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# SOLVABILITY OF NONLINEAR DIFFERENCE EQUATIONS OF FOURTH ORDER

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Abstract. In this article we show the existence of solutions to the nonlinear difference equation

$$x_n = \frac{x_{n-3}x_{n-4}}{x_{n-1}(a_n + b_n x_{n-2}x_{n-3}x_{n-4})}, \quad n \in \mathbb{N}_0,$$

where the sequences $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$, and initial the values $x_{-j}$, $j = \overline{1,4}$, are real numbers. Also we find the set of initial values for which solutions are undefinable when $a_n \neq 0$ and $b_n \neq 0$ for every $n \in \mathbb{N}_0$. When these two sequences are constant, we describe the long-term behavior of the solutions in detail.

## 1. Introduction

From the very beginning of the study of difference equations, a special attention was paid on the solvable ones. Some old results in the topic can be found, for example, in [9] and [18]. The publication of [24], in which Stević gave a theoretical explanation for the formula to solutions of the following difference equation

$$x_n = \frac{x_{n-2}}{1 + x_{n-1}x_{n-2}}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

presented in [10], trigged a renewed interest in the area (see, e.g., [1]-[4], [8, 21, 25], [28]-[42], [44]-[49]). There are also some equations and systems which are recently studied by using some solvable equations (see, e.g., [5, 23, 27, 43]).

In several papers were later studied some special cases of the following extension of equation (1.1)

$$x_n = \frac{x_{n-2}}{a_n + b_n x_{n-1}x_{n-2}}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

where $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$, and the initial values $x_{-2}, x_{-1}$ are real numbers, as well as some other extensions, by using the main idea in [24] (see, e.g., [1, 2, 4, 21, 29, 32, 46]). Some systems of difference equations which are extensions of equation (1.1) were studied, in [28, 30, 35, 36, 37, 39, 44]. For related results see [6, 8, 25, 31, 33, 38, 40, 41, 42, 45, 47, 48, 49].

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Note that, if $(x_n)_{n\ge -2}$ is a solution to equation (1.2) such that $x_n\ne 0$, $n\ge -2$, then we have that

$$x_n=\frac{x_{n-1}x_{n-2}}{x_{n-1}(a_n+b_nx_{n-1}x_{n-2})}.$$

This form of equation (1.2) suggests investigation of the related equations which in the numerators have more that one factor, after cancelling the same ones.

Motivated by this idea, here we will study the next difference equation

$$x_n=\frac{x_{n-3}x_{n-4}}{x_{n-1}(a_n+b_nx_{n-2}x_{n-3}x_{n-4})},\quad n\in\mathbb{N}_0, \tag{1.3}$$

where $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$ and the initial values $x_{-j}$, $j\in\{1,2,3,4\}$, are real numbers, which is naturally imposed for further studies in this direction.

For a solution $(x_n)_{n\ge -s}$ of the difference equation

$$x_n=f(x_{n-1},\dots,x_{n-s}),\quad n\in\mathbb{N}_0, \tag{1.4}$$

where $f:\mathbb{R}^s\to\mathbb{R}$, $s\in\mathbb{N}$, is said that it is periodic with period $p$, if there is an $n_0\ge -s$ such that

$$x_{n+p}=x_n,\quad \text{for } n\ge n_0.$$

If $n_0\ne -s$, sometimes is said that the solution is eventually periodic. For some results in the area (mostly on classes of equations not related to differential ones), see, e.g. [7, 11, 12, 13, 14, 15, 16, 17, 19, 20, 22, 25, 26] and the references therein.

This article is organized as follows. First, we will show that equation (1.3) can be solved in closed form. Then, we will study in detail the long-term behavior of their solutions for the case when $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$ are constant sequences. Finally, we will find the domain of undefinable solutions of the equation for the case when $a_n\ne 0\ne b_n$, for every $n\in\mathbb{N}_0$.

## 2. Closed form solutions for (1.3)

Let $(x_n)_{n\ge -4}$ be a solution to equation (1.3). If $x_{-j}=0$ for some $j\in\{3,4\}$, then clearly $x_0=0$, so that $x_1$ is not defined. If $x_{-2}=0$, then $x_1=0$, so that $x_2$ is not defined. If $x_{-1}=0$, then clearly $x_0$ is not defined. So, if $x_{-j}=0$ for some $j\in\{1,2,3,4\}$, then the solution is not defined.

On the other hand, if there is an $n\in\mathbb{N}_0$, say $n=n_0$, such that $x_{n_0}=0$ and $x_n\ne 0$ for $0\le n\le n_0-1$. Then $x_{n_0-3}=0$ or $x_{n_0-4}=0$, so that it must be $n_0\le 3$. If $n_0\in\{0,1,2\}$, then clearly $x_{-j}=0$ for some $j\in\{1,2,3,4\}$. If $n_0=3$, then $x_0=0$ (the case already treated) or $x_{-1}=0$. Hence, in all the cases there is a $j\in\{1,2,3,4\}$ such that $x_{-j}=0$, so that according to the first part of the consideration such solutions are not defined.

Therefore, for every well-defined solution of equation (1.3)

$$x_{-j}\ne 0,\quad 1\le j\le 4, \tag{2.1}$$

is equivalent to $x_n\ne 0$, $n\ge -4$.

Hence, for solutions satisfying (2.1), the change of variables

$$y_n=\frac{1}{x_nx_{n-1}x_{n-2}},\quad n\ge -2, \tag{2.2}$$

is possible and the sequence $(y_n)_{n\ge -2}$ satisfies the equation

$$y_n=a_ny_{n-2}+b_n,\quad n\in\mathbb{N}_0, \tag{2.3}$$

which means that

$$y_{2m+i} = a_{2m+i} y_{2(m-1)+i} + b_{2m+i}, \tag{2.4}$$

for every $m \in \mathbb{N}_0$ and $i \in \{0,1\}$, that is, $(y_{2m+i})_{m\geq -1}$, $i \in \{0,1\}$, are solutions to the difference equations

$$z_m = a_{2m+i} z_{m-1} + b_{2m+i}, \quad m \in \mathbb{N}_0, \tag{2.5}$$

$i \in \{0,1\}$.

