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CONTROLLABILITY FOR SEMILINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS WITHOUT UNIQUENESS

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Abstract. We study the controllability for a class of semilinear control problems in Hilbert spaces, for which the uniqueness is unavailable. Using the fixed point theory for multivalued maps with nonconvex values, we show that the nonlinear problem is approximately controllable provided that the corresponding linear problem is. We also obtain some results on the continuity of solution map and the topological structure of the solution set of the mentioned problem.

1. Introduction

We are concerned with the control problem

$$x'(t) = Ax(t) + F(t, x(t), x_t) + Bu(t), \quad t \in J := [0, T], \tag{1.1}$$

$$x(s) = \varphi(s), \quad s \in [-h, 0] \tag{1.2}$$

where the state function $x(t)$ takes values in a Hilbert space X, $u(t)$ belongs to a Hilbert space V, $F : J \times X \times C([-h, 0]; X) \to X$ is a nonlinear mapping, A is the generator of a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ in X and $B : L^2(J; V) \to L^2(J; X)$ is a bounded linear operator. The history x_t is defined by $x_t(s) := x(t+s)$ for $s \in [-h, 0]$.

The controllability problems governed by differential equations have been presented in a great work of literature. In dealing with control systems involving abstract differential equations in Banach spaces, semigroup theory has been an effective tool. Following this approach, one can find the comprehensive researches in the monographs [1, 3, 8, 12]. To solve nonlinear control problems like (1.1)-(1.2), a typical assumption used in many works (see, e.g. [5, 6, 16, 19]) is that the operator

$$\mathcal{W}u = \int_0^T S(T - s)Bu(s)ds \quad \text{has a pseudo-inverse} \tag{1.3}$$

(see [23] for the definition). This requires $\mathcal{W}$ to be surjective. In [19], the authors pointed out an affirmative example, in which the semigroup $S(\cdot)$ is not compact. However, for many application models, in which X is infinite dimensional and the

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semigroup $S(\cdot)$ is compact (or B is compact), the operator $\mathcal{W}$ is not surjective (see [21, 22] for details). Then (1.3) has been seen as a too strong assumption.

In [18], Naito introduced an approach to study the approximate controllability for control systems in Hilbert spaces. The goal is to find suitable conditions ensuring the positive invariance of reachable set of control systems under nonlinear perturbations; that is

$$\mathcal{R}_T(0) \subset \overline{\mathcal{R}_T(F)}, \tag{1.4}$$

where $\mathcal{R}_T(F)$ and $\mathcal{R}_T(0)$ are the reachable sets of the nonlinear system and corresponding linear system, respectively. Using a similar approach, there were some works, see e.g. [13, 15, 17, 24], that proved the controllability results for numerous semilinear differential and integro-differential equations. A crucial assumption in these works is that, control systems have a unique solution and the solution operator $W(u) = x(\cdot; u)$ is compact. This is usually obtained by the compactness of the semigroup $S(\cdot)$ and the Lipschitz continuity of nonlinearity. Then some well-known fixed point theorems, such as the Schauder fixed point theorem and Banach contraction principle, were employed to demonstrate (1.4).

In this article, we address the case when the uniqueness of (1.1)-(1.2) is unavailable and this turns out that W, in general, becomes a multivalued mapping. This fact leads to a difficulty in proving (1.4), namely, fixed point theory for singlevalued maps does not work in this situation. To overcome the obstacle, we will show that W is an R_δ-map. This property enables us to deploy the fixed point theory for nonconvex valued maps to obtain the inclusion $\mathcal{R}_T(0) \subset \mathcal{R}_T(F)$, that derives the controllability results.

The rest of our paper is as follows. In the next section, we prove the existence result in general case. Specifically, for the nonlinearity, the Lipschitz condition is replaced by a regular assumption expressed by measures of noncompactness. Sect. 3 shows the result describing the structure of the solution set. In fact, the solution set is compact and, moreover, it is an R_δ-set. This enable us to prove the controllability results in Sect. 4. Finally, we give a simple example to show that our approach allows relaxing the Lipschitz condition and uniform boundedness of the nonlinearity.

2. Existence results

In this section, we consider system (1.1)-(1.2) under some regular conditions which guarantee the existence but not provide the uniqueness of solutions. To show the existence results, we make use of the fixed point theory for condensing maps (see e.g. [14]).

Let $\mathcal{E}$ be a Banach space. Denote by $\mathcal{B}(\mathcal{E})$ the collection of nonempty bounded subsets of $\mathcal{E}$. We will use the following definition of measure of noncompactness.

