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LIMIT OF MINIMAX VALUES UNDER Γ-CONVERGENCE

MARCO DEGIOVANNI, MARCO MARZOCCHI

Abstract. We consider a sequence of minimax values related to a class of even functionals. We show the continuous dependence of these values under the Γ-convergence of the functionals.

1. Introduction

Let X be a Banach space and $f, g : X \to \mathbb{R}$ two functions of class C^1. Assume also that f and g are even and positively homogeneous of the same degree.

Several results of critical point theory (see [4, 15, 22, 25]) are based on the construction of a sequence of minimax values (c_m) given by

$$c_m = \inf_{K \in \mathcal{K}_s^{(m)}} \max_{u \in K} f(u)\,,$$

where $\mathcal{K}_s^{(m)}$ is the family of compact and symmetric subsets K of

$$\{u \in X : g(u) = 1\}$$

such that $\mathrm{i}(K) \geq m$ and i is a topological index which takes into account the symmetry of f and g. Typical examples are the Krasnosel'skiĭ genus (see e.g. [15, 22, 25]) and the $\mathbb{Z}_2$-cohomological index (see [11, 12]). More general examples are contained in [4].

A natural question concerns the behavior of the minimax values c_m when f and g are substituted by two sequences (f_h) and (g_h) converging in a suitable sense. This problem has been recently treated (see [5, 16, 21] and references therein) in the setting of homogenization problems and limit behavior of the p-Laplace operator.

As pointed out in [5], one has

$$c_m = \inf_{K \in \mathcal{K}} \mathcal{F}^{(m)}(K)\,,$$

where $\mathcal{K}$ is the family of nonempty compact subsets K of X and $\mathcal{F}^{(m)} : \mathcal{K} \to \overline{\mathbb{R}}$ is defined as

$$\mathcal{F}^{(m)}(K) = \begin{cases} \max_{u \in K} f(u) & \text{if } K \in \mathcal{K}_s^{(m)}\,, \\ +\infty & \text{otherwise}\,. \end{cases}$$

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In this way the behavior of minimax values of f is reduced to that of infimum values for the related functionals $\mathcal{F}^{(m)}$ and the convergence of infima has been extensively studied in the setting of Γ-convergence of functionals (see e.g. [3, 7]).

Let us mention that the behavior of critical values under Γ-convergence has been already studied also in [1, 9, 13, 14].

A goal of this article is to answer a question raised in [5, Remark 5.2], concerning the relation between the Γ-convergence of the functionals (f_h) and that of the related functionals $(\mathcal{F}_h^{(m)})$ (see the next Corollaries 4.4 and 6.3). By the way, [5, Remark 5.2] seemed to suggest a negative answer, while we will show that it is affirmative.

In particular, our results allow to treat the convergence of the minimax eigenvalues λ associated to nonlinear problems of the form

$$\begin{gathered} -\Delta_p u = \lambda V_p |u|^{p-2} u \quad \text{in } \Omega\,, \\ u = 0 \quad \text{on } \partial\Omega\,, \end{gathered}$$

where Ω is a (possibly unbounded) open subset on $\mathbb{R}^N$, $1 \leq p < N$ and the weight V_p is possibly indefinite. As usual, in the case $p = 1$ a suitable relaxed interpretation of the problem has to be introduced. For $1 < p < N$ fixed, eigenvalue problems of this kind have been treated in [17, 24]. For $p = 1$ with Ω bounded and $V_1(x) = 1$, we refer the reader to [6, 10, 18, 19, 20].

In Theorem 6.4 we will show the right continuity with respect to p of the minimax eigenvalues. When Ω is bounded and $V_p(x) = 1$, the problem has been already treated in [5, 16, 21].

A related question concerns, for f and g fixed, the dependence of the minimax values on the topology of the space. Actually, in the setting of classical critical point theory the topology is chosen so that f and g are of class C^1, while minimization methods and Γ-convergence techniques prefer weaker topologies in which the sets

$$\{u \in X : f(u) \leq b\,,\ g(u) = 1\}$$

are compact, but then f cannot be continuous.

In Corollary 3.3 we prove, under quite general assumptions, that the minimax values are not affected by a change of topology. Then in Theorem 5.2 we show an application in the setting of functionals of the Calculus of variations.

2. Review on variational convergence

Throughout this section, X will denote a metrizable topological space.

Definition 2.1. Let (f_h) be a sequence of functions from X to $\overline{\mathbb{R}}$. According to [7, Definition 4.1], we define two functions

$$\Big(\Gamma - \liminf_{h\to\infty} f_h\Big) : X \to \overline{\mathbb{R}}\,, \quad \Big(\Gamma - \limsup_{h\to\infty} f_h\Big) : X \to \overline{\mathbb{R}}\,,$$

as

$$\begin{aligned} \Big(\Gamma - \liminf_{h\to\infty} f_h\Big)(u) &= \sup_{U\in\mathcal{N}(u)} \Big[\liminf_{h\to\infty}\big(\inf\{f_h(v) : v \in U\}\big)\Big]\,, \\ \Big(\Gamma - \limsup_{h\to\infty} f_h\Big)(u) &= \sup_{U\in\mathcal{N}(u)} \Big[\limsup_{h\to\infty}\big(\inf\{f_h(v) : v \in U\}\big)\Big]\,, \end{aligned}$$

where $\mathcal{N}(u)$ denotes the family of neighborhoods of u.

If at some $u \in X$ we have

$$\Big(\Gamma - \liminf_{h\to\infty} f_h\Big)(u) = \Big(\Gamma - \limsup_{h\to\infty} f_h\Big)(u)\,,$$

we simply write

$$\Big(\Gamma - \lim_{h\to\infty} f_h\Big)(u)\,.$$

Let us also recall [7, Propositions 8.1 and 7.1].

Proposition 2.2. *The following facts hold:*

(a) *for every $u \in X$ and every sequence (u_h) converging to u in X, it holds*

$$\Big(\Gamma - \liminf_{h\to\infty} f_h\Big)(u) \le \liminf_{h\to\infty} f_h(u_h)\,;$$

(b) *for every $u \in X$ there exists a sequence (u_h) converging to u in X such that*

$$\Big(\Gamma - \liminf_{h\to\infty} f_h\Big)(u) = \liminf_{h\to\infty} f_h(u_h)\,;$$

(c) *for every $u \in X$ and every sequence (u_h) converging to u in X, it holds*

$$\Big(\Gamma - \limsup_{h\to\infty} f_h\Big)(u) \le \limsup_{h\to\infty} f_h(u_h)\,;$$

(d) *for every $u \in X$ there exists a sequence (u_h) converging to u in X such that*

$$\Big(\Gamma - \limsup_{h\to\infty} f_h\Big)(u) = \limsup_{h\to\infty} f_h(u_h)\,;$$

(e) *we have*

$$\inf_X \Big(\Gamma - \limsup_{h\to\infty} f_h\Big) \ge \limsup_{h\to\infty}\Big(\inf_X f_h\Big)\,.$$

Now let us recall from [8, Definition 5.2] a variant of the notion of equicoercivity.

Definition 2.3. A sequence (f_h) of functions from X to $\overline{\mathbb{R}}$ is said to be *asymptotically equicoercive* if, for every strictly increasing sequence (h_n) in $\mathbb{N}$ and every sequence (u_n) in X satisfying

$$\sup_{n\in\mathbb{N}} f_{h_n}(u_n) < +\infty\,,$$

there exists a subsequence (u_{n_j}) converging in X.

The next result is a simple variant of [7, Proposition 7.2]. We prove it for reader's convenience.