By a known formula, it follows that

$$y_{2m+i} = y_{i-2} \prod_{j=0}^{m} a_{2j+i} + \sum_{l=0}^{m} b_{2l+i} \prod_{j=l+1}^{m} a_{2j+i}, \quad m \in \mathbb{N}_0, \tag{2.6}$$

$i \in \{0,1\}$, are general solutions to the equations in (2.5). From (2.2) it follows that

$$x_{3m+i} = \frac{1}{y_{3m+i} x_{3m+i-1} x_{3m+i-2}} = \frac{y_{3m+i-1}}{y_{3m+i}} x_{3(m-1)+i},$$

$i \in \{0,1,2\}$, and consequently

$$x_{3m+i} = \frac{y_{3m+i-1}}{y_{3m+i}} \frac{y_{3m+i-4}}{y_{3m+i-3}} x_{3(m-2)+i},$$

$i \in \{0,1,2\}$, so by using the change $m \to 2m+j$, $m \in \mathbb{N}_0$, $j \in \{0,1\}$, is obtained

$$x_{6m+3j+i} = \frac{y_{6m+3j+i-1}}{y_{6m+3j+i}} \frac{y_{6m+3j+i-4}}{y_{6m+3j+i-3}} x_{6(m-1)+3j+i},$$

$i \in \{0,1,2\}$, $j \in \{0,1\}$, which can be written in the form

$$x_{6m+j} = \frac{y_{6m+j-1}}{y_{6m+j}} \frac{y_{6m+j-4}}{y_{6m+j-3}} x_{6(m-1)+j}, \quad m \in \mathbb{N}_0, \tag{2.7}$$

$j \in \overline{0,5}$, as far as $6m+j \geq 2$. From (2.7) it follows that

$$x_{6m+l} = x_{l-6} \prod_{s=0}^{m} \frac{y_{6s+l-1}}{y_{6s+l}} \frac{y_{6s+l-4}}{y_{6s+l-3}}, \quad m \geq -1, \tag{2.8}$$

for $l = \overline{2,7}$.

Employing the formulas in (2.6), in equalities (2.8) for $l$ even and odd separately, we have

$$\begin{aligned}
x_{6m+2i} &= x_{2i-6} \prod_{s=0}^{m} \frac{y_{6s+2i-1}}{y_{6s+2i}} \frac{y_{6s+2i-4}}{y_{6s+2i-3}} \\
&= x_{2i-6} \prod_{s=0}^{m} \frac{y_{-1} \prod_{j=0}^{3s+i-1} a_{2j+1} + \sum_{l=0}^{3s+i-1} b_{2l+1} \prod_{j=l+1}^{3s+i-1} a_{2j+1}}{y_{-2} \prod_{j=0}^{3s+i} a_{2j} + \sum_{l=0}^{3s+i} b_{2l} \prod_{j=l+1}^{3s+i} a_{2j}} \\
&\quad \times \frac{y_{-2} \prod_{j=0}^{3s+i-2} a_{2j} + \sum_{l=0}^{3s+i-2} b_{2l} \prod_{j=l+1}^{3s+i-2} a_{2j}}{y_{-1} \prod_{j=0}^{3s+i-2} a_{2j+1} + \sum_{l=0}^{3s+i-2} b_{2l+1} \prod_{j=l+1}^{3s+i-2} a_{2j+1}} \\
&= x_{2i-6} \prod_{s=0}^{m} \frac{(x_{-1}x_{-2}x_{-3})^{-1} \prod_{j=0}^{3s+i-1} a_{2j+1} + \sum_{l=0}^{3s+i-1} b_{2l+1} \prod_{j=l+1}^{3s+i-1} a_{2j+1}}{(x_{-2}x_{-3}x_{-4})^{-1} \prod_{j=0}^{3s+i} a_{2j} + \sum_{l=0}^{3s+i} b_{2l} \prod_{j=l+1}^{3s+i} a_{2j}} \\
&\quad \times \frac{(x_{-2}x_{-3}x_{-4})^{-1} \prod_{j=0}^{3s+i-2} a_{2j} + \sum_{l=0}^{3s+i-2} b_{2l} \prod_{j=l+1}^{3s+i-2} a_{2j}}{(x_{-1}x_{-2}x_{-3})^{-1} \prod_{j=0}^{3s+i-2} a_{2j+1} + \sum_{l=0}^{3s+i-2} b_{2l+1} \prod_{j=l+1}^{3s+i-2} a_{2j+1}}
\end{aligned}$$

$$
\begin{aligned}
&= x_{2i-6}\prod_{s=0}^{m}\frac{\prod_{j=0}^{3s+i-1}a_{2j+1}+x_{-1}x_{-2}x_{-3}\sum_{l=0}^{3s+i-1}b_{2l+1}\prod_{j=l+1}^{3s+i-1}a_{2j+1}}{\prod_{j=0}^{3s+i}a_{2j}+x_{-2}x_{-3}x_{-4}\sum_{l=0}^{3s+i}b_{2l}\prod_{j=l+1}^{3s+i}a_{2j}}\\
&\quad\times\frac{\prod_{j=0}^{3s+i-2}a_{2j}+x_{-2}x_{-3}x_{-4}\sum_{l=0}^{3s+i-2}b_{2l}\prod_{j=l+1}^{3s+i-2}a_{2j}}{\prod_{j=0}^{3s+i-2}a_{2j+1}+x_{-1}x_{-2}x_{-3}\sum_{l=0}^{3s+i-2}b_{2l+1}\prod_{j=l+1}^{3s+i-2}a_{2j+1}},
\end{aligned}
$$