Definition 2.1. Let $(\mathcal{A}, \leq)$ be a partially ordered set. A function $\beta : \mathcal{B}(\mathcal{E}) \to \mathcal{A}$ is called a measure of noncompactness (MNC) in $\mathcal{E}$ if

$$\beta(\overline{\text{co}}\, \Omega) = \beta(\Omega) \quad \text{for } \Omega \in \mathcal{P}(\mathcal{E}),$$

where $\overline{\text{co}}\, \Omega$ is the closure of the convex hull of Ω. An MNC β is called

(i) monotone, if $\Omega_0, \Omega_1 \in \mathcal{P}(\mathcal{E})$, $\Omega_0 \subset \Omega_1$ implies $\beta(\Omega_0) \leq \beta(\Omega_1)$;
(ii) nonsingular, if $\beta(\{a\} \cup \Omega) = \beta(\Omega)$ for any $a \in \mathcal{E}, \Omega \in \mathcal{P}(\mathcal{E})$;
(iii) invariant with respect to union with compact sets, if $\beta(K \cup \Omega) = \beta(\Omega)$ for every relatively compact set $K \subset \mathcal{E}$ and $\Omega \in \mathcal{P}(\mathcal{E})$;

If $\mathcal{A}$ is a cone in a normed space, we say that β is

- (iv) algebraically semi-additive, if $\beta(\Omega_0+\Omega_1) \leq \beta(\Omega_0)+\beta(\Omega_1)$ for any $\Omega_0, \Omega_1 \in \mathcal{P}(\mathcal{E})$;
- (v) regular, if $\beta(\Omega) = 0$ is equivalent to the relative compactness of Ω.

An important example of MNC is *the Hausdorff* MNC $\chi(\cdot)$, which is defined as follows

$$\chi(\Omega) = \inf\{\varepsilon : \Omega \text{ has a finite } \varepsilon\text{-net}\}.$$

We also make use of the following MNCs: for any bounded set $D \in C(J;\mathcal{E})$,

- the modulus of fiber noncompactness of D is given by
$$\gamma_L(D) = \sup_{t\in J} e^{-Lt}\chi(D(t)), \tag{2.1}$$
where L is a positive constant.
- the modulus of equicontinuity of D is
$$\omega_C(D) = \lim_{\delta\to 0} \sup_{y\in D} \max_{|t_1-t_2|<\delta} \|y(t_1) - y(t_2)\|_{\mathcal{E}}; \tag{2.2}$$

As remarked in [14], these MNCs satisfy all properties stated in Definition 2.1, but regularity. Consider now the function

$$\chi^* : \mathcal{B}(C(J;\mathcal{E})) \to \mathbb{R}^2_+, \quad \chi^*(\Omega) = \max_{D\in\Delta(\Omega)} (\gamma_L(D), \omega_C(D)), \tag{2.3}$$

where the MNCs γ_L and ω_C are defined in (2.1) and (2.2) respectively, $\Delta(\Omega)$ is the collection of all countable subsets of Ω and the maximum is taken in the sense of the partial order in the cone $\mathbb{R}^2_+$. By the arguments given in [14], χ^* is well defined. That is, the maximum is achieved in $\Delta(\Omega)$ and χ^* is an MNC in the space $C(J;\mathcal{E})$, which satisfies all properties in Definition 2.1 (see [14, Example 2.1.3] for details).

Definition 2.2. A continuous map $\mathcal{F} : Z \subseteq \mathcal{E} \to \mathcal{E}$ is said to be condensing with respect to an MNC β (β-condensing) if for any bounded set $\Omega \subset Z$, the relation

$$\beta(\Omega) \leq \beta(\mathcal{F}(\Omega))$$

implies the relative compactness of Ω.

Let β be a monotone nonsingular MNC in $\mathcal{E}$. The application of the topological degree theory for condensing maps (see, e.g., [14]) yields the following fixed point principle.

Theorem 2.3. [14, Corollary 3.3.1] *Let $\mathcal{M}$ be a bounded convex closed subset of $\mathcal{E}$ and let $\mathcal{F} : \mathcal{M} \to \mathcal{M}$ be a β-condensing map. Then the fixed point set of $\mathcal{F}$,* $\mathrm{Fix}(\mathcal{F}) := \{x = \mathcal{F}(x)\}$*, is a nonempty compact set.*

Now we consider the nonlinearity $F : J \times X \times C([-h, 0]; X) \to X$ in our problem (1.1)-(1.2). Denote

$$\|\psi\|_h = \|\psi\|_{C([-h,0];X)} := \sup_{\tau\in[-h,0]} \|\psi(\tau)\|,$$

here $\|\cdot\| := \|\cdot\|_X$. We assume that, F satisfies the following:

- (F1) $F(\cdot, \zeta, \psi)$ is measurable for each $\zeta \in X, \psi \in C([-h, 0]; X)$ and $F(t, \cdot, \cdot)$ is continuous for each $t \in J$;

(F2) there exist functions $a, b, c \in L^1(J)$ such that
$$\|F(t,\zeta,\psi)\| \le a(t)\|\zeta\| + b(t)\|\psi\|_h + c(t),$$
for all $(\zeta,\psi) \in X \times C([-h,0];X)$;

(F3) there are nonnegative functions $h, k : J \times J \to \mathbb{R}$ such that $h(t,\cdot), k(t,\cdot) \in L^1(0,t)$ for each $t > 0$ and
$$\chi(S(t-s)F(s,\Omega,\mathcal{Q})) \le h(t,s)\chi(\Omega) + k(t,s) \sup_{-h\le\theta\le 0} \chi(\mathcal{Q}(\theta)),$$
for all bounded subsets $\Omega \subset X$, $\mathcal{Q} \subset C([-h,0];X)$ and for a.e. $t, s \in J$.