Proposition 2.4. *If (f_h) is asymptotically equicoercive, we have*

$$\inf_X \Big(\Gamma - \liminf_{h\to\infty} f_h\Big) \le \liminf_{h\to\infty}\Big(\inf_X f_h\Big)\,.$$

Proof. Without loss of generality, we may assume that

$$\liminf_{h\to\infty}\Big(\inf_X f_h\Big) < +\infty\,.$$

Let

$$b > \liminf_{h\to\infty}\Big(\inf_X f_h\Big)$$

and let (f_{h_n}) be a subsequence such that

$$\sup_{n\in\mathbb{N}}\Big(\inf_X f_{h_n}\Big) < b\,.$$

Let $u_n \in X$ be such that

$$f_{h_n}(u_n) < b\,.$$

Then a subsequence (u_{n_j}) is convergent to some u in X. We infer that

$$\inf_X \Big(\Gamma - \liminf_{h\to\infty} f_h\Big) \le \Big(\Gamma - \liminf_{h\to\infty} f_h\Big)(u) \le \liminf_{j\to\infty} f_{h_{n_j}}(u_{n_j}) \le b$$

and the assertion follows by the arbitrariness of b. □

In the following, we denote by $\mathcal{K}$ be the family of nonempty compact subsets of X. If d is a compatible distance on X, the associated *Hausdorff distance* $d_{\mathcal{H}}$ is defined on $\mathcal{K}$ as

$$d_{\mathcal{H}}(K_1, K_2) = \max\Big\{ \max_{u\in K_1} d(u, K_2)\,,\, \max_{v\in K_2} d(v, K_1)\Big\}\,.$$

The $\mathcal{H}$*-topology* is the topology on $\mathcal{K}$ induced by $d_{\mathcal{H}}$. Recall that the $\mathcal{H}$-topology just depends on the topology of X, not on the distance d. Therefore $\mathcal{K}$ has an intrinsic structure of metrizable topological space.

Proposition 2.5. *Let (f_h) be a sequence of functions from X to $\overline{\mathbb{R}}$ and define $\mathcal{F}_h : \mathcal{K} \to \overline{\mathbb{R}}$ as*

$$\mathcal{F}_h(K) = \sup_K f_h\,.$$

Then (f_h) is asymptotically equicoercive if and only if $(\mathcal{F}_h)$ is asymptotically equicoercive with respect to the $\mathcal{H}$-topology.

Proof. Assume that (f_h) is asymptotically equicoercive and let (h_n) be a strictly increasing sequence in $\mathbb{N}$ and (K_n) a sequence in $\mathcal{K}$ such that

$$\sup_{n\in\mathbb{N}} \mathcal{F}_{h_n}(K_n) < +\infty\,.$$

We claim that $\overline{\cup_{n\in\mathbb{N}} K_n}$ is compact.

Actually, given a compatible distance d on X, let (u_j) be a sequence in this set and let $v_j \in K_{n_j}$ be such that $d(v_j, u_j) \to 0$. Up to a subsequence, either (n_j) is constant or (n_j) is strictly increasing. In the former case it is obvious that (v_j) admits a convergent subsequence, while in the latter case this is due to the asymptotic equicoercivity of (f_h). In any case, (u_j) also admits a convergent subsequence.

By Blaschke's theorem (see e.g. [2, Theorem 4.4.15]) we infer that the image of the sequence (K_n) is included in a compact subset of $\mathcal{K}$ and the assertion follows.

Conversely, assume that $(\mathcal{F}_h)$ is asymptotically equicoercive and let (h_n) and (u_n) be such that

$$\sup_{n\in\mathbb{N}} f_{h_n}(u_n) < +\infty\,.$$

If we set $K_n = \{u_n\}$, then (K_n) is a sequence in $\mathcal{K}$ with

$$\sup_{n\in\mathbb{N}} \mathcal{F}_{h_n}(K_n) < +\infty\,.$$

If (K_{n_j}) is convergent in $\mathcal{K}$, then (u_{n_j}) is convergent in X. □

3. Index theory and minimax values

In this article, we consider an index i with the following properties:

(i) $\mathrm{i}(K)$ is an integer greater or equal than 1 and is defined whenever K is a nonempty, compact and symmetric subset of a topological vector space such that $0 \notin K$;

(ii) if X is a topological vector space and $K \subseteq X \setminus \{0\}$ is compact, symmetric and nonempty, then there exists an open subset U of $X \setminus \{0\}$ such that $K \subseteq U$ and

$$\mathrm{i}(\widehat{K}) \leq \mathrm{i}(K) \text{ for any compact, symmetric and nonempty } \widehat{K} \subseteq U\,;$$

(iii) if X, Y are two topological vector spaces, $K \subseteq X \setminus \{0\}$ is compact, symmetric and nonempty and $\pi : K \to Y \setminus \{0\}$ is continuous and odd, we have

$$\mathrm{i}(\pi(K)) \geq \mathrm{i}(K)\,.$$

Well known examples are the Krasnosel'skiĭ genus (see e.g. [15, 22]) and the $\mathbb{Z}_2$-cohomological index (see [11, 12]). More general examples are contained in [4].

In the following, if X is a topological vector space we will denote by $\mathcal{K}_s$ the family of nonempty, compact and symmetric subsets of $X \setminus \{0\}$.

If X is just a vector space, we denote by $\mathcal{K}_{s,F}$ the family of nonempty, compact and symmetric subsets K of some finite dimensional subspace of X such that $0 \notin K$. Of course, we mean that the subspace is endowed with the unique topology which makes it a topological vector space.

Let us point out a situation in which the behavior of i on $\mathcal{K}_s$ is completely determined by that on $\mathcal{K}_{s,F}$.

Proposition 3.1. *If X is a metrizable and locally convex topological vector space, the following facts hold:*

(a) *for every $K \in \mathcal{K}_s$ and every sequence (K_h) in $\mathcal{K}_s$ converging to K with respect to the $\mathcal{H}$-topology, it holds*

$$\mathrm{i}(K) \geq \limsup_{h\to\infty} \mathrm{i}(K_h)\,;$$

(b) *for every $K \in \mathcal{K}_s$ there exists a sequence (K_h) in $\mathcal{K}_{s,F}$ converging to K with respect to the $\mathcal{H}$-topology such that*

$$\mathrm{i}(K) = \lim_{h\to\infty} \mathrm{i}(K_h)\,.$$

Proof. Assertion (a) easily follows from property (ii) of the index i. To prove (b), consider a compatible distance d on X such that $d(-u,-v) = d(u,v)$ and such that $B_r(u)$ is convex for any $u \in X$ and $r > 0$ (see e.g. [23]).

Given $K \in \mathcal{K}_s$, let $r > 0$ with $K \cap B_r(0) = \emptyset$ and let $F \subseteq K$ be a finite set such that

$$K \subseteq \cup_{v\in F} B_r(v)\,.$$

By substituting F with $F \cup (-F)$, we may assume that F is symmetric. For every $v \in F$, let $\vartheta_v : X \to [0,1]$ be a continuous function such that

$$\vartheta_v(u) = 0 \quad \text{whenever } u \notin B_r(v)\,,$$

$$\sum_{v\in F} \vartheta_v(u) = 1 \quad \text{for all } u \in K\,,$$

$$\sum_{v\in F}\vartheta_v(u)\le 1 \quad \text{for all } u\in X\,,$$
$$\vartheta_{-v}(u)=\vartheta_v(-u) \quad \text{for all } v\in F \text{ and } u\in X\,.$$

Since $0\in \operatorname{conv}(F)$, we can define an odd and continuous map $\pi : X\to \operatorname{conv}(F)$ as

$$\pi(u)=\sum_{v\in F}\vartheta_v(u)\,v\,.$$

For every $u\in K$ and $v\in F$, we have either $\vartheta_v(u)=0$ or $d(v,u)<r$, whence

$$\pi(u)\in \operatorname{conv}(\{v\in F: d(v,u)<r\}) \quad \text{for all } u\in K\,,$$

which implies

$$d(\pi(u),u)<r \quad \text{for all } u\in K\,.$$

In particular, we have $0\notin \pi(K)$, $\pi(K)\in \mathcal{K}_{s,F}$, $d_{\mathcal{H}}(\pi(K),K)<r$ and

$$\mathrm{i}(\pi(K))\ge \mathrm{i}(K)$$

by property (iii) of the index i. Then assertion (b) follows. □

In an equivalent way, one can say that $\mathrm{i}:\mathcal{K}_s\to [1,+\infty[$ is the upper semicontinuous envelope of its restriction to $\mathcal{K}_{s,F}$.