for $m\ge -1$, $i\in\{1,2,3\}$, and

$$
\begin{aligned}
x_{6m+2i+1} &= x_{2i-5}\prod_{s=0}^{m}\frac{y_{6s+2i}}{y_{6s+2i+1}}\frac{y_{6s+2i-3}}{y_{6s+2i-2}}\\
&= x_{2i-5}\prod_{s=0}^{m}\frac{y_{-2}\prod_{j=0}^{3s+i}a_{2j}+\sum_{l=0}^{3s+i}b_{2l}\prod_{j=l+1}^{3s+i}a_{2j}}{y_{-1}\prod_{j=0}^{3s+i}a_{2j+1}+\sum_{l=0}^{3s+i}b_{2l+1}\prod_{j=l+1}^{3s+i}a_{2j+1}}\\
&\quad\times\frac{y_{-1}\prod_{j=0}^{3s+i-2}a_{2j+1}+\sum_{l=0}^{3s+i-2}b_{2l+1}\prod_{j=l+1}^{3s+i-2}a_{2j+1}}{y_{-2}\prod_{j=0}^{3s+i-1}a_{2j}+\sum_{l=0}^{3s+i-1}b_{2l}\prod_{j=l+1}^{3s+i-1}a_{2j}}\\
&= x_{2i-5}\prod_{s=0}^{m}\frac{(x_{-2}x_{-3}x_{-4})^{-1}\prod_{j=0}^{3s+i}a_{2j}+\sum_{l=0}^{3s+i}b_{2l}\prod_{j=l+1}^{3s+i}a_{2j}}{(x_{-1}x_{-2}x_{-3})^{-1}\prod_{j=0}^{3s+i}a_{2j+1}+\sum_{l=0}^{3s+i}b_{2l+1}\prod_{j=l+1}^{3s+i}a_{2j+1}}\\
&\quad\times\frac{(x_{-1}x_{-2}x_{-3})^{-1}\prod_{j=0}^{3s+i-2}a_{2j+1}+\sum_{l=0}^{3s+i-2}b_{2l+1}\prod_{j=l+1}^{3s+i-2}a_{2j+1}}{(x_{-2}x_{-3}x_{-4})^{-1}\prod_{j=0}^{3s+i-1}a_{2j}+\sum_{l=0}^{3s+i-1}b_{2l}\prod_{j=l+1}^{3s+i-1}a_{2j}}\\
&= x_{2i-5}\prod_{s=0}^{m}\frac{\prod_{j=0}^{3s+i}a_{2j}+x_{-2}x_{-3}x_{-4}\sum_{l=0}^{3s+i}b_{2l}\prod_{j=l+1}^{3s+i}a_{2j}}{\prod_{j=0}^{3s+i}a_{2j+1}+x_{-1}x_{-2}x_{-3}\sum_{l=0}^{3s+i}b_{2l+1}\prod_{j=l+1}^{3s+i}a_{2j+1}}\\
&\quad\times\frac{\prod_{j=0}^{3s+i-2}a_{2j+1}+x_{-1}x_{-2}x_{-3}\sum_{l=0}^{3s+i-2}b_{2l+1}\prod_{j=l+1}^{3s+i-2}a_{2j+1}}{\prod_{j=0}^{3s+i-1}a_{2j}+x_{-2}x_{-3}x_{-4}\sum_{l=0}^{3s+i-1}b_{2l}\prod_{j=l+1}^{3s+i-1}a_{2j}},
\end{aligned}
$$

for $m\ge -1$, $i\in\{1,2,3\}$.

Hence the following theorem holds.

**Theorem 2.1.** *If $(x_n)_{n\ge -4}$ is a well-defined solution of equation (1.3), then it can be represented in the form*

$$
\begin{aligned}
x_{6m+2i} &= x_{2i-6}\prod_{s=0}^{m}\frac{\prod_{j=0}^{3s+i-1}a_{2j+1}+x_{-1}x_{-2}x_{-3}\sum_{l=0}^{3s+i-1}b_{2l+1}\prod_{j=l+1}^{3s+i-1}a_{2j+1}}{\prod_{j=0}^{3s+i}a_{2j}+x_{-2}x_{-3}x_{-4}\sum_{l=0}^{3s+i}b_{2l}\prod_{j=l+1}^{3s+i}a_{2j}}\\
&\quad\times\frac{\prod_{j=0}^{3s+i-2}a_{2j}+x_{-2}x_{-3}x_{-4}\sum_{l=0}^{3s+i-2}b_{2l}\prod_{j=l+1}^{3s+i-2}a_{2j}}{\prod_{j=0}^{3s+i-2}a_{2j+1}+x_{-1}x_{-2}x_{-3}\sum_{l=0}^{3s+i-2}b_{2l+1}\prod_{j=l+1}^{3s+i-2}a_{2j+1}},
\end{aligned}
\tag{2.9}
$$

*and*

$$
\begin{aligned}
x_{6m+2i+1} &= x_{2i-5}\prod_{s=0}^{m}\frac{\prod_{j=0}^{3s+i}a_{2j}+x_{-2}x_{-3}x_{-4}\sum_{l=0}^{3s+i}b_{2l}\prod_{j=l+1}^{3s+i}a_{2j}}{\prod_{j=0}^{3s+i}a_{2j+1}+x_{-1}x_{-2}x_{-3}\sum_{l=0}^{3s+i}b_{2l+1}\prod_{j=l+1}^{3s+i}a_{2j+1}}\\
&\quad\times\frac{\prod_{j=0}^{3s+i-2}a_{2j+1}+x_{-1}x_{-2}x_{-3}\sum_{l=0}^{3s+i-2}b_{2l+1}\prod_{j=l+1}^{3s+i-2}a_{2j+1}}{\prod_{j=0}^{3s+i-1}a_{2j}+x_{-2}x_{-3}x_{-4}\sum_{l=0}^{3s+i-1}b_{2l}\prod_{j=l+1}^{3s+i-1}a_{2j}},
\end{aligned}
\tag{2.10}
$$

*for $m\ge -1$, $i\in\{1,2,3\}$.*

**Remark 2.2.** The formulas in (2.9) and (2.10) can be regarded as an integral formula for general solution of equation (1.3). In fact, they include non-defined solutions, which will be described in detail in the last section of this article.

