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ANOTHER PROOF OF THE REGULARITY OF HARMONIC MAPS FROM A RIEMANNIAN MANIFOLD TO THE UNIT SPHERE

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Abstract. We shall consider harmonic maps from n-dimensional compact connected Riemannian manifold with boundary to the unit sphere under the Dirichlet boundary condition. We claim that if the Dirichlet data is smooth and so-called "small", all minimizers of the energy functional are also smooth and "small".

1. Introduction

Let (M, g) be a n-dimensional Riemannian manifold with boundary ∂M endowed with a smooth Riemannian metric g. For any $p \in M$, let $(x_1, \dots, x_n)$ be a coordinate system near p. Then g can be represented by

$$g = \sum_{\alpha,\beta=1}^{n} g_{\alpha\beta} dx_\alpha \otimes dx_\beta$$

where $(g_{\alpha\beta})$ is a positive definite symmetric $n \times n$ matrix. We write the inverse matrix of $(g_{\alpha\beta})$ by $(g^{\alpha\beta})$ and the volume element of (M, g) by $dv_g = \sqrt{g}dx$ where $g = \det(g_{\alpha\beta})$, and we use the notations that for any vector fields $\mathbf{u}, \mathbf{v}$, $\langle \mathbf{u}, \mathbf{v}\rangle_g = g(\mathbf{u}, \mathbf{v})$ and $|\mathbf{u}|_g^2 = \langle \mathbf{u}, \mathbf{u}\rangle_g$. We view maps from M into a k-dimensional unit sphere $\mathbb{S}^k \subset \mathbb{R}^{k+1}$, extrinsically. The Sobolev space $W^{1,2}(M, \mathbb{R}^{k+1})$ is standardly defined and the space $W^{1,2}(M, \mathbb{S}^k)$ is defined by

$$W^{1,2}(M, \mathbb{S}^k) = \big\{\mathbf{u} = (u^1, \dots, u^{k+1}) \in W^{1,2}(M, \mathbb{R}^{k+1});\ \mathbf{u}(x) \in \mathbb{S}^k \text{ a.e. } x \in M\big\}.$$

For any $\mathbf{u} \in W^{1,2}(M, \mathbb{S}^k)$, the Dirichlet energy density is defined by

$$e(\mathbf{u}) = \frac{1}{2}|\nabla \mathbf{u}|_g^2 \tag{1.1}$$

where $|\nabla \mathbf{u}|_g^2 = \sum_{i=1}^{k+1} |\nabla u^i|_g^2$. In any local coordinate system $x = (x_1, \dots, x_n)$, we see that

$$e(\mathbf{u}) = \frac{1}{2} \sum_{\alpha,\beta=1}^{n} \sum_{i=1}^{k+1} g^{\alpha\beta} \frac{\partial u^i}{\partial x_\alpha} \frac{\partial u^i}{\partial x_\beta},$$

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and the Dirichlet energy is defined by

$$E(\mathbf{u}, M) = \int_M e(\mathbf{u}) dv_g. \tag{1.2}$$

We say $\mathbf{u} \in W^{1,2}(M, \mathbb{S}^k)$ is weakly harmonic map, if

$$\int_M \sum_{\alpha,\beta=1}^{n} g^{\alpha\beta}\Big(\frac{\partial \mathbf{u}}{\partial x_\alpha} \cdot \frac{\partial \boldsymbol{\phi}}{\partial x_\beta} + \Big(\frac{\partial \mathbf{u}}{\partial x_\alpha} \cdot \frac{\partial \mathbf{u}}{\partial x_\beta}\Big)\mathbf{u} \cdot \boldsymbol{\phi}\Big) dv_g = 0 \tag{1.3}$$

for any $\boldsymbol{\phi} \in C_0^\infty(M, \mathbb{R}^{k+1})$ where $\cdot$ denotes the Euclidean inner product in $\mathbb{R}^{k+1}$. Then $\mathbf{u}$ satisfies the harmonic map equation in the sense of distribution

$$\Delta_g \mathbf{u} + \sum_{\alpha,\beta=1}^{n} g^{\alpha\beta} \frac{\partial \mathbf{u}}{\partial x_\alpha} \cdot \frac{\partial \mathbf{u}}{\partial x_\beta} \mathbf{u} = \mathbf{0} \quad \text{in } M \tag{1.4}$$

where Δ_g is the Laplace-Beltrami operator on (M, g) given by

$$\Delta_g = \frac{1}{\sqrt{g}} \sum_{\alpha,\beta=1}^{n} \frac{\partial}{\partial x_\alpha}\Big(\sqrt{g} g^{\alpha\beta} \frac{\partial}{\partial x_\beta}\Big).$$