Now let X be a metrizable and locally convex topological vector space and let $f:X\to[0,+\infty]$ and $g:X\setminus\{0\}\to\mathbb{R}$ be two functions such that:

(a) f and g are even and positively homogeneous of degree 1;
(b) f is convex;
(c) for every $b\in\mathbb{R}$, the restriction of g to $\{u\in X\setminus\{0\}: f(u)\le b\}$ is continuous.

For every $m\ge 1$, one can define a minimax value c_m as

$$c_m=\inf_{K\in\mathcal{K}_s^{(m)}}\ \sup_K f\,,$$

where $\mathcal{K}_s^{(m)}$ is the family K's in $\mathcal{K}_s$ such that

$$K\subseteq\{u\in X\setminus\{0\}: g(u)=1\}\,,\quad \mathrm{i}(K)\ge m\,,$$

with the convention

$$\inf_{K\in\mathcal{K}_s^{(m)}}\ \sup_K f=+\infty \quad \text{if } \mathcal{K}_s^{(m)}=\emptyset.$$

One can also consider

$$\inf_{K\in\mathcal{K}_{s,F}^{(m)}}\ \sup_K f\,,$$

where $\mathcal{K}_{s,F}^{(m)}$ is the family K's in $\mathcal{K}_{s,F}$ such that

$$K\subseteq\{u\in X\setminus\{0\}: g(u)=1\}\,,\quad \mathrm{i}(K)\ge m\,,$$

with analogous convention if $\mathcal{K}_{s,F}^{(m)}=\emptyset$.

We aim to show that the two values agree, so that the topology of X plays a role just in assumption (c).

Theorem 3.2. *For every integer $m\ge 1$ we have*

$$\inf_{K\in\mathcal{K}_s^{(m)}}\ \sup_K f=\inf_{K\in\mathcal{K}_{s,F}^{(m)}}\ \sup_K f\,.$$

Proof. Of course, we have

$$\inf_{K\in\mathcal{K}_s^{(m)}} \sup_K f \le \inf_{K\in\mathcal{K}_{s,F}^{(m)}} \sup_K f\,.$$

To prove the converse, let $K \in \mathcal{K}_s^{(m)}$ with

$$\sup_K f < +\infty$$

and let $b \in \mathbb{R}$ with

$$b > \sup_K f\,.$$

Consider a compatible distance d on X as in the proof of Proposition 3.1. By assumption (c) we can find $r > 0$ such that $K \cap B_r(0) = \emptyset$ and

$$\begin{gathered} g(w) > 0\,, \quad \sup_K f < b\, g(w) \\ \text{whenever } w \in X \text{ with } d(w,K) < r \text{ and } f(w) < b\,. \end{gathered} \tag{3.1}$$

Now let F, ϑ_v and π be as in the proof of Proposition 3.1, so that $\pi(K) \in \mathcal{K}_{s,F}$ with $\mathrm{i}(\pi(K)) \ge \mathrm{i}(K) \ge m$ and $d(\pi(u), u) < r$ with

$$\pi(u) \in \mathrm{conv}(\{v \in F : d(v,u) < r\}) \quad \text{for all } u \in K\,.$$

Since f is convex, for every $u \in K$ there exists $v \in F$ such that $d(v,u) < r$ and $f(\pi(u)) \le f(v) < b$, whence $g(\pi(u)) > 0$ and

$$\frac{f(\pi(u))}{g(\pi(u))} \le \frac{f(v)}{g(\pi(u))} < b$$

by (3.1). Since g is even and continuous on $\pi(K)$ by assumption (c), if we set

$$\widehat{K} = \Big\{ \frac{\pi(u)}{g(\pi(u))} : u \in K \Big\}\,,$$

we have $\widehat{K} \in \mathcal{K}_{s,F}^{(m)}$ with

$$\sup_{\widehat{K}} f \le b$$

and the assertion follows by the arbitrariness of b. □

Corollary 3.3. *Under the assumptions of Theorem 3.2, let Y be a vector subspace of X such that*

$$\{u \in X \setminus \{0\} : g(u) > 0 \text{ and } f(u) < +\infty\} \subseteq Y$$

and let τ_Y be any topology on Y which makes Y a metrizable and locally convex topological vector space such that, for every $b \in \mathbb{R}$, the restriction of g to

$$\{u \in Y \setminus \{0\} : f(u) \le b\}$$

is τ_Y-continuous.

Then the minimax values defined in the space Y agree with those defined in the originary space X.

Proof. First of all, there is no change if X is substituted by Y endowed with the topology of X. By Theorem 3.2 it is equivalent to consider the classes $\mathcal{K}_{s,F}^{(m)}$ which do not change, when passing from the topology of X to τ_Y. □

4. Variational convergence of functions and sup-functions

Let X be a metrizable and locally convex topological vector space and, for every $h \in \mathbb{N}$, let $f_h : X \to [0, +\infty]$ and $g_h : X \setminus \{0\} \to \mathbb{R}$ be two functions such that:

(a) f_h and g_h are both even and positively homogeneous of degree 1;
(b) f_h is convex;
(c) for every $b \in \mathbb{R}$, the restriction of g_h to $\{u \in X \setminus \{0\} : f_h(u) \leq b\}$ is continuous.

For any integer $m \geq 1$, denote by $\mathcal{K}_{s,h}^{(m)}$ the family of nonempty, compact and symmetric subsets K of

$$\{u \in X \setminus \{0\} : g_h(u) = 1\}$$

such that $\mathrm{i}(K) \geq m$ and define $\mathcal{F}_h^{(m)} : \mathcal{K} \to [0, +\infty]$ as

$$\mathcal{F}_h^{(m)}(K) = \begin{cases} \sup_K f_h & \text{if } K \in \mathcal{K}_{s,h}^{(m)}\,, \\ +\infty & \text{otherwise}\,. \end{cases}$$

The set $\mathcal{K}$ will be endowed with the $\mathcal{H}$-topology.

Let also $f : X \to [0, +\infty]$ and $g : X \to \mathbb{R}$ be two even functions such that $g(0) = 0$ and define $\mathcal{K}_s^{(m)} \subseteq \mathcal{K}$ and $\mathcal{F}^{(m)} : \mathcal{K} \to [0, +\infty]$ in an analogous way.

Theorem 4.1. *Assume that*

$$f(u) \geq \Big(\Gamma - \limsup_{h\to\infty} f_h\Big)(u) \quad \textit{for all } u \in X$$

and that, for every strictly increasing sequence (h_n) in $\mathbb{N}$ and every sequence (u_n) in $X \setminus \{0\}$ converging to $u \neq 0$ such that

$$\sup_{n\in\mathbb{N}} f_{h_n}(u_n) < +\infty\,,$$

it holds

$$g(u) = \lim_{n\to\infty} g_{h_n}(u_n)\,.$$

Then, for every $m \geq 1$, we have

$$\mathcal{F}^{(m)}(K) \geq \Big(\Gamma - \limsup_{h\to\infty} \mathcal{F}_h^{(m)}\Big)(K) \quad \textit{for all } K \in \mathcal{K}\,,$$

$$\inf_{K\in\mathcal{K}} \mathcal{F}^{(m)}(K) \geq \limsup_{h\to\infty} \Big(\inf_{K\in\mathcal{K}} \mathcal{F}_h^{(m)}(K)\Big)\,,$$

$$\inf_{K\in\mathcal{K}_s^{(m)}} \sup_K f \geq \limsup_{h\to\infty} \Big(\inf_{K\in\mathcal{K}_{s,h}^{(m)}} \sup_K f_h\Big)\,.$$

Proof. Let $m \geq 1$ and let $K \in \mathcal{K}$ with $\mathcal{F}^{(m)}(K) < +\infty$. Then K is a nonempty, compact and symmetric subset of $\{u \in X \setminus \{0\} : g(u) = 1\}$ with $\mathrm{i}(K) \geq m$. Consider a compatible distance d on X as in the proof of Proposition 3.1.