## 3. Constant coefficients case

In this section we study equation (1.3) when

$$a_n = a, \quad b_n = b, \quad n \in \mathbb{N}_0,$$

where $a$ and $b$ are some real constants. In this case, equation (1.3) becomes

$$x_n = \frac{x_{n-3}x_{n-4}}{x_{n-1}(a + bx_{n-2}x_{n-3}x_{n-4})}, \quad n \in \mathbb{N}_0. \tag{3.1}$$

If $x_{-j} \neq 0$, $j = \overline{1,4}$, from (2.9) and (2.10) we have

$$\begin{aligned} x_{6m+2i} &= x_{2i-6} \prod_{s=0}^{m} \frac{(x_{-1}x_{-2}x_{-3})^{-1} \prod_{j=0}^{3s+i-1} a + \sum_{l=0}^{3s+i-1} b \prod_{j=l+1}^{3s+i-1} a}{(x_{-2}x_{-3}x_{-4})^{-1} \prod_{j=0}^{3s+i} a + \sum_{l=0}^{3s+i} b \prod_{j=l+1}^{3s+i} a} \\ &\quad \times \frac{(x_{-2}x_{-3}x_{-4})^{-1} \prod_{j=0}^{3s+i-2} a + \sum_{l=0}^{3s+i-2} b \prod_{j=l+1}^{3s+i-2} a}{(x_{-1}x_{-2}x_{-3})^{-1} \prod_{j=0}^{3s+i-2} a + \sum_{l=0}^{3s+i-2} b \prod_{j=l+1}^{3s+i-2} a} \\ &= x_{2i-6} \prod_{s=0}^{m} \frac{(x_{-1}x_{-2}x_{-3})^{-1} a^{3s+i} + b \sum_{l=0}^{3s+i-1} a^{3s+i-1-l}}{(x_{-2}x_{-3}x_{-4})^{-1} a^{3s+i+1} + b \sum_{l=0}^{3s+i} a^{3s+i-l}} \\ &\quad \times \frac{(x_{-2}x_{-3}x_{-4})^{-1} a^{3s+i-1} + b \sum_{l=0}^{3s+i-2} a^{3s+i-2-l}}{(x_{-1}x_{-2}x_{-3})^{-1} a^{3s+i-1} + b \sum_{l=0}^{3s+i-2} a^{3s+i-2-l}}, \end{aligned} \tag{3.2}$$

$m \geq -1$, $i \in \{1, 2, 3\}$, and

$$\begin{aligned} x_{6m+2i+1} &= x_{2i-5} \prod_{s=0}^{m} \frac{(x_{-2}x_{-3}x_{-4})^{-1} \prod_{j=0}^{3s+i} a + \sum_{l=0}^{3s+i} b \prod_{j=l+1}^{3s+i} a}{(x_{-1}x_{-2}x_{-3})^{-1} \prod_{j=0}^{3s+i} a + \sum_{l=0}^{3s+i} b \prod_{j=l+1}^{3s+i} a} \\ &\quad \times \frac{(x_{-1}x_{-2}x_{-3})^{-1} \prod_{j=0}^{3s+i-2} a + \sum_{l=0}^{3s+i-2} b \prod_{j=l+1}^{3s+i-2} a}{(x_{-2}x_{-3}x_{-4})^{-1} \prod_{j=0}^{3s+i-1} a + \sum_{l=0}^{3s+i-1} b \prod_{j=l+1}^{3s+i-1} a} \\ &= x_{2i-5} \prod_{s=0}^{m} \frac{(x_{-2}x_{-3}x_{-4})^{-1} a^{3s+i+1} + b \sum_{l=0}^{3s+i} a^{3s+i-l}}{(x_{-1}x_{-2}x_{-3})^{-1} a^{3s+i+1} + b \sum_{l=0}^{3s+i} a^{3s+i-l}} \\ &\quad \times \frac{(x_{-1}x_{-2}x_{-3})^{-1} a^{3s+i-1} + b \sum_{l=0}^{3s+i-2} a^{3s+i-2-l}}{(x_{-2}x_{-3}x_{-4})^{-1} a^{3s+i} + b \sum_{l=0}^{3s+i-1} a^{3s+i-1-l}}, \end{aligned} \tag{3.3}$$

for $m \geq -1$, $i \in \{1, 2, 3\}$.

If $a \neq 1$, then from (3.2) and (3.3) we have

$$\begin{aligned} x_{6m+2i} &= x_{2i-6} \prod_{s=0}^{m} \frac{(x_{-1}x_{-2}x_{-3})^{-1}(1-a)a^{3s+i} + b(1-a^{3s+i})}{(x_{-2}x_{-3}x_{-4})^{-1}(1-a)a^{3s+i+1} + b(1-a^{3s+i+1})} \\ &\quad \times \frac{(x_{-2}x_{-3}x_{-4})^{-1}(1-a)a^{3s+i-1} + b(1-a^{3s+i-1})}{(x_{-1}x_{-2}x_{-3})^{-1}(1-a)a^{3s+i-1} + b(1-a^{3s+i-1})} \\ &= x_{2i-6} \prod_{s=0}^{m} \frac{((x_{-1}x_{-2}x_{-3})^{-1}(1-a)-b)a^{3s+i} + b}{((x_{-2}x_{-3}x_{-4})^{-1}(1-a)-b)a^{3s+i+1} + b} \\ &\quad \times \frac{((x_{-2}x_{-3}x_{-4})^{-1}(1-a)-b)a^{3s+i-1} + b}{((x_{-1}x_{-2}x_{-3})^{-1}(1-a)-b)a^{3s+i-1} + b}, \end{aligned} \tag{3.4}$$

for $m \geq -1$, $i \in \{1,2,3\}$, and

$$\begin{aligned} x_{6m+2i+1} &= x_{2i-5} \prod_{s=0}^{m} \frac{(x_{-2}x_{-3}x_{-4})^{-1}(1-a)a^{3s+i+1} + b(1-a^{3s+i+1})}{(x_{-1}x_{-2}x_{-3})^{-1}(1-a)a^{3s+i+1} + b(1-a^{3s+i+1})} \\ &\quad \times \frac{(x_{-1}x_{-2}x_{-3})^{-1}(1-a)a^{3s+i-1} + b(1-a^{3s+i-1})}{(x_{-2}x_{-3}x_{-4})^{-1}(1-a)a^{3s+i} + b(1-a^{3s+i})} \\ &= x_{2i-5} \prod_{s=0}^{m} \frac{((x_{-2}x_{-3}x_{-4})^{-1}(1-a)-b)a^{3s+i+1} + b}{((x_{-1}x_{-2}x_{-3})^{-1}(1-a)-b)a^{3s+i+1} + b} \\ &\quad \times \frac{((x_{-1}x_{-2}x_{-3})^{-1}(1-a)-b)a^{3s+i-1} + b}{((x_{-2}x_{-3}x_{-4})^{-1}(1-a)-b)a^{3s+i} + b}, \end{aligned} \tag{3.5}$$

for $m \geq -1$, $i \in \{1,2,3\}$.