Remark 2.4. Let us give a note on assumption (F3). In the case $X = \mathbb{R}^m$, (F3) can be deduced from (F2). That is the locally bounded property implies that the set $S(t-s)F(s,\Omega,\mathcal{Q})$ is bounded in $\mathbb{R}^m$ and then it is precompact for each $t, s \in J$. Especially, if $S(t)$ is compact for $t > 0$ then (F3) is testified obviously with $k = h = 0$.

Definition 2.5. A function $x \in C([-h,T];X)$ is called a mild solution of (1.1)-(1.2) corresponding to control u if
$$x(t) = \begin{cases} \varphi(t), & \text{for } t \in [-h,0], \\ S(t)\varphi(0) + \int_0^t S(t-s)[F(s,x(s),x_s) + Bu(s)]ds, & \text{for } t \in J. \end{cases}$$

Let
$$\mathcal{C}_\varphi = \{y \in C(J;X) :\ y(0) = \varphi(0)\},$$
and for $y \in \mathcal{C}_\varphi$, we set
$$y[\varphi](t) = \begin{cases} \varphi(t), & \text{for } t \in [-h,0], \\ y(t), & \text{for } t \in J. \end{cases}$$
For each $u \in L^2(J;V)$, denote by $\mathcal{F}_u$ the operator acting on $\mathcal{C}_\varphi$ such that
$$\mathcal{F}_u(x)(t) = S(t)\varphi(0) + \int_0^t S(t-s)[F(s,x(s),x[\varphi]_s) + Bu(s)]ds. \tag{2.4}$$
Define a mapping $\mathbf{S} : L^1(J;X) \to C(J;X)$ as
$$\mathbf{S}(f)(t) = \int_0^t S(t-s)f(s)ds. \tag{2.5}$$
Additionally, putting
$$N_F(x)(t) = F(t,x(t),x[\varphi]_t), \tag{2.6}$$
we have
$$\mathcal{F}_u(x) = S(\cdot)\varphi(0) + \mathbf{S}(N_F(x) + Bu).$$
It is evident that $x \in \mathcal{C}_\varphi$ is a fixed point of $\mathcal{F}_u$ if and only if $x[\varphi]$ is a mild solution of (1.1)-(1.2). We have the first property of the solution operator $\mathcal{F}_u$.

Lemma 2.6. *Let* (F1) *and* (F2) *hold. Then $\mathcal{F}_u(\{y_n\})$ is relatively compact for all bounded sequence $\{y_n\} \subset \mathcal{C}_\varphi$ satisfying $\gamma_L(\{y_n\}) = 0$. In particular, we have $\omega_C(\mathcal{F}_u(\{y_n\})) = 0$.*

Proof. We use an assertion that, if $\{f_n\} \subset L^1(J;X)$ is a semicompact sequence, that is

- there exists $q \in L^1(J)$ such that $\|f_n(t)\| \le q(t)$ for all n and for a.e. $t \in J$;
- $\chi(\{f_n(t)\}) = 0$ for a.e. $t \in J$,

then $\mathbf{S}(f_n)$ is relatively compact in $C(J;X)$ (see [14]). Now let $\{y_n\} \subset \mathcal{C}_\varphi$ be a bounded sequence such that $\gamma_L(\{y_n\}) = 0$. Then it follows from (F2) that, $f_n(t) = F(t, y_n(t), y_n[\varphi]_t)$ satisfies the estimate

$$\begin{aligned}\|f_n(t)\| &\le a(t)\|y_n(t)\| + b(t)(\sup_{s\in[0,t]} \|y_n(s)\| + \|\varphi\|_h) + c(t)\\ &\le q(t) := M[a(t) + b(t)] + b(t)\|\varphi\|_h + c(t),\end{aligned}$$

where M is an upper bound for $\{y_n\}$ in $C(J;X)$. As $\gamma_L(\{y_n\}) = 0$ one has $\chi(\{y_n(t)\}) = 0$ for all $t \in J$; i.e. $\{y_n(t)\}$ is relatively compact for each $t \in J$. Then it is readily seen that $\{y_n[\varphi]_t\}$ is a relatively compact set in $C([-h,0];X)$. Since $F(t,\cdot,\cdot)$ is continuous, we get that $\{F(t, y_n(t), y_n[\varphi]_t)\}$ is relatively compact for a.e. $t \in J$. Thus $\{f_n\}$ is a semicompact sequence and then $\mathcal{F}_u(\{y_n\}) = \mathbf{S}(\{f_n\}) + \mathbf{S}(Bu) + S(t)\varphi(0)$ is relatively compact in $C(J;X)$. In particular, $\mathcal{F}_u(\{y_n\})$ is an equicontinuous set, or equivalently, $\omega_C(\mathcal{F}_u(\{y_n\})) = 0$. The proof is complete. $\square$

Now choosing L in the definition of γ_L in (2.1) such that

$$\ell := 2\sup_{t\in J}\int_0^t e^{-L(t-s)}[h(t,s) + k(t,s)]ds < 1, \tag{2.7}$$

we will show that $\mathcal{F}_u$ is χ^*-condensing. To this aim, we need the following assertion, whose proof can be found in [14].