Next we say $\mathbf{u} \in W^{1,2}(M, \mathbb{S}^k)$ is a minimizing harmonic map, if for any $\Omega \subset M$,

$$E(\mathbf{u}, \Omega) := \int_\Omega e(\mathbf{u}) dv_g \le E(\mathbf{v}, \Omega) \tag{1.5}$$

for all $\mathbf{v} \in W^{1,2}(\Omega, \mathbb{S}^k)$ with $\mathbf{v}|_{\partial M} = \mathbf{u}|_{\partial M}$.

The regularity of minimizing harmonic maps has been studied by many authors for a general target Riemannian manifold N instead of $\mathbb{S}^k$. For the case where $\dim M = 2$, Morrey [13] showed that if $\mathbf{u} \in W^{1,2}(M, N)$ is a minimizing harmonic map, then $\mathbf{u} \in C^\infty(M, N)$. For $n \ge 3$, Schoen and Uhlenbeck [14] have shown that if we define the singular set of any minimizing map $\mathbf{u} \in W^{1,2}(M, N)$ by

$$\operatorname{sing}(\mathbf{u}) = \{x \in M; \mathbf{u} \text{ is discontinuous at } x\},$$

then $\operatorname{sing}(\mathbf{u})$ is a closed set, and it is discrete for $n = 3$, and

$$\dim_H(\operatorname{sing}(\mathbf{u})) \le n - 3$$

for $n \ge 4$ where $\dim_H(\operatorname{sing}(\mathbf{u}))$ is the Hausdorff dimension of $\operatorname{sing}(\mathbf{u})$. Moreover, it is well known that $\mathbf{u}$ is analytic in $M \setminus \operatorname{sing}(\mathbf{u})$ (cf. Borchers and Garber [5]).

For $p \in N, r > 0$, let $B_r(p) = \{q \in N; \operatorname{dist}_N(q, p) \le r\}$ be the closed geodesic ball with center p and radius r, and let $C(p)$ be the cut locus of p. We call $B_r(p)$ is a regular ball if the following two conditions hold.

(i) $\sqrt{\kappa} r < \pi/2$ where $\kappa = \max\{0, \sup_{B_r(p)} K^N\}$, K^N is the sectional curvature of N.
(ii) $C(p) \cap B_r(p) = \emptyset$.

Hildebrandt et al. [9] have established the following existence theorem of smooth harmonic maps with given boundary data contained in a regular ball. (see also Lin and Wang [12, Theorem 3.1.7]).

Theorem 1.1 ([9]). *Suppose that $B_r(p) \subset N$ is a regular ball and $\Omega \subset M$ is a bounded domain and $\mathbf{g} : \Omega \to B_r(p)$ is continuous map and has finite energy. Then there exists a harmonic map $\mathbf{u} \in C^{2+\alpha}(\Omega, N) \cap C^0(\overline{\Omega}, N)$ with $\mathbf{u}|_{\partial M} = \mathbf{g}$.*

As the first step of their proof, they considered the following variational problem. Find a minimizer of

$$\inf_{\mathbf{u}\in V}\int_\Omega e(\mathbf{u})dv_g$$

where the admissible space V is as follows. Choose $r_1 \in (r, \pi/2\sqrt{\kappa})$ such that $B_{r_1}(p) \subset N$ is also regular ball, and define

$$V = \{\mathbf{u} \in W^{1,2}(\Omega, B_{r_1}(p)); \mathbf{u}|_{\partial M} = \mathbf{g}\}.$$

This admissible space seems to be restrictive. Thus in the present paper, we report that in order to get the same result for the target manifold $N = \mathbb{S}^k$, we can take the admissible space $V = W^{1,2}(M, \mathbb{S}^k, \mathbf{g}) := \{\mathbf{u} \in W^{1,2}(M, \mathbb{S}^k); \mathbf{u}|_{\partial M} = \mathbf{g}\}$.