**Case $a = 1$.** From (3.2) and (3.3) we have

$$\begin{aligned} &x_{6m+2i} \\ &= x_{2i-6} \prod_{s=0}^{m} \frac{(x_{-1}x_{-2}x_{-3})^{-1} + b(3s+i)}{(x_{-2}x_{-3}x_{-4})^{-1} + b(3s+i+1)} \frac{(x_{-2}x_{-3}x_{-4})^{-1} + b(3s+i-1)}{(x_{-1}x_{-2}x_{-3})^{-1} + b(3s+i-1)}, \end{aligned} \tag{3.6}$$

for $m \geq -1$, $i \in \{1,2,3\}$, and

$$\begin{aligned} &x_{6m+2i+1} \\ &= x_{2i-5} \prod_{s=0}^{m} \frac{(x_{-2}x_{-3}x_{-4})^{-1} + b(3s+i+1)}{(x_{-1}x_{-2}x_{-3})^{-1} + b(3s+i+1)} \frac{(x_{-1}x_{-2}x_{-3})^{-1} + b(3s+i-1)}{(x_{-2}x_{-3}x_{-4})^{-1} + b(3s+i)}, \end{aligned} \tag{3.7}$$

for $m \geq -1$, $i \in \{1,2,3\}$.

## 4. Long-term behavior of solutions to (3.1)

Before we formulate and prove the main results in this section, we want to introduce the following notation

$$y_{-1} = (x_{-1}x_{-2}x_{-3})^{-1}, \quad y_{-2} = (x_{-2}x_{-3}x_{-4})^{-1},$$

which are consistent with the considerations and notation in the previous section (see the change of variables (2.2)). Set

$$p_m^{2i} = \frac{((y_{-1}(1-a)-b)a^{3m+i} + b)((y_{-2}(1-a)-b)a^{3m+i-1} + b)}{((y_{-2}(1-a)-b)a^{3m+i+1} + b)((y_{-1}(1-a)-b)a^{3m+i-1} + b)} \tag{4.1}$$

and

$$p_m^{2i+1} = \frac{((y_{-2}(1-a)-b)a^{3m+i+1}+b)((y_{-1}(1-a)-b)a^{3m+i-1}+b)}{((y_{-1}(1-a)-b)a^{3m+i+1}+b)((y_{-2}(1-a)-b)a^{3m+i}+b)}, \tag{4.2}$$

for $m \ge -1$ and $i \in \{1,2,3\}$.

**Case $a \neq -1$, $b \neq 0$.** First we describe the long-term behavior of well-defined solution of equation (3.1) for the case $a \neq -1$, $b \neq 0$.

**Theorem 4.1.** *Assume that $a \neq -1$, $b \neq 0$ and $(x_n)_{n\ge -4}$ is a well-defined solution of equation* (3.1)*. Then the following statements are true.*

(a) *If $|a| > 1$, $y_{-1} \neq b/(1-a) \neq y_{-2}$, then $x_n \to 0$ as $n \to +\infty$.*
(b) *If $|a| > 1$, $y_{-1} = b/(1-a) \neq y_{-2}$, then $x_{6m+2i} \to 0$, $i \in \{1,2,3\}$ as $m \to +\infty$.*
(c) *If $|a| > 1$, $y_{-1} = b/(1-a) \neq y_{-2}$, then $|x_{6m+2i+1}| \to \infty$, $i \in \{1,2,3\}$ as $m \to +\infty$.*
(d) *If $|a| > 1$, $y_{-1} \neq b/(1-a) = y_{-2}$, then $|x_{6m+2i}| \to \infty$, $i \in \{1,2,3\}$ as $m \to +\infty$.*
(e) *If $|a| > 1$, $y_{-1} \neq b/(1-a) = y_{-2}$, then $x_{6m+2i+1} \to 0$, $i \in \{1,2,3\}$ as $m \to +\infty$.*
(f) *If $|a| < 1$, then the sequences $(x_{6m+j})_{m\in\mathbb{N}_0}$ converge for every $j = \overline{0,5}$.*
(g) *If $y_{-1} = b/(1-a) = y_{-2}$ or $a = 0$, then $x_{6m+j} = x_{j-6}$, $m \in \mathbb{N}_0$, $j = \overline{2,7}$.*
(h) *If $a = 1$, then $x_n \to 0$ as $n \to +\infty$.*

*Proof.* (a): From (4.1) and (4.2), we have

$$p_m^{2i} = \frac{((y_{-1}(1-a)-b)+(b/a^{3m+i}))((y_{-2}(1-a)-b)+(b/a^{3m+i-1}))}{((y_{-2}(1-a)-b)a+(b/a^{3m+i}))((y_{-1}(1-a)-b)+(b/a^{3m+i-1}))} \to \frac{1}{a}$$

and

$$p_m^{2i+1} = \frac{((y_{-2}(1-a)-b)+(b/a^{3m+i+1}))((y_{-1}(1-a)-b)+(b/a^{3m+i-1}))}{((y_{-1}(1-a)-b)+(b/a^{3m+i+1}))((y_{-2}(1-a)-b)a+(b/a^{3m+i-1}))} \to \frac{1}{a},$$

as $m \to +\infty$, for every $i \in \{1,2,3\}$, which means that

$$\lim_{m\to+\infty} p_m^j = \frac{1}{a}, \tag{4.3}$$

for every $j = \overline{2,7}$. From (3.4), (3.5), (4.3) and the assumption $|a| > 1$, statement (a) follows easily.

