e^{\beta t}}\Big)^{\frac{1}{p-1}} = \Big(\frac{e^{-\beta T}}{p-1}\Big)^{\frac{1}{p-1}}. \tag{3.24}$$

On the other hand,

$$\begin{aligned} u_t(x,t) &= \sum_{y\in V} \omega(x,y)\left(u(y,t) - u(x,t)\right) + e^{\beta t} u^p(x,t) \\ &\ge -ku(x,t) + e^{\beta t} u^p(x,t) \\ &= u^p(x,t)\left(e^{\beta t} - ku^{1-p}(x,t)\right), \end{aligned} \tag{3.25}$$

where $k = \max_{x\in S} \sum_{y\in V} \omega(x,y)$. In particular, we have

$$\begin{aligned} U'(t) &\ge U^p(x,t)\big(e^{\beta t} - kU^{1-p}(x,t)\big) \\ &\ge U^p(x,t)\Big(e^{\beta t} - k\frac{p-1}{\beta}(e^{\beta T} - e^{\beta t})\Big). \end{aligned} \tag{3.26}$$

Integrating as before over (t,T), we have

$$U^{1-p}(t) \ge \frac{(p-1)(\beta + k(p-1))}{\beta^2}(e^{\beta T} - e^{\beta t}) - \frac{k(p-1)^2}{\beta}(T-t)e^{\beta T},$$

then

$$U(t) \le \Big(\frac{(p-1)\left(\beta + k(p-1)\right)}{\beta^2}(e^{\beta T} - e^{\beta t}) - \frac{k(p-1)^2}{\beta}(T-t)e^{\beta T}\Big)^{\frac{1}{1-p}},$$

so

$$\lim_{t\to T^-} (T-t)^{\frac{1}{p-1}} \max_{x\in V} u(x,t) \le \Big(\frac{e^{-\beta T}}{p-1}\Big)^{\frac{1}{p-1}}. \tag{3.27}$$

From (3.24) and (3.27), we complete the proof. □

4. Examples and numerical experiments

In this section, we give two examples to illustrate our results from section 3. First, we consider the special graph G_1 for problem (1.1). The graph G_1 has two nodes x_1 and x_2, where x_1 is boundary and x_2 is interior. Then, problem (1.1) can be rewritten as

$$\begin{gathered} u_t(x_2,t) = -\omega u(x_2,t) + e^{\beta t}u^p(x_2,t), \quad t>0, \\ u(x_2,0)=u_0>0, \end{gathered} \tag{4.1}$$

where ω, $\beta>0$ and $p>1$ are real constants.

Figure 1. Graph G_1

Equation (4.1) is the well-known Bernoulli equation. The explicit solution of (4.1) is

$$u(x_2,t) = \Big(e^{\omega(p-1)t}\Big(u_0^{1-p} - \frac{1-p}{\beta-(p-1)\omega} + \frac{1-p}{\beta-(p-1)\omega}e^{(\beta-\omega(p-1))t}\Big)\Big)^{\frac{1}{1-p}}. \tag{4.2}$$

Consider $1<p<p_\beta$ and $\omega=\lambda_1$, then $\frac{1-p}{\beta-(p-1)\omega}<0$, so we can get that

$$u_0^{1-p} - \frac{1-p}{\beta-(p-1)\omega} + \frac{1-p}{\beta-(p-1)\omega}e^{(\beta-\omega(p-1))t}$$

is decreasing on t. Notice that $u_0>0$ and $\frac{1-p}{\beta-(p-1)\omega}<0$, then the solution $u(t)$ will blow up in finite time. The blow-up time on L^∞-norm to problem (4.1) is

$$T = \frac{\ln\big(1-\frac{(\beta-\omega(p-1))u_0^{1-p}}{1-p}\big)}{\beta-\omega(p-1)}.$$

Then we have the limit,

$$\lim_{t\to T^-}(T-t)^{\frac{1}{p-1}}u(x_2,t) = \Big(\frac{e^{-\beta T}}{p-1}\Big)^{\frac{1}{p-1}}.$$

We remark that if $\frac{1-p}{\beta-(p-1)\omega}>0$ and $u_0^{1-p}-\frac{1-p}{\beta-(p-1)\omega}>0$, then the only solution (4.2) to problem (4.1) is global.