Now, let $b \in \mathbb{R}$ with

$$b > \mathcal{F}^{(m)}(K) = \sup_K f$$

and let $\delta > 0$. Let $\sigma \in]0, 1[$ be such that

$$\sup_K f + \sigma < bs \quad \text{whenever } |s - 1| < \sigma\,, \tag{4.1}$$

$$d\big(s^{-1} w, u\big) < \delta \quad \text{whenever } u \in K,\ w \in X \text{ with } d(w, u) < \sigma \text{ and } |s - 1| < \sigma\,. \tag{4.2}$$

Then let $\overline{h} \in \mathbb{N}$ and $r \in]0, \sigma/2]$ be such that $K \cap B_{2r}(0) = \emptyset$ and

$$|g_h(w) - 1| < \sigma \tag{4.3}$$

for any $h \geq \overline{h}$ and any $w \in X$ with $d(w, K) < 2r$ and $f_h(w) < b + \sigma$.

Again, let F and ϑ_v be as in the proof of Proposition 3.1. Since F is a finite set, by (d) of Proposition 2.2 we can define, for every $h \in \mathbb{N}$, an odd map $\psi_h : F \to X$ such that

$$\lim_{h\to\infty} \psi_h(v) = v \quad \text{for all } v \in F\,,$$

$$f(v) \geq \limsup_{h\to\infty} f_h(\psi_h(v)) \quad \text{for all } v \in F\,.$$

Without loss of generality, we assume that

$$d(\psi_h(v), v) < r \text{ and } f_h(\psi_h(v)) < f(v) + \sigma \text{ for any } h \geq \overline{h} \text{ and } v \in F\,.$$

Then define an odd and continuous map $\pi_h : X \to \mathrm{conv}(\psi_h(F))$ as

$$\pi_h(u) = \sum_{v\in F} \vartheta_v(u)\, \psi_h(v)\,.$$

For every $u \in K$ and $v \in F$, we have either $\vartheta_v(u) = 0$ or $d(v, u) < r$, hence $d(\psi_h(v), u) < 2r$. Therefore,

$$\pi_h(u) \in \mathrm{conv}\left(\{\psi_h(v) : v \in F\,,\ d(\psi_h(v), u) < 2r\}\right) \quad \text{for all } u \in K\,,$$

whence

$$d(\pi_h(u), u) < 2r \leq \sigma \quad \text{for all } h \geq \overline{h} \text{ and } u \in K\,.$$

Moreover, since f_h is convex, for every $u \in K$ there exists $v \in F$ such that $d(\psi_h(v), u) < 2r$ and $f_h(\pi_h(u)) \leq f_h(\psi_h(v)) < f(v) + \sigma$, whence

$$f_h(\pi_h(u)) < b + \sigma \quad \text{for all } h \geq \overline{h} \text{ and } u \in K\,.$$

From (4.3), it follows

$$\pi_h(u) \neq 0 \text{ and } |g_h(\pi_h(u)) - 1| < \sigma \quad \text{for all } h \geq \overline{h} \text{ and } u \in K$$

and $\pi_h(K)$ is a compact and symmetric subset of $X \setminus \{0\}$ with

$$\mathrm{i}(\pi_h(K)) \geq \mathrm{i}(K) \geq m\,.$$

Moreover,

$$\frac{f_h(\pi_h(u))}{g_h(\pi_h(u))} < \frac{f(v) + \sigma}{g_h(\pi_h(u))} < b$$

by (4.1) and g_h is continuous and even on $\pi_h(K)$. If we set

$$K_h = \Big\{\frac{\pi_h(u)}{g_h(\pi_h(u))} : u \in K\Big\}\,,$$

we have $K_h \in \mathcal{K}_{s,h}^{(m)}$ and

$$f_h(w) < b \quad \text{for all } h \geq \overline{h} \text{ and } w \in K_h\,,$$

whence

$$\mathcal{F}_h^{(m)}(K_h) \leq b \quad \text{for all } h \geq \overline{h}\,.$$

Moreover, we have

$$d\left(\frac{\pi_h(u)}{g_h(\pi_h(u))}, u\right) < \delta \quad \text{for all } h \geq \overline{h} \text{ and } u \in K$$

by (4.2) and (4.3), whence

$$d_{\mathcal{H}}\left(K_h, K\right) < \delta \quad \text{for all } h \geq \overline{h}\,.$$

It follows

$$\limsup_{h\to\infty}\left(\inf\left\{\mathcal{F}_h^{(m)}(\widehat{K}) : d_{\mathcal{H}}\left(\widehat{K}, K\right) < \delta\right\}\right) \leq b\,,$$

hence

$$\Bigl(\Gamma - \limsup_{h\to\infty}\mathcal{F}_h^{(m)}\Bigr)(K) \leq b$$

by the arbitrariness of δ. We conclude that

$$\mathcal{F}^{(m)}(K) \geq \Bigl(\Gamma - \limsup_{h\to\infty}\mathcal{F}_h^{(m)}\Bigr)(K)$$

by the arbitrariness of b.

From (e) of Proposition 2.2 we infer that

$$\inf_{K\in\mathcal{K}}\mathcal{F}^{(m)}(K) \geq \limsup_{h\to\infty}\Bigl(\inf_{K\in\mathcal{K}}\mathcal{F}_h^{(m)}(K)\Bigr)$$

and the last assertion is just a reformulation of this fact. □

Theorem 4.2. *Assume that*

$$f(u) \leq \Bigl(\Gamma - \liminf_{h\to\infty} f_h\Bigr)(u) \quad \textit{for all } u \in X$$

and that, for every strictly increasing sequence (h_n) *in* $\mathbb{N}$ *and every sequence* (u_n) *in* $X \setminus \{0\}$ *such that*

$$\sup_{n\in\mathbb{N}} f_{h_n}(u_n) < +\infty\,, \quad \lim_{n\to\infty}(u_n, g_{h_n}(u_n)) = (u, c) \quad \textit{with } c > 0\,,$$

it holds

$$u \neq 0 \textit{ and } g(u) = c\,.$$

Then, for every $m \geq 1$*, we have*

$$\mathcal{F}^{(m)}(K) \leq \Bigl(\Gamma - \liminf_{h\to\infty}\mathcal{F}_h^{(m)}\Bigr)(K) \quad \textit{for all } K \in \mathcal{K}\,.$$

Proof. Let $m \geq 1$, let $K \in \mathcal{K}$ and let (K_h) be a sequence converging to K in $\mathcal{K}$ such that

$$\Bigl(\Gamma - \liminf_{h\to\infty}\mathcal{F}_h^{(m)}\Bigr)(K) = \liminf_{h\to\infty}\mathcal{F}_h^{(m)}(K_h)\,.$$

Without loss of generality, we may assume that this value is not $+\infty$. Let $b \in \mathbb{R}$ with

$$b > \liminf_{h\to\infty}\mathcal{F}_h^{(m)}(K_h)\,.$$

Then there exists a subsequence (K_{h_n}) such that

$$\sup_{n\in\mathbb{N}}\sup_{K_{h_n}} f_{h_n} = \sup_{n\in\mathbb{N}}\mathcal{F}_{h_n}^{(m)}(K_{h_n}) < b\,.$$

In particular, $K_{h_n} \in \mathcal{K}_{s,h_n}^{(m)}$ so that K also is symmetric.