(b) and (c): In this case we have

$$p_m^{2i} = \frac{(y_{-2}(1-a)-b)a^{3m+i-1}+b}{(y_{-2}(1-a)-b)a^{3m+i+1}+b} \to \frac{1}{a^2}, \tag{4.4}$$

$$p_m^{2i+1} = \frac{(y_{-2}(1-a)-b)a^{3m+i+1}+b}{(y_{-2}(1-a)-b)a^{3m+i}+b} \to a, \tag{4.5}$$

as $m \to +\infty$, for every $i \in \{1,2,3\}$, From (3.4), (3.5), (4.4), (4.5) and the assumption $|a| > 1$, these two statements follow easily.

(d) and (e): In this case we have

$$p_m^{2i} = \frac{(y_{-1}(1-a)-b)a^{3m+i}+b}{(y_{-1}(1-a)-b)a^{3m+i-1}+b} \to a, \tag{4.6}$$

$$p_m^{2i+1} = \frac{(y_{-1}(1-a)-b)a^{3m+i-1}+b}{(y_{-1}(1-a)-b)a^{3m+i+1}+b} \to \frac{1}{a^2}, \tag{4.7}$$

as $m \to +\infty$, for every $i \in \{1,2,3\}$, From (3.4), (3.5), (4.6), (4.7) and the assumption $|a| > 1$, these two statements follow easily.

(f): Using the asymptotic relation

$$(1+x)^{-1} = 1 - x + O(x^2), \tag{4.8}$$

when $x$ is in a neighborhood of zero, we have

$$\begin{aligned} p_m^{2i} &= \frac{(1+(y_{-1}(1-a)-b)a^{3m+i}/b)(1+(y_{-2}(1-a)-b)a^{3m+i-1}/b)}{(1+(y_{-2}(1-a)-b)a^{3m+i+1}/b)(1+(y_{-1}(1-a)-b)a^{3m+i-1}/b)} \\ &= 1 + \frac{1}{b}\Big((y_{-1}(1-a)-b)\Big(1-\frac{1}{a}\Big) + (y_{-2}(1-a)-b)\Big(\frac{1}{a}-a\Big)\Big)a^{3m+i} \\ &\quad + o(a^{3m}) \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} p_m^{2i+1} &= \frac{(1+(y_{-2}(1-a)-b)a^{3m+i+1}/b)(1+(y_{-1}(1-a)-b)a^{3m+i-1}/b)}{(1+(y_{-1}(1-a)-b)a^{3m+i+1}/b)(1+(y_{-2}(1-a)-b)a^{3m+i}/b)} \\ &= 1 + \frac{1}{b}\Big((y_{-2}(1-a)-b)(a-1) + (y_{-1}(1-a)-b)\Big(\frac{1}{a}-a\Big)\Big)a^{3m+i} \\ &\quad + o(a^{3m}), \end{aligned} \tag{4.10}$$

for every $i \in \{1,2,3\}$ and sufficiently large $m$. From (4.9), (4.10), the assumption $|a| < 1$, and by a known result on the convergence of products the result follows easily.

(g): The result follows from direct calculations and formulas (3.4) and (3.5).

(h): Let

$$\begin{gathered} r_m^{2i} = \frac{y_{-1}+bi+3bm}{y_{-2}+b(i+1)+3bm}\,\frac{y_{-2}+b(i-1)+3bm}{y_{-1}+b(i-1)+3bm}, \\ r_m^{2i+1} = \frac{y_{-2}+b(i+1)+3bm}{y_{-1}+b(i+1)+3bm}\,\frac{y_{-1}+b(i-1)+3bm}{y_{-2}+bi+3bm}, \end{gathered}$$

for $i \in \{1,2,3\}$. Then we have

$$r_m^{2i} = \frac{\Big(1+\frac{y_{-1}+bi}{3bm}\Big)}{\Big(1+\frac{y_{-2}+b(i+1)}{3bm}\Big)}\,\frac{\Big(1+\frac{y_{-2}+b(i-1)}{3bm}\Big)}{\Big(1+\frac{y_{-1}+b(i-1)}{3bm}\Big)} = 1 - \frac{1}{3m} + O\Big(\frac{1}{m^2}\Big) \tag{4.11}$$

and

$$r_m^{2i+1} = \frac{\Big(1+\frac{y_{-2}+b(i+1)}{3bm}\Big)}{\Big(1+\frac{y_{-1}+b(i+1)}{3bm}\Big)}\,\frac{\Big(1+\frac{y_{-1}+b(i-1)}{3bm}\Big)}{\Big(1+\frac{y_{-2}+bi}{3bm}\Big)} = 1 - \frac{1}{3m} + O\Big(\frac{1}{m^2}\Big). \tag{4.12}$$

From (4.11) and (4.12), we have that the products in (3.6), (3.7) are equiconvergent with the product

$$\prod_{j=1}^{n}\Big(1-\frac{1}{3j}+O\Big(\frac{1}{j^2}\Big)\Big),$$

that is, with the sequence

$$\exp\Big(\sum_{j=1}^{n}\ln\Big(1-\frac{1}{3j}+O\Big(\frac{1}{j^2}\Big)\Big)\Big)=\exp\Big(-\frac{1}{3}\sum_{j=1}^{n}\Big(\frac{1}{j}+O\Big(\frac{1}{j^2}\Big)\Big)\Big). \tag{4.13}$$

From (4.13), and the fact that $\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{j}=+\infty$, the statement follows.