Now, we consider a complicated graph G_2 which has six nodes, where x_1,x_4 and x_6 are boundary and x_2,x_3 and x_5 are interior. Moreover, we only consider the weight function ω=1. Thus, the problem can be rewritten as

$$\begin{gathered} u_t(x_2,t) = u(x_3,t)+u(x_5,t)-3u(x_2,t)+e^{\beta t}u^p(x_2,t), \\ u_t(x_3,t) = u(x_2,t)+u(x_5,t)-3u(x_3,t)+e^{\beta t}u^p(x_3,t), \\ u_t(x_5,t) = u(x_2,t)+u(x_3,t)-3u(x_5,t)+e^{\beta t}u^p(x_5,t), \\ u(x_2,0)=\alpha>0, \\ u(x_3,0)=\zeta>0, \\ u(x_5,0)=\gamma>0. \end{gathered} \tag{4.3}$$

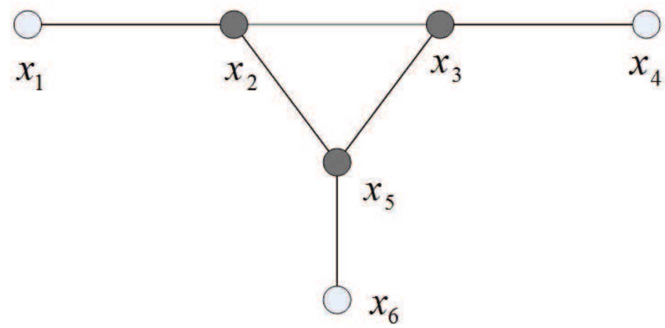

Figure 2. Graph G_2

Let $U = (u(x_2,t), u(x_3,t), u(x_5,t))^{\mathrm{T}}$, and the coefficient matrix is

$$A = \begin{pmatrix} -3 & 1 & 1 \\ 1 & -3 & 1 \\ 1 & 1 & -3 \end{pmatrix}.$$

Thus, (4.3) can be rewritten as

$$\begin{gathered} U_t = A * U + e^{\beta t} U^p, \\ U_0 = (\alpha, \zeta, \gamma)^{\mathrm{T}}. \end{gathered} \tag{4.4}$$

Because of nonlinearity, it is hard to handle system (4.4) by exact analysis technique. Instead, we calculate the solution by difference method. Then the explicit scheme is

$$\begin{gathered} \frac{U^{n+1} - U^n}{\Delta t} = A * U^n + (U^n)^p, \\ U^0 = (\alpha, \zeta, \gamma)^{\mathrm{T}}, \end{gathered} \tag{4.5}$$

where $U^n = (u(x_2, n\Delta t), u(x_3, n\Delta t), u(x_5, n\Delta t))^{\mathrm{T}}$ and Δt is the time step. Moreover, we set $\alpha = 0.4, \zeta = 0.6, \gamma = 0.7, \beta = 3$ and $p = 2$, respectively. The numerical experiment result is shown in Figure 3. We observe that the solution blows up in finite time.

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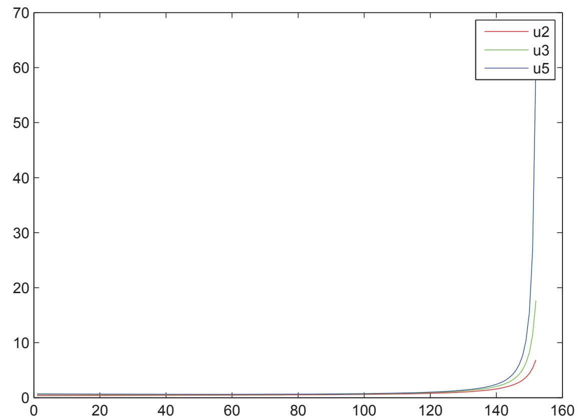

FIGURE 3. Blow-up phenomenon for the equation (4.3)

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