Proposition 2.7. *If $\{w_n\} \subset L^1(J;X)$ such that $\|w_n(t)\| \le \nu(t)$, for a.e. $t \in J$, and for some $\nu \in L^1(J)$, then*

$$\chi(\{\int_0^t w_n(s)ds\}) \le 2\int_0^t \chi(\{w_n(s)\})ds, \quad \text{for } t \in J.$$

Lemma 2.8. *Let* (F1)–(F3) *hold. Then the solution operator $\mathcal{F}_u$ is χ^*-condensing.*

Proof. By (F1)–(F2), it is clear that $\mathcal{F}_u$ is a continuous mapping. Let $\Omega \subset \mathcal{C}_\varphi$ be a bounded set such that

$$\chi^*(\Omega) \le \chi^*(\mathcal{F}_u(\Omega)). \tag{2.8}$$

Then we show that Ω is relatively compact in $C(J;X)$. By the definition of χ^*, there exists a sequence $\{y_n\} \subset \Omega$ such that

$$\chi^*(\mathcal{F}_u(\Omega)) = \Big(\gamma_L(\mathcal{F}_u(\{y_n\})), \omega_C(\mathcal{F}_u(\{y_n\}))\Big) \ge \Big(\gamma_L(\{y_n\}), \omega_C(\{y_n\})\Big). \tag{2.9}$$

We first give an estimate for $\gamma_L(\mathcal{F}_u(\{y_n\}))$. By Proposition 2.7 and (F3), we have

$$\begin{aligned}\chi(\mathcal{F}_u(\{y_n\})(t)) &\le \chi\Big(\{\int_0^t S(t-s)F(t, y_n(s), y_n[\varphi]_s)ds\}\Big)\\ &\le 2\int_0^t \chi(\Big\{S(t-s)F(t, y_n(s), y_n[\varphi]_s)\}\Big)ds\\ &\le 2\int_0^t [h(t,s)\chi(\{y_n(s)\}) + k(t,s)\sup_{\tau\in[-h,0]}\chi(\{y_n[\varphi](s+\tau)\})]ds\\ &\le 2\int_0^t [h(t,s)\chi(\{y_n(s)\}) + k(t,s)\sup_{\rho\in[0,s]}\chi(\{y_n(\rho)\})]ds\\ &\le 2\int_0^t [h(t,s) + k(t,s)]\sup_{\rho\in[0,s]}\chi(\{y_n(\rho)\})ds.\end{aligned}$$

Then

$$\begin{aligned} e^{-Lt}\chi(\mathcal{F}_u(\{y_n\})(t)) &\le 2\int_0^t e^{-L(t-s)}[h(t,s)+k(t,s)]\sup_{\rho\in[0,s]} e^{-L\rho}\chi(\{y_n(\rho)\})ds \\ &\le 2\gamma_L(\{y_n\})\int_0^t e^{-L(t-s)}[h(t,s)+k(t,s)]ds. \end{aligned}$$

The last inequality implies

$$\gamma_L(\mathcal{F}_u(\{y_n\})) \le \ell\gamma_L(\{y_n\}).$$

Taking into account (2.9), we have $\gamma_L(\{y_n\}) \le \ell\gamma_L(\{y_n\})$. Then $\gamma_L(\{y_n\}) = 0$ due to the fact that $\ell < 1$ as chosen in (2.7). This turns out that $\gamma_L(\mathcal{F}_u(\{y_n\})) = 0$ and by Lemma 2.6, we have $\omega_C(\mathcal{F}_u(\{y_n\})) = 0$. Using (2.9) again, we have $\chi^*(\mathcal{F}_u(\Omega)) = 0$. Hence $\chi^*(\Omega) = 0$ due to (2.8). The proof is complete. □

We are in a position to state the existence result.

Theorem 2.9. *Assume hypotheses* (F1)–(F3). *Then the solution set of problem* (1.1)-(1.2) *is nonempty and compact. In addition, any solution of* (1.1)-(1.2) *obeys the following estimate*

$$\sup_{\tau\in[0,t]}\|x(\tau)\|_X \le C_0(M_\varphi + \|Bu\|_{L^1(J;X)})\exp\{C_0\int_0^t[a(s)+b(s)]ds\}, t\in J, \tag{2.10}$$

where $C_0 = \sup_{t\in J}\|S(t)\|_{X\to X}, M_\varphi = \|\varphi(0)\| + \|\varphi\|_h\|b\|_{L^1(J)} + \|c\|_{L^1(J)}$.