On the other hand, for every $u \in K$, there exists $u_h \in K_h$ with $u_h \to u$. Since $f_{h_n}(u_{h_n}) < b$ and $g_{h_n}(u_{h_n}) = 1$, it follows that

$$f(u) \leq \liminf_{h\to\infty} f_h(u_h) \leq \liminf_{n\to\infty} f_{h_n}(u_{h_n}) \leq b \quad \text{for all } u \in K,$$

$$K \subseteq \{u \in X \setminus \{0\} : g(u) = 1\}\,.$$

Let U be an open subset of $X \setminus \{0\}$ such that $K \subseteq U$ and

$$\mathrm{i}(\widehat{K}) \leq \mathrm{i}(K)$$

for any nonempty, compact and symmetric subset $\widehat{K}$ of U. Since $K_{h_n} \subseteq U$ eventually as $n \to \infty$, we have $\mathrm{i}(K_{h_n}) \leq \mathrm{i}(K)$ eventually as $n \to \infty$, whence $\mathrm{i}(K) \geq m$. Therefore,

$$\mathcal{F}^{(m)}(K) = \sup_K f \leq b \,.$$

By the arbitrariness of b, the assertion follows. □

Corollary 4.3. *Assume that*

$$f(u) \leq \Big(\Gamma - \liminf_{h\to\infty} f_h\Big)(u) \quad \textit{for all } u \in X$$

and that for every strictly increasing sequence (h_n) in $\mathbb{N}$ and every sequence (u_n) in $X \setminus \{0\}$ such that

$$\sup_{n\in\mathbb{N}} f_{h_n}(u_n) < +\infty \,, \qquad \lim_{n\to\infty} g_{h_n}(u_n) = c \quad \textit{with } c > 0 \,,$$

there exists a subsequence (u_{n_j}) such that

$$\lim_{j\to\infty} u_{n_j} = u \quad \textit{with } u \neq 0 \textit{ and } g(u) = c \,.$$

Then, for every $m \geq 1$, the sequence $(\mathcal{F}_h^{(m)})$ is asymptotically equicoercive and

$$\mathcal{F}^{(m)}(K) \leq \Big(\Gamma - \liminf_{h\to\infty} \mathcal{F}_h^{(m)}\Big)(K) \quad \textit{for all } K \in \mathcal{K} \,,$$

$$\inf_{K\in\mathcal{K}} \mathcal{F}^{(m)}(K) \leq \liminf_{h\to\infty} \Big(\inf_{K\in\mathcal{K}} \mathcal{F}_h^{(m)}(K)\Big) \,,$$

$$\inf_{K\in\mathcal{K}_s^{(m)}} \sup_K f \leq \liminf_{h\to\infty} \Big(\inf_{K\in\mathcal{K}_{s,h}^{(m)}} \sup_K f_h\Big) \,.$$

Proof. If we define $\tilde{f}_h : X \to [0,+\infty]$ and $\widetilde{\mathcal{F}}_h : \mathcal{K} \to [0,+\infty]$ as

$$\tilde{f}_h(u) = \begin{cases} f_h(u) & \text{if } g_h(u) = 1 \,, \\ +\infty & \text{otherwise} \,, \end{cases}$$

$$\widetilde{\mathcal{F}}_h(K) = \sup_K \tilde{f}_h \,,$$

it is easily seen that $(\tilde{f}_h)$ is asymptotically equicoercive. By Proposition 2.5 $(\widetilde{\mathcal{F}}_h)$ also is asymptotically equicoercive. In turn, from $\mathcal{F}_h^{(m)} \geq \widetilde{\mathcal{F}}_h$ it follows that $(\mathcal{F}_h^{(m)})$ is asymptotically equicoercive.

From Theorem 4.2 we infer that

$$\mathcal{F}^{(m)}(K) \leq \Big(\Gamma - \liminf_{h\to\infty} \mathcal{F}_h^{(m)}\Big)(K) \quad \text{for all } K \in \mathcal{K}$$

and the other assertions follow from Proposition 2.4. □

Corollary 4.4. *Assume that*

$$f(u) = \Big(\Gamma - \lim_{h\to\infty} f_h\Big)(u) \quad \textit{for all } u \in X$$

and that, for every strictly increasing sequence (h_n) in $\mathbb{N}$ and every sequence (u_n) in $X \setminus \{0\}$ such that

$$\sup_{n\in\mathbb{N}} f_{h_n}(u_n) < +\infty \,,$$

there exists a subsequence (u_{n_j}) *converging to some* u *in* X *with*

$$\lim_{j\to\infty} g_{h_{n_j}}(u_{n_j}) = g(u)\,.$$

Then, for every $m \geq 1$*, the sequence* $(\mathcal{F}_h^{(m)})$ *is asymptotically equicoercive and*

$$\mathcal{F}^{(m)}(K) = \Big(\Gamma - \lim_{h\to\infty} \mathcal{F}_h^{(m)}\Big)(K) \quad \textit{for all } K \in \mathcal{K}\,,$$

$$\inf_{K\in\mathcal{K}} \mathcal{F}^{(m)}(K) = \lim_{h\to\infty}\Big(\inf_{K\in\mathcal{K}} \mathcal{F}_h^{(m)}(K)\Big)\,,$$

$$\inf_{K\in\mathcal{K}_s^{(m)}} \sup_K f = \lim_{h\to\infty}\Big(\inf_{K\in\mathcal{K}_{s,h}^{(m)}} \sup_K f_h\Big)\,.$$

Proof. Since $g(0) = 0$, if (u_{n_j}) is convergent to some u in X with

$$\sup_{n\in\mathbb{N}} f_{h_n}(u_n) < +\infty\,, \qquad \lim_{n\to\infty} g_{h_n}(u_n) = c > 0\,,$$

it follows that $u \neq 0$ and $g(u) = c$. Then the assertion is just a combination of Theorem 4.1 and Corollary 4.3. □

5. Minimax values and functionals of calculus of variations

Throughout this section, Ω denotes an open subset of $\mathbb{R}^N$ with $N \geq 2$ and, for any $q \in [1,\infty]$, $\|\cdot\|_q$ the usual norm in L^q. Since Ω is allowed to be unbounded, for any $p \in]1,N[$ we will consider the Banach space $D_0^{1,p}(\Omega)$ (see e.g. [17]) endowed with the norm

$$\|u\| = \|\nabla u\|_p = \Big(\int_\Omega |\nabla u|^p\,dx\Big)^{1/p}\,.$$

Recall that $D_0^{1,p}(\Omega)$ is continuously embedded in $L^{p^*}(\Omega)$, where $p^* = Np/(N-p)$, and contains $C_c^\infty(\Omega)$ as a dense vector subspace. For any $p \in]1,N[$, define $\mathcal{E}_p : L^1_{\rm loc}(\Omega) \to [0,+\infty]$ as

$$\mathcal{E}_p(u) = \begin{cases} \|\nabla u\|_p & \text{if } u \in D_0^{1,p}(\Omega)\,,\\ +\infty & \text{otherwise}\,.\end{cases}$$

In the case $p = 1$, define first $\widehat{\mathcal{E}}_1 : L^1_{\rm loc}(\Omega) \to [0,+\infty]$ as

$$\widehat{\mathcal{E}}_1(u) = \begin{cases} \int_\Omega |\nabla u|\,dx & \text{if } u \in C_c^1(\Omega)\,,\\ +\infty & \text{otherwise}\,,\end{cases}$$

then denote by $\mathcal{E}_1 : L^1_{\rm loc}(\Omega) \to [0,+\infty]$ the lower semicontinuous envelope of $\widehat{\mathcal{E}}_1$ with respect to the $L^1_{\rm loc}(\Omega)$-topology. If Ω is bounded and has Lipschitz boundary, then $\mathcal{E}_1$ has a well known integral representation (see e.g. [7, Example 3.14]).

In any case, $\mathcal{E}_1$ is convex, even and positively homogeneous of degree 1. Moreover,

$$X_1 = \{u \in L^1_{\rm loc}(\Omega) : \mathcal{E}_1(u) < +\infty\}$$

is a vector subspace of $L^1_{\rm loc}(\Omega)$ and $\mathcal{E}_1$ is a norm on X_1 which makes X_1 a normed space continuously embedded in $L^{1^*}(\Omega) = L^{\frac{N}{N-1}}(\Omega)$.

More precisely, if we set

$$S(N,p) = \inf\Big\{\frac{\int_{\mathbb{R}^N} |\nabla u|^p\,dx}{\big(\int_{\mathbb{R}^N} |u|^{p^*}\,dx\big)^{p/p^*}} : u \in C_c^1(\mathbb{R}^N)\setminus\{0\}\Big\} \quad \text{whenever } 1 \leq p < N\,,$$

then we have

$$\inf_{1\le p\le q} S(N,p) > 0 \quad \text{for all } q \in]1,N[\,,$$

$$S(N,p)^{1/p}\,\|u\|_{p^*} \le \mathcal{E}_p(u) \quad \text{whenever } 1\le p<N \text{ and } \mathcal{E}_p(u)<+\infty\,.$$

It follows easily that, for every $q \in]1,N[$ and $b\in\mathbb{R}$, the set

$$\cup_{1\le p\le q}\big\{u\in L^1_{\rm loc}(\Omega) : \mathcal{E}_p(u)\le b\big\}$$

has compact closure in $L^1_{\rm loc}(\Omega)$.