We note that in the case where $N = \mathbb{S}^k$, since $K^N = 1$ and $C(p) = \{-p\}$, if $0 < r < \pi/2$, then the ball $B_r(p)$ is regular.

2. Preliminaries

Let M be a n-dimensional connected compact Riemannian manifold with smooth boundary ∂M and $\mathbb{S}^k \subset \mathbb{R}^{k+1}$ the unit sphere in $\mathbb{R}^{k+1}$ $(k \geq 2)$. For every $p \in \mathbb{S}^k$ and $r > 0$, we denote the closed geodesic ball in $\mathbb{S}^k$ with center p and radius r by $B_r(p)$. Throughout this paper we treat the $B_r(p)$ which is an closed ball with $0 < r < \pi/2$, so $B_r(p)$ is a regular ball in this case. We denote the standard Sobolev space by $W^{1,2}(\Omega, \mathbb{R}^{k+1})$, and define

$$W^{1,2}(\Omega, \mathbb{S}^k) = \{\mathbf{u} \in W^{1,2}(\Omega, \mathbb{R}^{k+1}); \mathbf{u}(x) \in \mathbb{S}^k \text{ a.e. } x \in M\}.$$

Let $\mathbf{e} : \partial M \to \mathbb{S}^k$ be a smooth given vector field, for instance, $\mathbf{e} \in C^{2+\alpha}(\partial M, \mathbb{S}^k)$, and define

$$W^{1,2}(M, \mathbb{S}^k, \mathbf{e}) = \{\mathbf{u} \in W^{1,2}(M, \mathbb{S}^k); \mathbf{u}|_{\partial M} = \mathbf{e}\}.$$

Here we assume the hypotheses

(H1) $\mathbf{e} \in C^{2+\alpha}(\partial M, \mathbb{S}^k)$ has a finite energy extension $\widetilde{e} \in W^{1,2}(M, \mathbb{S}^k)$ such that $\widetilde{e}|_{\partial M} = e$.

Remark 2.1. It is not trivial that $W^{1,2}(M, \mathbb{S}^k, \mathbf{e}) \neq \emptyset$. However if $M = \Omega \subset \mathbb{R}^n$ is a bounded C^2 domain, Hardt and Lin [8, Theorem 6.2] (cf. [12, Lemma 2.2.10]) have proved the fact in the case where the target space is a more general simply connected Riemannian manifold N (i.e., $\Pi_0(N) = \Pi_1(N) = 0$) that any map $\mathbf{e} \in W^{1/2,2}(\partial M, N)$ admits a finite energy extension $\widetilde{\mathbf{e}} \in W^{1,2}(\Omega, N)$. Recall that $N = \mathbb{S}^k$ has $\Pi_0(\mathbb{S}^k) = \Pi_1(\mathbb{S}^k) = 0$, unless $k = 1$.

$\mathbf{u} \in W^{1,2}(M, \mathbb{S}^k)$ is called weakly harmonic map in the sense of Introduction with boundary data $\mathbf{e}$ if for any $\mathbf{v} \in W_0^{1,2}(M, \mathbb{R}^{k+1})$,

$$\int_M (\langle \nabla\mathbf{u}, \nabla\mathbf{v}\rangle_g - |\nabla\mathbf{u}|_g^2 \mathbf{u}\cdot\mathbf{v})dx = 0, \tag{2.1}$$

and $\mathbf{u}|_{\partial M} = \mathbf{e}$ where

$$\langle \nabla\mathbf{u}, \nabla\mathbf{v}\rangle_g = \sum_{i=1}^{k+1}\langle \nabla u^i, \nabla v^i\rangle_g$$

for $\mathbf{u} = (u^1, \dots, u^{k+1}), \mathbf{v} = (v^1, \dots, v^{k+1})$ and $\mathbf{u}\cdot\mathbf{v}$ is the standard Euclidean inner product. Then $\mathbf{u}$ satisfies the following equations, in the sense of distributions,

$$\begin{gathered} \Delta_g \mathbf{u} + |\nabla\mathbf{u}|_g^2 \mathbf{u} = \mathbf{0} \quad \text{in } M, \\ \mathbf{u} = \mathbf{e} \quad \text{on } \partial M. \end{gathered} \tag{2.2}$$