Proof. By the hypotheses, the solution operator $\mathcal{F}_u$ is χ^*-condensing due to Lemma 2.8. Let $\xi \in C(J;X)$ be the solution of the integral equation

$$\begin{aligned} \xi(t) &= C_0\|\varphi(0)\| + C_0\|\varphi\|_h\|b\|_{L^1(J)} + C_0\|c\|_{L^1(J)} \\ &\quad + C_0\|Bu\|_{L^1(J;X)} + C_0\int_0^t[a(s)+b(s)]\xi(s)ds, \end{aligned}$$

and

$$\mathcal{M} = \{y\in\mathcal{C}_\varphi : \sup_{s\in[0,t]}\|y(s)\| \le \xi(t), t\in J\},$$

where $C_0 = \sup_{t\in J}\|S(t)\|_{L(X)}$ ($\|\cdot\|_{L(X)}$ stands for operator norm). Then it is easy to check that $\mathcal{M}$ is a bounded, closed and convex set. In addition, if $y\in\mathcal{M}$ then

$$\begin{aligned} &\|\mathcal{F}_u(y)(t)\| \\ &\le \|S(t)\varphi(0)\| + \int_0^t\|S(t-s)[F(s,y(s),y[\varphi]_s)+Bu(s)]\|ds \\ &\le C_0\|\varphi(0)\| + C_0\|Bu\|_{L^1(J;X)} + C_0\int_0^t[a(s)\|y(s)\| + b(s)\|y[\varphi]_s\|_h + c(s)]ds \\ &\le C_0\|\varphi(0)\| + C_0\|Bu\|_{L^1(J;X)} + C_0\|c\|_{L^1(J)} \\ &\quad + C_0\int_0^t[a(s)\|y(s)\| + b(s)(\sup_{\tau\in[0,s]}\|y(\tau)\| + \|\varphi\|_h)]ds \\ &\le C_0\|\varphi(0)\| + C_0\|Bu\|_{L^1(J;X)} + C_0\|c\|_{L^1(J)} + C_0\|\varphi\|_h\|b\|_{L^1(J)} \\ &\quad + C_0\int_0^t[a(s)+b(s)]\sup_{\tau\in[0,s]}\|y(\tau)\|ds \\ &\le C_0\|\varphi(0)\| + C_0\|Bu\|_{L^1(J;X)} + C_0\|c\|_{L^1(J)} + C_0\|\varphi\|_h\|b\|_{L^1(J)} \end{aligned}$$

$$+ C_0 \int_0^t [a(s) + b(s)]\xi(s)ds = \xi(t).$$

Due to the fact that ξ is increasing, we have

$$\|\mathcal{F}_u(y)(\rho)\| \le \xi(\rho) \le \xi(t),$$

for all $0 \le \rho \le t$. Thus $\mathcal{F}_u(y) \in \mathcal{M}$; that is $\mathcal{F}_u(\mathcal{M}) \subset \mathcal{M}$. The application of Theorem 2.3 yields the conclusion of existence result. Now let x be a solution of (1.1)-(1.2) then by the same estimate as that for $\mathcal{F}_u$, we have

$$\begin{aligned} \|x(t)\| &\le C_0\|\varphi(0)\| + C_0\|Bu\|_{L^1(J;X)} + C_0\|c\|_{L^1(J)} + C_0\|\varphi\|_h\|b\|_{L^1(J)} \\ &\quad + C_0 \int_0^t [a(s) + b(s)] \sup_{\tau\in[0,s]} \|x(\tau)\|ds \\ &\le C_0(M_\varphi + \|Bu\|_{L^1(J;X)}) + C_0 \int_0^t [a(s) + b(s)] \sup_{\tau\in[0,s]} \|x(\tau)\|ds. \end{aligned}$$

Since the right hand side is increasing with respect to t, one has

$$\sup_{\tau\in[0,t]} \|x(\tau)\| \le C_0(M_\varphi + \|Bu\|_{L^1(J;X)}) + C_0 \int_0^t [a(s) + b(s)] \sup_{\tau\in[0,s]} \|x(\tau)\|ds.$$

Hence we obtain estimate (2.10) by using the Gronwall inequality. The proof is complete. □

3. Topological structure of the solution set

Let Y and Z be metric spaces. A multi-valued map (multimap) $G : Y \to \mathcal{P}(Z)$ is said to be: (i) *upper semi-continuous (u.s.c.)* if the set

$$G_+^{-1}(V) = \{y \in Y : G(y) \subset V\}$$

is open for any open set $V \subset Z$; (ii) *closed* if its graph $\Gamma_G \subset Y \times Z$,

$$\Gamma_G = \{(y, z) : z \in G(y)\}$$

is a closed subset of $Y \times Z$.

The multimap G is called *quasi-compact* if its restriction to any compact set is compact. The following statement gives a sufficient condition for upper semi-continuity.

Lemma 3.1 ([14]). *Let Y and Z be metric spaces and $G : Y \to \mathcal{P}(Z)$ a closed quasi-compact multimap with compact values. Then G is u.s.c.*

Consider *the solution multimap*

$$W : L^2(J; V) \to \mathcal{P}(C(J; X)), \quad W(u) = \{x : x = \mathcal{F}_u(x)\}. \tag{3.1}$$

We need an additional assumption on the semigroup $S(\cdot)$:

- (A1) The semigroup $S(\cdot)$ generated by A is compact, i.e. $S(t)$ is compact for all $t > 0$.