Now, given $p\in[1,N[$, consider $V_p\in L^{N/p}(\Omega)$. Let $\varrho_p:\mathbb{R}\to\mathbb{R}$ be the odd function such that

$$\varrho_p(s) = s^{1/p} \quad \text{for all } s\ge 0$$

and define $g_p : L^1_{\rm loc}(\Omega)\to\mathbb{R}$ as

$$g_p(u) = \begin{cases} \varrho_p\Big(\int_\Omega V_p\,|u|^p\,dx\Big) & \text{if } u\in L^{p^*}(\Omega)\,,\\ 0 & \text{otherwise}\,.\end{cases} \tag{5.1}$$

Proposition 5.1. *The following facts hold:*

(a) *g_p is even and positively homogeneous of degree 1;*
(b) *for every $b\in\mathbb{R}$, the restriction of g_p to $\big\{u\in L^1_{\rm loc}(\Omega):\mathcal{E}_p(u)\le b\big\}$ is continuous.*

Proof. Assertion (a) is obvious. If (u_n) is convergent to u in $L^1_{\rm loc}(\Omega)$ with $\mathcal{E}_p(u_n)\le b$, then (u_n) is bounded in $L^{p^*}(\Omega)$ and assertion (b) also follows (see also [25, Lemma 2.13]). □

We aim to compare the minimax values with respect to the $L^1_{\rm loc}(\Omega)$-topology with those with respect to a stronger topology. As before, denote by $\mathcal{K}^{(m)}_{s,p}$ the family of compact and symmetric subsets K of

$$\{u\in L^1_{\rm loc}(\Omega) : g_p(u)=1\}$$

such that $\mathrm{i}(K)\ge m$, with respect to the topology of $L^1_{\rm loc}(\Omega)$.

If $1<p<N$, denote also by $\mathcal{V}^{(m)}_p$ the family of compact and symmetric subsets K of

$$\Big\{u\in D^{1,p}_0(\Omega) : \int_\Omega V_p\,|u|^p\,dx = 1\Big\}$$

such that $\mathrm{i}(K)\ge m$, with respect to the norm topology of $D^{1,p}_0(\Omega)$.

If $p=1$, denote by $\mathcal{V}^{(m)}_1$ the family of compact and symmetric subsets K of

$$\Big\{u\in L^{\frac{N}{N-1}}(\Omega) : \int_\Omega V_1\,|u|\,dx = 1\Big\}$$

such that $\mathrm{i}(K)\ge m$, with respect to the norm topology of $L^{\frac{N}{N-1}}(\Omega)$.

Theorem 5.2. *Let $f_p : L^1_{\rm loc}(\Omega)\to[0,+\infty]$ be convex, even and positively homogeneous of degree 1. Moreover, suppose there exists $\nu>0$ such that*

$$f_p(u)\ge \nu\,\mathcal{E}_p(u) \quad \textit{for all } u\in L^1_{\rm loc}(\Omega)\,.$$

Then, for every $m\ge 1$, we have

$$\inf_{K\in\mathcal{K}^{(m)}_{s,p}}\ \sup_K f_p = \inf_{K\in\mathcal{V}^{(m)}_p}\ \sup_K f_p\,.$$

Proof. From Proposition 5.1 and the lower estimate on f_p we infer that, for every $b \in \mathbb{R}$, the restriction of g_p to $\{u \in L^1_{\rm loc}(\Omega) : f_p(u) \leq b\}$ is $L^1_{\rm loc}(\Omega)$-continuous. Of course, the same is true if we consider a stronger topology. Then the assertion follows from Corollary 3.3. $\square$

Now, in view of the convergence results of the next section, let us prove some further basic facts concerning $\mathcal{E}_p$ and g_p. The authors want to thank Lorenzo Brasco for pointing out that a previous version of this theorem was incorrect.

Theorem 5.3. *For every sequence (p_h) decreasing to p in $[1, N[$, we have*

$$\mathcal{E}_p(u) = \Big(\Gamma - \lim_{h\to\infty} \mathcal{E}_{p_h}\Big)(u) \quad \text{for all } u \in L^1_{\rm loc}(\Omega)\,.$$

Proof. Let us prove only the case $p = 1 < p_h$. The other cases are similar and even simpler. Let d be a compatible distance on $L^1_{\rm loc}(\Omega)$ and let $u \in L^1_{\rm loc}(\Omega)$. Let $b \in \mathbb{R}$ with

$$b > \Big(\Gamma - \liminf_{h\to\infty} \mathcal{E}_{p_h}\Big)(u)$$

and let (u_h) be a sequence converging to u in $L^1_{\rm loc}(\Omega)$ such that

$$\Big(\Gamma - \liminf_{h\to\infty} \mathcal{E}_{p_h}\Big)(u) = \liminf_{h\to\infty} \mathcal{E}_{p_h}(u_h)\,.$$

Let $(\mathcal{E}_{p_{h_n}})$ be such that

$$\sup_{n\in\mathbb{N}} \mathcal{E}_{p_{h_n}}(u_{h_n}) < b\,.$$

First of all,

$$\sup_{n\in\mathbb{N}} \int_\Omega |u_{h_n}|^{p^*_{h_n}}\, dx < +\infty\,,$$

so that $u \in L^{\frac{N}{N-1}}(\Omega)$. Let $v_n \in C^1_c(\Omega)$ be such that

$$d(v_n, u_{h_n}) < \frac{1}{n}\,, \quad \mathcal{E}_{p_{h_n}}(v_n) < b\,.$$

Then (v_n) also converges to u in $L^1_{\rm loc}(\Omega)$ and is bounded in $L^{\frac{N}{N-1}}_{\rm loc}(\Omega)$. For every $\vartheta \in C^1_c(\mathbb{R}^N)$ with $0 \leq \vartheta \leq 1$, we have

$$\begin{aligned}
b &> \|\nabla v_n\|_{p_{h_n}} \geq \|\vartheta \nabla v_n\|_{p_{h_n}} \\
&\geq \|\nabla(\vartheta v_n)\|_{p_{h_n}} - \|v_n \nabla\vartheta\|_{p_{h_n}} \\
&\geq \mathcal{L}^n(\operatorname{supp}(\vartheta))^{\frac{1-p_{h_n}}{p_{h_n}}}\, \|\nabla(\vartheta v_n)\|_1 - \|v_n \nabla\vartheta\|_{p_{h_n}} \\
&\geq \mathcal{L}^n(\operatorname{supp}(\vartheta))^{\frac{1-p_{h_n}}{p_{h_n}}}\, \mathcal{E}_1(\vartheta v_n) - \|v_n \nabla\vartheta\|_{p_{h_n}}\,,
\end{aligned}$$

where $\mathcal{L}^n$ denotes the Lebesgue measure. Passing to the lower limit as $n \to \infty$, we obtain

$$b \geq \mathcal{E}_1(\vartheta u) - \|u \nabla\vartheta\|_1\,.$$

Let $\vartheta : \mathbb{R}^N \to [0, 1]$ be a C^1-function such that $\vartheta(x) = 1$ if $|x| \leq 1$ and $\vartheta(x) = 0$ if $|x| \geq 2$ and let $\vartheta_k(x) = \vartheta(x/k)$. Then

$$b \geq \mathcal{E}_1(\vartheta_k u) - \int_\Omega |u|\, |\nabla\vartheta_k|\, dx\,.$$

It is easily seen that $(\vartheta_k u)$ is convergent to u in $L^1_{\rm loc}(\Omega)$, while $(|\nabla\vartheta_k|)$ is bounded in $L^N(\Omega)$ and convergent to 0 a.e. in Ω. Passing to the lower limit as $k \to \infty$, we obtain $b \geq \mathcal{E}_1(u)$, hence

$$\mathcal{E}_1(u) \leq \Big(\Gamma - \liminf_{h\to\infty} \mathcal{E}_{p_h}\Big)(u)$$

by the arbitrariness of b.