**Case $a=-1$, $b\neq 0$.** Here we describe long-term behavior of well-defined solutions of (3.1) for the case $a=-1$, $b\neq 0$, by using the next two formulas

$$x_{6m+2i}=x_{2i-6}\prod_{s=0}^{m}\frac{(2y_{-1}-b)(-1)^{3s+i}+b}{(2y_{-2}-b)(-1)^{3s+i+1}+b}\cdot\frac{(2y_{-2}-b)(-1)^{3s+i-1}+b}{(2y_{-1}-b)(-1)^{3s+i-1}+b}$$

and

$$x_{6m+2i+1}=x_{2i-5}\prod_{s=0}^{m}\frac{(2y_{-2}-b)(-1)^{3s+i+1}+b}{(2y_{-1}-b)(-1)^{3s+i+1}+b}\cdot\frac{(2y_{-1}-b)(-1)^{3s+i-1}+b}{(2y_{-2}-b)(-1)^{3s+i}+b},$$

for $m\geq -1$ and $i\in\{1,2,3\}$, which are obtained from (3.4) and (3.5) with $a=-1$.

Employing these formulas we obtain

$$\begin{aligned} x_{12m+2i}&=x_{2i-6}\prod_{s=0}^{2m}\frac{(2y_{-1}-b)(-1)^{3s+i}+b}{(2y_{-1}-b)(-1)^{3s+i-1}+b}\\ &=x_{2i-6}\frac{(2y_{-1}-b)(-1)^{i}+b}{(2y_{-1}-b)(-1)^{i-1}+b}\prod_{s=0}^{m-1}\frac{b^2-(2y_{-1}-b)^2}{b^2-(2y_{-1}-b)^2}\\ &=x_{2i-6}\frac{(2y_{-1}-b)(-1)^{i}+b}{(2y_{-1}-b)(-1)^{i-1}+b}, \end{aligned} \tag{4.14}$$

$$\begin{aligned} x_{12m+6+2i}&=x_{2i-6}\prod_{s=0}^{2m+1}\frac{(2y_{-1}-b)(-1)^{3s+i}+b}{(2y_{-1}-b)(-1)^{3s+i-1}+b}\\ &=x_{2i-6}\prod_{s=0}^{m}\frac{b^2-(2y_{-1}-b)^2}{b^2-(2y_{-1}-b)^2}\\ &=x_{2i-6}, \end{aligned} \tag{4.15}$$

$$\begin{aligned} x_{12m+2i+1}&=x_{2i-5}\prod_{s=0}^{2m}\frac{(2y_{-2}-b)(-1)^{3s+i+1}+b}{(2y_{-2}-b)(-1)^{3s+i}+b}\\ &=x_{2i-5}\frac{(2y_{-2}-b)(-1)^{i+1}+b}{(2y_{-2}-b)(-1)^{i}+b}\prod_{s=0}^{m-1}\frac{b^2-(2y_{-2}-b)^2}{b^2-(2y_{-2}-b)^2}\\ &=x_{2i-5}\frac{(2y_{-2}-b)(-1)^{i+1}+b}{(2y_{-2}-b)(-1)^{i}+b}, \end{aligned} \tag{4.16}$$

$$\begin{aligned} x_{12m+6+2i+1}&=x_{2i-5}\prod_{s=0}^{2m+1}\frac{(2y_{-2}-b)(-1)^{3s+i+1}+b}{(2y_{-2}-b)(-1)^{3s+i}+b}\\ &=x_{2i-5}\prod_{s=0}^{m}\frac{b^2-(2y_{-2}-b)^2}{b^2-(2y_{-2}-b)^2}\\ &=x_{2i-5} \end{aligned} \tag{4.17}$$

for $m\geq -1$ and $i\in\{1,2,3\}$.

From (4.14)-(4.17) the following theorem follows. □

**Theorem 4.2.** *Assume that $a = -1$, $b \neq 0$. Then every well-defined solution $(x_n)_{n\geq -4}$ of equation* (3.1) *is twelve-periodic and is given by formulas* (4.14)-(4.17).

The twelve-periodicity of every well-defined solution $(x_n)_{n\geq -4}$ of equation (3.1) in the case $a = -1$, $b \neq 0$, can be proved also without calculations in the following way. First note that the sequence

$$y_n = \frac{1}{x_n x_{n-1} x_{n-2}}, \quad n \geq -2,$$

satisfies the recurrence relation

$$y_n = b - y_{n-2}, \quad n \in \mathbb{N}_0,$$

from which it follows that

$$y_n = y_{n-4}, \quad n \geq 2;$$

that is, sequence $(y_n)_{n\geq -2}$ is four-periodic, and consequently the sequence $u_n = 1/y_n$, $n \geq -2$, is also four-periodic. Further, we have

$$x_n = \frac{u_n}{x_{n-1}x_{n-2}} = \frac{u_n}{u_{n-1}} x_{n-3}, \quad n \geq -1. \tag{4.18}$$

By using relation (4.18) four times, we obtain

$$x_n = \frac{u_n}{u_{n-1}} \frac{u_{n-3}}{u_{n-4}} \frac{u_{n-6}}{u_{n-7}} \frac{u_{n-9}}{u_{n-10}} x_{n-12}, \quad n \geq 8.$$

This along with four-periodicity of $(u_n)_{n\geq -2}$ implies twelve-periodicity of $(x_n)_{n\geq -4}$.

**Case $a \neq 0$, $b = 0$.** If $a \neq 0$ and $b = 0$ then equation (3.1) becomes

$$x_n = \frac{x_{n-3}x_{n-4}}{x_{n-1}a}, \quad n \in \mathbb{N}_0,$$

and formulas (3.4)-(3.7) also hold, from which we obtain

$$x_{6m+2i} = \frac{x_{2i-6}}{a^{m+1}},$$

for $m \geq -1$, $i \in \{1, 2, 3\}$, and

$$x_{6m+2i+1} = \frac{x_{2i-5}}{a^{m+1}},$$

for $m \geq -1$, $i \in \{1, 2, 3\}$, which means that

$$x_{6m+j} = \frac{x_{j-6}}{a^{m+1}}, \tag{4.19}$$

for every $m \geq -1$ and $j = \overline{2, 7}$. Using (4.19) we obtain the following theorem.