Proposition 3.2. *Under assumption* (A1)*, the restriction of operator* $\mathbf{S}$*, given by* (2.5)*, on $L^2(J; X)$ is compact; i.e., if $\Omega \subset L^2(J; X)$ is a bounded set, then $\mathbf{S}(\Omega)$ is relatively compact in $C(J; X)$.*

The proof of the above proposition is standard and we omit it. To obtain further properties of the solution multimap W, we justify (F2) as follows

(F2A) The nonlinearity F satisfies (F2) with $a, b, c \in L^2(J)$.

Lemma 3.3. *Under assumptions* (F1), (F2A), (A1)*, the solution multimap* W *defined by* (3.1) *is completely continuous; i.e., it is u.s.c. and sends each bounded set into a relatively compact set.*

Proof. Let $\mathcal{Q} \subset L^2(J;V)$ be a bounded set. We prove that $W(\mathcal{Q})$ is relatively compact in $C(J;X)$. Suppose that $\{x_n\} \subset W(\mathcal{Q})$. Then there exists a sequence $\{u_n\} \subset \mathcal{Q}$ such that

$$x_n(t) = S(t)\varphi(0) + \int_0^t S(t-s)[F(s, x_n(s), x_n[\varphi]_s) + Bu_n(s)]ds. \tag{3.2}$$

Equivalently, we can write

$$x_n(t) = S(t)\varphi(0) + \mathbf{S}(f_n + Bu_n)(t), \tag{3.3}$$

where $\mathbf{S}$ is the operator defined in (2.5), $f_n(t) = F(t, x_n(t), x_n[\varphi]_t)$. We observe that $\{Bu_n\}$ is a bounded set in $L^2(J;X)$ since B is a bounded linear operator. This implies, by Proposition 3.2, that $\{\mathbf{S}(Bu_n)\}$ is relatively compact in $C(J;X)$. On the other hand, by some standard estimates we can obtain that $\{x_n\}$ is a bounded sequence in $C(J;X)$. This together with (F2A) deduce that $\{f_n\}$ is also bounded in $L^2(J;X)$ and one ensures that $\{\mathbf{S}(f_n + Bu_n)\}$ is compact. In view of (3.3), we conclude that $\{x_n\}$ is compact as well.

To prove that W is u.s.c., it remains to show, according to Lemma 3.1, that W has a closed graph. Let $u_n \to u$ in $L^2(J;V)$ and $x_n \in W(u_n), x_n \to x$ in $C(J;X)$. We claim that $x \in W(u)$. Indeed, one has

$$x_n(t) = S(t)\varphi(0) + \int_0^t S(t-s)[F(s, x_n(s), x_n[\varphi]_s) + Bu_n(s)]ds. \tag{3.4}$$

Since $F(t,\cdot,\cdot)$ is a continuous function, we have $f_n(s) = F(s, x_n(s), x_n[\varphi]_s)$ converging to $f(s) = F(s, x(s), x[\varphi]_s)$ for a.e. $s \in J$. Due to the fact that $\{f_n\}$ is integrably bounded, the Lebesgue dominated convergence theorem implies

$$f_n - f \to 0 \quad \text{in } L^1(J;X).$$

In addition, since B is bounded, one can assert that

$$Bu_n - Bu \to 0 \quad \text{in } L^1(J;X).$$

Therefore, taking (3.4) into account, we arrive at

$$x(t) = S(t)\varphi(0) + \int_0^t S(t-s)[F(s, x(s), x[\varphi]_s) + Bu(s)]ds, t \geq 0.$$

The proof is complete. □

Let us recall some notions which will be used in the sequel.

Definition 3.4. A subset B of a metric space Y is said to be contractible in Y if the inclusion map $i_B : B \to Y$ is null-homotopic; i.e., there exists $y_0 \in Y$ and a continuous map $h : B \times [0,1] \to Y$ such that $h(y,0) = y$ and $h(y,1) = y_0$ for every $y \in B$.

Definition 3.5. A subset B of a metric space Y is called an R_δ-set if B can be represented as the intersection of decreasing sequence of compact contractible sets.

A multimap $G : X \to \mathcal{P}(Y)$ is said to be an R_δ-map if G is u.s.c. and for each $x \in X$, $G(x)$ is an R_δ-set in Y. Every single-valued continuous map can be seen as an R_δ-map. The following lemma gives us a condition for a set of being R_δ.

Lemma 3.6 ([4]). *Let X be a metric space, E a Banach space and $g : X \to E$ a proper map; i.e., g is continuous and $g^{-1}(K)$ is compact for each compact set $K \subset E$. If there exists a sequence $\{g_n\}$ of mappings from X into E such that*

(1) *g_n is proper and $\{g_n\}$ converges to g uniformly on X;*
(2) *for a given point $y_0 \in E$ and for all y in a neighborhood $\mathcal{N}(y_0)$ of y_0 in E, there exists exactly one solution x_n of the equation $g_n(x) = y$.*

Then $g^{-1}(y_0)$ is an R_δ-set.

To use this Lemma, we need the following result, which is called *the Lasota-Yorke Approximation Theorem* (see e.g., [9]).

Lemma 3.7. *Let E be a normed space and $f : X \to E$ a continuous map. Then for each $\epsilon > 0$, there is a locally Lipschitz map $f_\epsilon : X \to E$ such that:*

$$\|f_\epsilon(x) - f(x)\|_E < \epsilon$$

for each $x \in X$.