Now let $u \in L^1_{\rm loc}(\Omega)$, let $b \in \mathbb{R}$ with $b > \mathcal{E}_1(u)$ and let $\delta > 0$. Let $w \in C^1_c(\Omega)$ with $d(w,u) < \delta$ and $\|\nabla w\|_1 < b$. Then

$$b > \lim_{h\to\infty} \mathcal{E}_{p_h}(w)\,,$$

whence

$$b > \limsup_{h\to\infty}\big(\inf\{\mathcal{E}_{p_h}(v) : d(v,u) < \delta\}\big)\,.$$

By the arbitrariness of δ, it follows that

$$b \geq \Big(\Gamma - \limsup_{h\to\infty} \mathcal{E}_{p_h}\Big)(u)\,,$$

hence

$$\mathcal{E}_1(u) \geq \Big(\Gamma - \limsup_{h\to\infty} \mathcal{E}_{p_h}\Big)(u)$$

by the arbitrariness of b. □

Theorem 5.4. *Let (p_h) be a sequence converging to p in $[1, N[$ and let $V_h \in L^{N/p_h}(\Omega)$ and $V \in L^{N/p}(\Omega)$ be such that*

$$\lim_{h\to\infty} V_h(x) = V(x) \quad \text{for a.e. } x \in \Omega\,,$$
$$\lim_{h\to\infty} \|V_h\|_{N/p_h} = \|V\|_{N/p}\,.$$

Define $g_h, g : L^1_{\rm loc}(\Omega) \to \mathbb{R}$ according to (5.1). Then, for every strictly increasing sequence (h_n) in $\mathbb{N}$ and (u_n) in $L^1_{\rm loc}(\Omega)$ such that

$$\sup_{n\in\mathbb{N}} \mathcal{E}_{p_{h_n}}(u_n) < +\infty\,,$$

there exists a subsequence (u_{n_j}) such that

$$\lim_{j\to\infty} u_{n_j} = u \quad \text{in } L^1_{\rm loc}(\Omega)\,,$$
$$\lim_{j\to\infty} g_{h_{n_j}}(u_{n_j}) = g(u)\,.$$

Proof. Up to a subsequence, (u_n) is convergent to some u in $L^1_{\rm loc}(\Omega)$ and a.e. in Ω. Moreover, for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ independent of n such that

$$\big|V_{h_n}\,|u_n|^{p_{h_n}} - V\,|u|^p\big| \leq C_\varepsilon |V_{h_n}|^{N/p_{h_n}} + \varepsilon |u_n|^{p^*_{h_n}} + |V|\,|u|^p\,,$$

whence

$$C_\varepsilon |V_{h_n}|^{N/p_{h_n}} + \varepsilon |u_n|^{p^*_{h_n}} - \big|V_{h_n}\,|u_n|^{p_{h_n}} - V\,|u|^p\big| \geq -|V|\,|u|^p\,.$$

From Fatou's lemma it follows that

$$\begin{aligned} &C_\varepsilon \int_\Omega |V|^{N/p}\,dx \\ &\leq C_\varepsilon \int_\Omega |V|^{N/p}\,dx + \varepsilon\Big(\sup_{n\in\mathbb{N}} \|u_n\|_{p^*_{h_n}}^{p^*_{h_n}}\Big) - \limsup_{n\to\infty} \int_\Omega \big|V_{h_n}\,|u_n|^{p_{h_n}} - V\,|u|^p\big|\,dx\,, \end{aligned}$$

whence

$$\limsup_{n\to\infty}\int_\Omega \left|V_{h_n}\,|u_n|^{p_{h_n}} - V\,|u|^p\right|\,dx \le \varepsilon\Big(\sup_{n\in\mathbb{N}}\|u_n\|_{p^*_{h_n}}^{p^*_{h_n}}\Big)\,.$$

Since $(\mathcal{E}_{p_{h_n}}(u_n))$ is bounded, we infer that

$$\sup_{n\in\mathbb{N}}\|u_n\|_{p^*_{h_n}}^{p^*_{h_n}} < +\infty$$

and the assertion follows by the arbitrariness of ε. □

6. Convergence of minimax values for functionals of calculus of variations

In this section, Ω still denotes an open subset of $\mathbb{R}^N$ with $N \ge 2$ and, for any $p \in [1, N[$, $\mathcal{E}_p : L^1_{\rm loc}(\Omega) \to [0, +\infty]$ the functional introduced in the previous section.

Assume that (p_h) is a sequence converging to p in $[1, N[$, $f : L^1_{\rm loc}(\Omega) \to [0, +\infty]$ is a functional, (f_h) is a sequence of functionals from $L^1_{\rm loc}(\Omega)$ to $[0, +\infty]$, $V \in L^{N/p}(\Omega)$ and (V_h) is a sequence with $V_h \in L^{N/p_h}(\Omega)$. Also suppose that:

- (H1) f is even;
- (H2) each f_h is convex, even and positively homogeneous of degree 1; moreover, there exists $\nu > 0$ such that
$$f_h(u) \ge \nu\mathcal{E}_{p_h}(u) \quad \text{for all } h \in \mathbb{N} \text{ and } u \in L^1_{\rm loc}(\Omega)\,;$$
- (H3) we have
$$\lim_{h\to\infty} V_h(x) = V(x) \quad \text{for a.e. } x \in \Omega\,,$$
$$\lim_{h\to\infty} \|V_h\|_{N/p_h} = \|V\|_{N/p}\,.$$

Let $\mathcal{K}$ be the family of nonempty compact subsets of $L^1_{\rm loc}(\Omega)$ endowed with the $\mathcal{H}$-topology and define $g_h, g : L^1_{\rm loc}(\Omega) \to \mathbb{R}$ according to (5.1). Then define $\mathcal{K}^{(m)}_{s,h}, \mathcal{K}^{(m)}_s \subseteq \mathcal{K}$ and $\mathcal{F}^{(m)}_h, \mathcal{F}^{(m)} : \mathcal{K} \to [0, +\infty]$ as in Section 4.

Theorem 6.1. *Assume that*

$$f(u) \ge \Big(\Gamma - \limsup_{h\to\infty} f_h\Big)(u) \quad \textit{for all } u \in L^1_{\rm loc}(\Omega)\,.$$

Then, for every $m \ge 1$, we have

$$\mathcal{F}^{(m)}(K) \ge \Big(\Gamma - \limsup_{h\to\infty}\mathcal{F}^{(m)}_h\Big)(K) \quad \textit{for all } K \in \mathcal{K}\,,$$

$$\inf_{K\in\mathcal{K}}\mathcal{F}^{(m)}(K) \ge \limsup_{h\to\infty}\Big(\inf_{K\in\mathcal{K}}\mathcal{F}^{(m)}_h(K)\Big)\,,$$

$$\inf_{K\in\mathcal{K}^{(m)}_s}\sup_K f \ge \limsup_{h\to\infty}\Big(\inf_{K\in\mathcal{K}^{(m)}_{s,h}}\sup_K f_h\Big)\,.$$

The proof of the above theorem follows from Theorem 4.1, Proposition 5.1 and Theorem 5.4.

Theorem 6.2. *Assume that*

$$f(u) \le \Big(\Gamma - \liminf_{h\to\infty} f_h\Big)(u) \quad \textit{for all } u \in L^1_{\rm loc}(\Omega)\,.$$

Then, for every $m \geq 1$, the sequence $(\mathcal{F}_h^{(m)})$ is asymptotically equicoercive and we have

$$\mathcal{F}^{(m)}(K) \leq \Big(\Gamma - \liminf_{h\to\infty} \mathcal{F}_h^{(m)}\Big)(K) \quad \text{for all } K \in \mathcal{K}\,,$$

$$\inf_{K\in\mathcal{K}} \mathcal{F}^{(m)}(K) \leq \liminf_{h\to\infty} \Big(\inf_{K\in\mathcal{K}} \mathcal{F}_h^{(m)}(K)\Big)\,,$$

$$\inf_{K\in\mathcal{K}_s^{(m)}} \sup_K f \leq \liminf_{h\to\infty} \Big(\inf_{K\in\mathcal{K}_{s,h}^{(m)}} \sup_K f_h\Big)\,.$$

The proof of the above theorem follows from Corollary 4.3, Proposition 5.1 and Theorem 5.4.