We also say that $\mathbf{u} \in W^{1,2}(M, \mathbb{S}^k)$ is a minimizing harmonic map with boundary data $\mathbf{e}$ if $\mathbf{u}$ is a minimizer of

$$\inf_{\mathbf{u}\in W^{1,2}(M,\mathbb{S}^k,\mathbf{e})} \int_M |\nabla \mathbf{u}|_g^2 dv_g. \tag{2.3}$$

Lemma 2.2. *Any minimizing harmonic map $\mathbf{u} \in W^{1,2}(M, \mathbb{S}^k)$ is a weakly harmonic map.*

The proof is well known. For example, see [12, Proposition 2.1.5].
We state the main theorem.

Theorem 2.3. *Assume that M is a $C^{2+\alpha}$ connected compact Riemannian manifold with boundary ∂M for some $0 < \alpha < 1$ and assume that a boundary data $\mathbf{e} \in C^{2+\alpha}(\partial M, \mathbb{S}^k)$ satisfying* (H1) *is given, and satisfies that $\mathbf{e}(\partial M) \subset B_r(p)$ for some point $p \in \mathbb{S}^k$ and $0 < r < \pi/2$. Then if $\mathbf{u}$ is any minimizer of*

$$\inf_{\mathbf{u}\in V} \int_M |\nabla \mathbf{u}|_g^2 dv_g$$

where $V = W^{1,2}(M, \mathbb{S}^k, \mathbf{e})$, then $\mathbf{u}(M) \subset B_r(p)$ and $\mathbf{u}$ is a unique harmonic map in $C^{2+\alpha}(M, \mathbb{S}^k)$.

Remark 2.4. *In* [9] *and* [12, Theorem 3.17]*, they took the admissible space V as $V = \{\mathbf{u} \in H^1(M, \mathbb{S}^k, \mathbf{e}); \mathbf{u}(M) \subset B_{r_1}(p)\}$ for some $r < r_1 < \pi/2$, and they call such solution a "small solution". However, we can remove the rather stronger condition. We emphasize that even if we take $W^{1,2}(M, \mathbb{S}^k, \mathbf{e})$ as the admissible space, we can get the same result as* [9]*, and we seem to make more natural. To do so, we shall use the weak Harnack inequality (cf. Gilbarg and Trudinger* [7, Theorem 8.18] *or Chen and Wu* [6, Chapter 4, Lemma 1.3]*) and the maximum principle for minimizing harmonic maps (cf. Jost* [11, Lemma 4.10.1]*). Such strategy also appear in the author's papers Aramaki* [1, 2, 3] *and Aramaki, Chinen, Ito and Ono* [4].

3. Proof of Theorem 2.3

For the proof we need the following lemma which is can be found for example in [12, Proposition 2.1.5].

Lemma 3.1. *Let $V = W^{1,2}(M, \mathbb{S}^k, \mathbf{e})$. Then*

$$\inf_{\mathbf{u}\in V} \int_M |\nabla \mathbf{u}|_g^2 dv_g$$

is achieved in V.