The following theorem is the main result in this section.

Theorem 3.8. *Assume the hypotheses of Lemma 3.3. Then for each $u \in L^2(J;V)$, $W(u)$ is an R_δ-set.*

Proof. Since the nonlinearity $F(t,\cdot,\cdot)$ in our problem is continuous, according to Lemma 3.7, one can take a sequence $\{F_n\}$ such that, $F_n(t,\cdot,\cdot)$ are locally Lipschitz functions and

$$\|F_n(t,\zeta,\psi) - F(t,\zeta,\psi)\| < \epsilon_n,$$

for a.e. $t \in J$ and for all $\zeta \in X, \psi \in C([-h,0];X)$, where $\epsilon_n \to 0$ as $n \to \infty$. Without loss of generality, we can assume that

$$\|F_n(t,\zeta,\psi)\| \le a(t)\|\zeta\| + b(t)\|\psi\|_h + c(t) + 1,$$

for all n. Consider the equation

$$x(t) = y^*(t) + \int_0^t S(t-s)[F_n(s,x(s),x[\varphi]_s) + Bu(s)]ds. \tag{3.5}$$

Using the same arguments as in the previous section, one obtains the existence result for (3.5). In addition, since $F_n(t,\cdot,\cdot)$ is locally Lipschitz, the solution of (3.5) is unique. Let

$$\mathcal{G}(x) = (I - \mathcal{F}_u)(x),$$
$$\mathcal{F}_{un}(x) = S(t)\varphi(0) + \int_0^t S(t-s)[F_n(s,x(s),x[\varphi]_s) + Bu(s)]ds,$$
$$\mathcal{G}_n(x)(t) = (I - \mathcal{F}_{un})(x).$$

Then one claims that the maps $\mathcal{G}$ and $\mathcal{G}_n$ are proper. Indeed, we will prove this assertion, e.g., for $\mathcal{G}$. Let us show that $\mathcal{G}^{-1}(K)$ is a compact set for each compact set $K \subset C(J;X)$. Assume that

$$(I - \mathcal{F}^u)(D) = K$$

and $\{x_n\} \subset D$ is any sequence. Then there exists a sequence $\{y_n\} \subset K$ such that

$$x_n - \mathcal{F}^u(x_n) = y_n.$$

That is,

$$x_n(t) = S(t)\varphi(0) + y_n(t) + \int_0^t S(t-s)[f_n(s) + Bu(s)]ds, \tag{3.6}$$

where $f_n(s) = F(s, x_n(s), x_n[\varphi]_s), s \in J$.

Using (F2A) and the fact that $\{y_n\}$ is bounded in $C(J;X)$, we see that $\{x_n\}$ is also bounded in $C(J;X)$. Then $\{f_n\}$, in turn, is bounded in $L^2(J;X)$. Thus $\{\mathbf{S}(f_n))\}$ is compact according to Proposition 3.2. Thus $\{x_n\}$ is relatively compact and therefore D is a compact set.

On the other hand, $\{\mathcal{G}_n\}$ converges to $\mathcal{G}$ uniformly in $C(J;X)$ and equation $\mathcal{G}_n(x) = y$ has a unique solution for each $y \in \mathcal{C}_\varphi$ since it is in the form of (3.5). Therefore, applying Lemma 3.6 we conclude that

$$W(u) = \mathcal{G}^{-1}(0)$$

is an R_δ-set. The proof is complete. □

4. Controllability results

Following the arguments in Section 2, the problem

$$y'(t) = Ay(t) + F(t, y(t), y_t) + z(t),\ t \in J, \tag{4.1}$$

$$y(s) = \varphi(s),\ s \in [-h, 0] \tag{4.2}$$

has at least one mild solution $y = y(\cdot; z)$, for each $z \in L^2(J;X)$. Furthermore, by the same arguments in Section 3, we see that the solution map

$$W(z) = \{y(\cdot; z) : \text{ the solution of (4.1)-(4.2)}\}$$

is an R_δ-map. Moreover, in view of (**F1**)-(F2), the map N_F defined by (2.6) is continuous. Therefore, N_F is also an R_δ-map.

Define a linear operator $\mathcal{S}_T : L^2(J;X) \to X$ by

$$\mathcal{S}_T(v) = \int_0^T S(T-s)v(s)ds.$$

Denoting $\mathcal{N} = \{v \in L^2(J;X) : \mathcal{S}_T v = 0\}$, one has that $\mathcal{N}$ is a closed subspace of $L^2(J;X)$. Let $\mathcal{N}^\perp$ be the orthogonal space of $\mathcal{N}$ in $L^2(J;X)$ and Q be the projection from $L^2(J;X)$ into $\mathcal{N}^\perp$. Let $R[B]$ be the range of B, we make use of the following assumption

(B1) For any $p \in L^2(J;X)$, there exists $q \in R[B]$ such that

$$\mathcal{S}_T(p) = \mathcal{S}_T(q).$$

This assumption implies that $\{y + \mathcal{N}\} \cap R[B] \neq \emptyset$ for any $y \in \mathcal{N}^\perp$. Hence, by the proof of [18, Lemma 1], the mapping P from $\mathcal{N}^\perp$ to $R[B]$ by

$$Py = \Big\{y^* : y^* \in \{y + \mathcal{N}\} \cap R[B], \text{and} \\ \|y^*\|_{L^2(J;X)} = \min\{\|x\|_{L^2(J;X)} : x \in \{y + \mathcal{N}\} \cap R[B]\}\Big\}$$

is well-defined. Moreover, P is linear and bounded.