Corollary 6.3. *Assume that*

$$f(u) = \Big(\Gamma - \lim_{h\to\infty} f_h\Big)(u) \quad \text{for all } u \in L^1_{\rm loc}(\Omega)\,.$$

Then, for every $m \geq 1$, the sequence $(\mathcal{F}_h^{(m)})$ is asymptotically equicoercive and we have

$$\mathcal{F}^{(m)}(K) = \Big(\Gamma - \lim_{h\to\infty} \mathcal{F}_h^{(m)}\Big)(K) \quad \text{for all } K \in \mathcal{K}\,,$$

$$\inf_{K\in\mathcal{K}} \mathcal{F}^{(m)}(K) = \lim_{h\to\infty} \Big(\inf_{K\in\mathcal{K}} \mathcal{F}_h^{(m)}(K)\Big)\,,$$

$$\inf_{K\in\mathcal{K}_s^{(m)}} \sup_K f = \lim_{h\to\infty} \Big(\inf_{K\in\mathcal{K}_{s,h}^{(m)}} \sup_K f_h\Big)\,.$$

The proof of the above corollary follows from Corollary 4.4, Proposition 5.1 and Theorem 5.4.

As an example, whenever $1 \leq p < N$ and $m \geq 1$, consider again $V_p \in L^{N/p}(\Omega)$ and the families $\mathcal{V}_p^{(m)}$ already defined in Section 5. Define

$$\lambda_p^{(m)} = \inf_{K\in\mathcal{V}_p^{(m)}} \sup_{u\in K} \big(\mathcal{E}_p(u)\big)^p\,.$$

In particular, if $1 < p < N$ we have

$$\lambda_p^{(m)} = \inf_{K\in\mathcal{V}_p^{(m)}} \sup_{u\in K} \int_\Omega |\nabla u|^p\,dx\,.$$

Theorem 6.4. *Let (p_h) be a sequence decreasing to p in $[1,N[$ and assume that*

$$\lim_{h\to\infty} V_{p_h}(x) = V_p(x) \quad \text{for a.e. } x \in \Omega\,,$$

$$\lim_{h\to\infty} \|V_{p_h}\|_{N/p_h} = \|V_p\|_{N/p}\,.$$

Then, for every $m \geq 1$, we have $\lim_{h\to\infty} \lambda_{p_h}^{(m)} = \lambda_p^{(m)}$.

Proof. Of course, it is equivalent to show that

$$\lim_{h\to\infty} \Big(\lambda_{p_h}^{(m)}\Big)^{1/p_h} = \Big(\lambda_p^{(m)}\Big)^{1/p}\,.$$

By Theorem 5.2 we get the same values $\lambda_p^{(m)}$ using the $L^1_{\rm loc}(\Omega)$-topology. Then the assertion follows from Corollary 6.3 and Theorem 5.3. □

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References

[1] A. Ambrosetti, C. Sbordone; *Γ-convergenza e G-convergenza per problemi non lineari di tipo ellittico*, Boll. Un. Mat. Ital., A (5), **13** (1976), no. 2, 352–362.

[2] L. Ambrosio, P. Tilli; *Topics on analysis in metric spaces*, Oxford Lecture Series in Mathematics and its Applications, **25**, Oxford University Press, Oxford, 2004.

[3] H. Attouch; *Variational convergence for functions and operators*, Applicable Mathematics Series, Pitman, Boston, 1984.

[4] T. Bartsch; *Topological methods for variational problems with symmetries*, Lecture Notes in Mathematics, **1560**, Springer-Verlag, Berlin, 1993.

[5] T. Champion, L. De Pascale; *Asymptotic behaviour of nonlinear eigenvalue problems involving p-Laplacian-type operators*, Proc. Roy. Soc. Edinburgh Sect. A, **137** (2007), no. 6, 1179–1195.

[6] K. C. Chang; *The spectrum of the 1-Laplace operator*, Commun. Contemp. Math., **11** (2009), no. 5, 865–894.

[7] G. Dal Maso; *An introduction to Γ-convergence*, Progress in Nonlinear Differential Equations and their Applications, **8**, Birkhäuser, Boston, 1993.

[8] M. Degiovanni; *Homotopical properties of a class of nonsmooth functions*, Ann. Mat. Pura Appl., (4) **156** (1990), 37–71.

[9] M. Degiovanni, J. Eisner; *Stability of essential values under Γ-convergence*, Ricerche Mat., **49** (2000), suppl., 319–339.

[10] M. Degiovanni, P. Magrone; *Linking solutions for quasilinear equations at critical growth involving the "1-Laplace" operator*, Calc. Var. Partial Differential Equations, **36** (2009), no. 4, 591–609.

[11] E. R. Fadell, P. H. Rabinowitz; *Bifurcation for odd potential operators and an alternative topological index*, J. Functional Analysis, **26** (1977), no. 1, 48–67.

[12] E. R. Fadell, P. H. Rabinowitz; *Generalized cohomological index theories for Lie group actions with an application to bifurcation questions for Hamiltonian systems*, Invent. Math., **45** (1978), no. 2, 139–174.

[13] A. Groli; *On Γ-convergence for problems of jumping type*, Electron. J. Differential Equations, (2003), No. 60, 16 pp.

[14] R. L. Jerrard, P. Sternberg; *Critical points via Γ-convergence: general theory and applications*, J. Eur. Math. Soc. (JEMS), **11** (2009), no. 4, 705–753.

[15] M. A. Krasnosel'skiĭ; *Topological methods in the theory of nonlinear integral equations*, A Pergamon Press Book, The Macmillan Co., New York, 1964.

[16] S. Littig, F. Schuricht; *Convergence of the eigenvalues of the p-Laplace operator as p goes to 1*, Calc. Var. Partial Differential Equations, **49** (2014), no. 1-2, 707–727.

[17] M. Lucia, F. Schuricht; *A class of degenerate elliptic eigenvalue problems*, Adv. Nonlinear Anal., **2** (2013), no. 1, 91–125.

[18] Z. Milbers, F. Schuricht; *Existence of a sequence of eigensolutions for the 1-Laplace operator*, J. Lond. Math. Soc. (2), **82** (2010), no. 1, 74–88.

[19] Z. Milbers, F. Schuricht; *Some special aspects related to the 1-Laplace operator*, Adv. Calc. Var., **4** (2011), no. 1, 101–126.

[20] Z. Milbers, F. Schuricht; *Necessary condition for eigensolutions of the 1-Laplace operator by means of inner variations*, Math. Ann., **356** (2013), no. 1, 147–177.

[21] E. Parini; *Continuity of the variational eigenvalues of the p-Laplacian with respect to p*, Bull. Aust. Math. Soc., **83** (2011), no. 3, 376–381.

[22] P. H. Rabinowitz; *Minimax methods in critical point theory with applications to differential equations*, CBMS Regional Conference Series in Mathematics, **65**, Published for the Conference Board of the Mathematical Sciences, Washington, 1986.

[23] W. Rudin; *Functional analysis*, International Series in Pure and Applied Mathematics, McGraw-Hill Inc., New York, 1991.

[24] A. Szulkin, M. Willem; *Eigenvalue problems with indefinite weight*, Studia Math., **135** (1999), no. 2, 191–201.

[25] M. Willem; *Minimax theorems*, Progress in Nonlinear Differential Equations and their Applications, **24**, Birkhäuser, Boston, 1996.

Marco Degiovanni
Dipartimento di Matematica e Fisica, Università Cattolica del Sacro Cuore, Via dei Musei 41, 25121 Brescia, Italy
E-mail address: marco.degiovanni@unicatt.it

Marco Marzocchi
Dipartimento di Matematica e Fisica, Università Cattolica del Sacro Cuore, Via dei Musei 41, 25121 Brescia, Italy
E-mail address: marco.marzocchi@unicatt.it