Let $\mathbf{u} \in W^{1,2}(M, \mathbb{S}^k, \mathbf{e})$ be a minimizer of (2.3). Then $\mathbf{u}$ satisfies the Euler-Lagrange equation in the sense of distribution

$$\begin{gathered} -\Delta_g \mathbf{u} = |\nabla \mathbf{u}|_g^2 \mathbf{u} \quad \text{in } M, \\ \mathbf{u} = \mathbf{e} \quad \text{on } \partial M\,. \end{gathered} \tag{3.1}$$

Proposition 3.2. *Let $\mathbf{e} \in C^{2+\alpha}(\partial M, \mathbb{S}^k)$ for some $0 < \alpha < 1$ and assume that $\mathbf{e}(\partial M) \subset B_r(p)$ for some $p \in \mathbb{S}^k$ and $0 < r < \pi/2$. Then for any minimizer $\mathbf{u}$ of* (2.3) *satisfies $\mathbf{u}(\Omega) \subset B_r(p)$.*

Proof. After the rotation of coordinate axis of $\mathbb{R}^{k+1}$, we can choose the center p of $B_r(p)$ so that $p = (1, 0, \dots, 0)$. We write $\mathbf{e}(x) = (e^1(x), \dots, e^{k+1}(x))$. The hypothesis means that $e^1(x) \geq \cos r$ for $x \in \partial M$. Let $\mathbf{u} = (u^1, \dots, u^{k+1})$ be any minimizer of (2.3). Since $u^1 \in W^{1,2}(M)$, it is well known that $|u^1| \in W^{1,2}(M)$ and $|\nabla |u^1|| = |\nabla u^1|$ a.e. in M. Define $\mathbf{w} = (w_1, \dots, w^{k+1}) = (|u^1|, u^2, \dots, u^{k+1}) \in W^{1,2}(M, \mathbb{R}^{k+1})$. Since $u^1 = e^1 > 0$ on ∂M, we can see that $\mathbf{w} \in W^{1,2}(M, \mathbb{S}^k, \mathbf{e})$, and $\mathbf{w}$ is also a minimizer of (2.3). Therefore $\mathbf{w}$ also satisfies (3.1), and $\mathbf{w} \in C^{2+\alpha}$ near the boundary (cf. Schoen and Uhlenbeck [15, Proposition 3.1]. In particular, w^1 satisfies $w^1 \geq 0$ and

$$\begin{gathered} -\Delta_g w^1 = |\nabla \mathbf{w}|_g^2 w^1 \quad \text{in } M, \\ w^1 = e^1 \quad \text{on } \partial M \end{gathered} \tag{3.2}$$

For any $q \in M$, choose a local coordinate neighborhood U_q and a local coordinate system $(x_1, \dots, x_n)$. Then w^1 is a bounded non-negative weak supersolution of

$$\Delta_g = \frac{1}{\sqrt{g}} \sum_{\alpha,\beta=1}^{n} \frac{\partial}{\partial x_\alpha}\Big(\sqrt{g} g^{\alpha\beta} \frac{\partial}{\partial x_\beta}\Big);$$

that is to say, $\Delta_g w^1 \leq 0$ in U_q. We can apply the weak Harnack inequality (cf. [7, Theorem 8.18] or [6, Chapter 4, Lemma 1.3]). Thus for any $1 \leq p < n/(n-2)$, $B_{2R} \subset U_q$

$$\operatorname{ess\,inf}_{B_R} w^1 \geq c\Big(\frac{1}{|B_{2R}|} \int_{B_{2R}} (w^1)^p dx\Big)^{1/p}$$

where $c > 0$ depends on n, p. Since $w^1 \in C^{2+\alpha}$ near the boundary and $w^1 = e^1 \geq \cos r > 0$ on ∂M, there exists $\delta > 0$ such that if we define $M_\delta = \{x \in M; \operatorname{dist}(x, \partial M) \leq \delta\}$, then $w^1 \geq c_0 := \cos r/2$ in M_δ. Since $\dim_H \operatorname{sing}(w^1) \leq n-3$ (in the case where $n = 3$, $\operatorname{sing}(w^1)$ is discrete), for any $x_0 \in M \setminus \operatorname{sing}(w^1)$, we can choose $x_1 \in M_\delta$ and a continuous curve l in M joining x_0 and x_1 such that $l \cap \operatorname{sing}(w^1) = \emptyset$. For every $x \in l$, there exists $R > 0$ such that $B_{2R}(x)$ is contained in a local coordinate neighborhood and

$$\operatorname{ess\,inf}_{B_R(x)} w^1 \geq c\Big(\frac{1}{|B_{2R}(x)|} \int_{B_{2R}(x)} (w^1)^p dx\Big)^{1/p}. \tag{3.3}$$