**Theorem 4.3.** *Assume that $a \neq 0$, $b = 0$, and $(x_n)_{n\geq -4}$ is a well-defined solution of equation* (3.1). *Then the following statements are true.*

(a) *If $|a| > 1$, then $x_n \to 0$ as $n \to +\infty$.*
(b) *If $|a| < 1$, then $|x_n| \to \infty$ as $n \to +\infty$.*
(c) *If $a = 1$, then the sequence $(x_n)_{n\geq -4}$ is six-periodic.*
(d) *If $a = -1$, then the sequence $(x_n)_{n\geq -4}$ is twelve-periodic.*

## 5. Domain of undefinable solutions for (1.3)

We have already shown that solutions of equation (1.3) are not defined if $x_{-j} = 0$ for some $j \in \{1, 2, 3, 4\}$. A natural problem is to describe the set of all initial values for which solutions to equation (1.3) are not defined.

**Definition 5.1** ([34])**.** Consider the difference equation

$$x_n = f(x_{n-1}, \dots, x_{n-s}, n), \quad n \in \mathbb{N}_0, \tag{5.1}$$

where $s \in \mathbb{N}$, and $x_{-i} \in \mathbb{R}$, $i = \overline{1, s}$. The string of numbers $x_{-s}, \dots, x_{-1}, x_0, \dots, x_{n_0}$ where $n_0 \geq -1$, is called an *undefined solution* of equation (5.1) if

$$x_j = f(x_{j-1}, \dots, x_{j-s}, j)$$

for $0 \leq j < n_0 + 1$, and $x_{n_0+1}$ is not a defined number; that is, the quantity $f(x_{n_0}, \dots, x_{n_0-s+1}, n_0 + 1)$ is not defined.

The set of all initial values $x_{-s}, \dots, x_{-1}$ which generate undefined solutions of equation (5.1) is called *domain of undefinable solutions* of the equation. This domain is characterized in the next theorem for the case $a_n \neq 0$, $b_n \neq 0$, $n \in \mathbb{N}_0$.

**Theorem 5.2.** *Assume that $a_n \neq 0$, $b_n \neq 0$, $n \in \mathbb{N}_0$. Then the domain of undefinable solutions of equation* (1.3) *is the set*

$$\begin{aligned} \mathcal{U} = \cup_{m \in \mathbb{N}_0} \cup_{i=0}^{1} \Big\{ (x_{-4}, \dots, x_{-1}) \in \mathbb{R}^4 : x_{i-2} x_{i-3} x_{i-4} = \frac{1}{c_m}, \ \text{where} \\ c_m := -\sum_{j=0}^{m} \frac{b_{2j+i}}{a_{2j+i}} \prod_{l=0}^{j-1} \frac{1}{a_{2l+i}} \neq 0 \Big\} \cup \cup_{j=1}^{4} \Big\{ (x_{-4}, \dots, x_{-1}) \in \mathbb{R}^4 : x_{-j} = 0 \Big\}. \end{aligned} \tag{5.2}$$

*Proof.* The considerations at the beginning of Section 2 show that the domain of undefinable solutions of equation (1.3) contains the set

$$\cup_{j=1}^{4} \big\{ (x_{-4}, \dots, x_{-1}) \in \mathbb{R}^4 : x_{-j} = 0 \big\}.$$

Now assume $x_{-j} \neq 0$, $j = \overline{1, 4}$ (i.e. $x_n \neq 0$ for every $n \geq -4$). If a solution $(x_n)_{n \geq -4}$ with such initial values is not defined then it must be

$$x_{n-2} x_{n-3} x_{n-4} = -\frac{a_n}{b_n} \tag{5.3}$$

for some $n \in \mathbb{N}_0$ (here we use the condition $b_n \neq 0$, $n \in \mathbb{N}_0$).

Now recall that the change of variables (2.2) implies that equation (1.3) is equivalent to the equations in (2.4). Hence, this along with (5.3) implies that solution $(x_n)_{n \geq -4}$ is not defined if

$$y_{2(m-1)+i} = -\frac{b_{2m+i}}{a_{2m+i}}$$

for some $m \in \mathbb{N}_0$ and $i \in \{0, 1\}$. Set

$$f_{2m+i}(t) := a_{2m+i} t + b_{2m+i}, \quad m \in \mathbb{N}_0, \ i \in \{0, 1\}.$$

Then $f_{2m+i}^{-1}(t) = (t - b_{2m+i})/a_{2m+i}$, $m \in \mathbb{N}_0$, $i \in \{0, 1\}$, and specially

$$f_{2m+i}^{-1}(0) = -\frac{b_{2m+i}}{a_{2m+i}}, \quad m \in \mathbb{N}_0, \ i \in \{0, 1\}. \tag{5.4}$$

Now write equations in (2.4) as

$$y_{2m+i} = f_{2m+i}(y_{2(m-1)+i}), \quad m \in \mathbb{N}_0,$$

for $i \in \{0,1\}$. Then, we have

$$y_{2m+i} = f_{2m+i} \circ f_{2(m-1)+i} \circ \cdots \circ f_i(y_{i-2}), \quad m \in \mathbb{N}_0,\ i \in \{0,1\}. \tag{5.5}$$

Equalities (5.4) and (5.5) imply that

$$y_{2(m-1)+i} = -\frac{b_{2m+i}}{a_{2m+i}}$$

for some $m \in \mathbb{N}_0$, $i \in \{0,1\}$, if and only if

$$y_{i-2} = f_i^{-1} \circ \cdots \circ f_{2m+i}^{-1}(0). \tag{5.6}$$

From (5.6) we obtain

$$y_{i-2} = -\sum_{j=0}^{m} \frac{b_{2j+i}}{a_{2j+i}} \prod_{l=0}^{j-1} \frac{1}{a_{2l+i}},$$

for some $m \in \mathbb{N}_0$ and $i \in \{0,1\}$, which along with the relations

$$y_{i-2} = \frac{1}{x_{i-2}x_{i-3}x_{i-4}}, \quad i \in \{0,1\},$$

implies that the first union in (5.2) belongs to the domain of undefinable solutions and consequently the result. □

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