Since l is compact, there exist finitely many $R_j > 0$ and $x_{(j)} \in l$ $(j = 1, 2, \dots, N)$ such that $\cup_{j=1}^N B_{R_j}(x_{(j)}) \supset l$ and $x_{(1)} = x_0, x_{(N)} = x_1$. Since $\operatorname{ess\,inf}_{B_R(x_{(N)})} w^1 > 0$, it follows from (3.3) that $\operatorname{ess\,inf}_{B_R(x_{(N-1)})} w^1 > 0$. Repeating this procedure, we have $\operatorname{ess\,inf}_{B_R(x_0)} w^1 > 0$. In particular, $w^1(x_0) > 0$. Thus we see that $w^1 > 0$ in $M \setminus \operatorname{sing}(w^1)$. Hence we see that $u^1 > 0$ in $M \setminus \operatorname{sing}(u^1)$ or $u^1 < 0$ in $M \setminus \operatorname{sing}(u^1)$. Since $u^1 = e_1 > 0$ on ∂M, we have $u^1 > 0$ in $M \setminus \operatorname{sing}(u^1)$. Since u^1 is continuous near ∂M, there exist $\delta > 0$ and $c_0 > 0$ such that $u^1 \geq c_0$ on M_δ. Define $M^\delta = \{x \in M; \operatorname{dist}(x, \partial M) \geq \delta\}$. Choose $R > 0$ so that $2R < \delta$ and fix $1 \leq p < n/(n-2)$. For any $y \in M^\delta$, there exists $c' = c'(n,p) > 0$ such that for any $B_{2R}(y)$ contained in a local coordinate neighborhood,

$$\operatorname{ess\,inf}_{B_R(y)} u^1 \geq c'\Big(\frac{1}{|B_{2R}(y)|} \int_{B_{2R}(y)} (u^1)^p dv_g\Big)^{1/p}.$$

Since M^δ is compact, there exists finitely many points y_i and positive numbers R_i $(i = 1, 2, \dots, L)$ such that $\cup_{i=1}^{L} B_{R_i}(y_i) \supset M^\delta$. If we define

$$c_i = c_i' \Big(\frac{1}{|B_{2R_i}(y_i)|} \int_{B_{2R_i}(y_i)} (u^1)^p dx \Big)^{1/p} \quad (i = 1, 2, \dots, L),$$

and $c = \min\{c_0, c_1, \dots, c_L\}$, we have $u^1 \geq c$ a.e. on M. Therefore we can find r' with $r < r' < \pi/2$ such that $n(M) \subset B_{r'}(p)$. $\square$

Next, we use the following maximum principle by Jost.

Lemma 3.3 ([11]). *Let B_0 and B_1 be closed subsets of $\mathbb{S}^k$ and $B_0 \subset B_1$. Suppose that there exists a C^1 retraction map $\Pi : B_1 \to B_0$ satisfying the condition*

$$|\nabla \Pi(x)(\mathbf{v})| < |\mathbf{v}| \quad \textit{for all } x \in B_1 \setminus B_0, \textit{ and all } \mathbf{v} \in T_x\mathbb{S}^k.$$

For any boundary data $\mathbf{e} : \partial M \to B_0$, if $\mathbf{u} \in W^{1,2}(M, \mathbb{S}^k, \mathbf{e}) : M \to B_1$ is an energy minimizing map of (2.3) with the boundary data $\mathbf{e}$, then $\mathbf{u}(x) \in B_0$ a.e. $x \in M$.

We apply this lemma with $B_0 = B_r(p), B_1 = B_{r'}(p)$, we see that $\mathbf{u}(M) \subset B_r(p)$. Then we can see that $\mathbf{u} \in C^{2+\alpha}(M, \mathbb{R}^{k+1})$ by the regularity theory in [14, 15] and [9]. The uniqueness of the solution follows from Jäger and Kaul [10]. This completes the proof.

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