Remark 4.1. We take assumption (B1) as the same one in [20]; i.e., it requires $q \in R[B]$, which is slightly stronger than that in [18] ($q \in \overline{R[B]}$). In fact, this requirement is necessary for our arguments when the solution to control system is not unique. Moreover, in application, (B1) is much easier to verify than that in [18]. In related works (e.g., [17, 18, 20, 24]), the authors checked (B1) only.

For given $u_0 \in L^2(J;V)$, we construct the operator $\mathcal{K} : \mathcal{N}^\perp \to \mathcal{P}(\mathcal{N}^\perp)$ given by

$$\mathcal{K}v = QBu_0 - QN_F WPv. \tag{4.3}$$

We will show that $\mathcal{K}$ has a fixed point. For this purpose, we need the following notions and facts in the sequel.

Definition 4.2. Let Y be a metric space.

(1) Y is called an absolute retract (AR-space) if for any metric space X and any closed $A \subset X$, every continuous function $f : A \to Y$ extends to a continuous $\tilde{f} : X \to Y$.
(2) Y is called an absolute neighborhood retract (ANR-space) if for any metric space X, any closed $A \subset X$ and continuous $f : A \to Y$, there exists a neighborhood $U \supset A$ and a continuous extension $\tilde{f} : U \to Y$ of f.

Obviously, if Y is an AR-space then Y is an ANR-space.

Proposition 4.3 ([7])**.** *Let C be a convex set in a locally convex linear space Y. Then C is an AR-space.*

In particular, the last proposition states that every Banach space and its convex subsets are AR-spaces. The following theorem is the main tool for this section. For related results on fixed point theory for ANR-spaces, we refer the reader to [9, 11].

Theorem 4.4 ([10, Corollary 4.3])**.** *Let Y be an AR-space. Assume that $\phi : Y \to \mathcal{P}(Y)$ can be factorized as*

$$\phi = \phi_n \circ \phi_{n-1} \circ \dots \circ \phi_1$$

where

$$\phi_i : Y_{i-1} \to \mathcal{P}(Y_i),\ i = 1, .., N$$

are R_δ-maps and $Y_i, i = 1, .., N-1$ are ANR-spaces, $Y_0 = Y_N = Y$ are AR-spaces. If there is a compact set D such that $\phi(Y) \subset D \subset Y$ then ϕ has a fixed point.

Using this theorem, we get the following result.

Theorem 4.5. *Let* (F1), (F2A), (A1) *hold. Assume that* (B1) *takes place. Then the operator $\mathcal{K}$ defined in* (4.3) *has a fixed point in $\mathcal{N}^\perp$ provided that*

$$C_0\sqrt{T}\|P\|\|a + b\|_{L^2(J;X)} e^{C_0\|a+b\|_{L^1(J;X)}} < 1. \tag{4.4}$$

Proof. The operator $\mathcal{K}$ can be factorized as

$$\mathcal{K} = \mathcal{T} \circ Q \circ N_F \circ W \circ P$$

where $\mathcal{T}(v) = QBu_0 - v$, a single-valued and continuous mapping. It is easy to see that all component in above presentation is R_δ-map. Therefore, in order to use Theorem 4.4, it suffices to show that there exists a convex subset $D \subset \mathcal{N}^\perp$ such that $\mathcal{K}(D) \subset D$ and $\mathcal{K}(D)$ is a compact set. We look for $R > 0$ such that $\|\mathcal{K}(v)\|_{L^2(J;X)} \le R$ provided $\|v\|_{L^2(J;X)} \le R$ and then take

$$D = \overline{B}_R \cap \mathcal{N}^\perp, \tag{4.5}$$

thanks to the fact that $\mathcal{N}^\perp$ is a convex subset of $L^2(J;X)$. For $y \in W(Pv)$, it follows from Theorem 2.9 that

$$\sup_{\tau\in[0,t]} \|y(\tau)\| \le C_0(M_\varphi + \|Pv\|_{L^1(J;X)}) \exp\{C_0 \int_0^t [a(s)+b(s)]ds\} \tag{4.6}$$

Now using (F2) and (4.6), we have

$$\begin{aligned}
\|N_F(y)(t)\| &= \|F(t,y(t),y_t)\| \le a(t)\|y(t)\| + b(t)\|y_t\|_h + c(t)\\
&\le [a(t)+b(t)] \sup_{\tau\in[0,t]} \|y(\tau)\| + b(t)\|\varphi\|_h + c(t)\\
&\le C_0\Big(M_\varphi + \|Pv\|_{L^1(J;X)}\Big)[a(t)+b(t)] \exp\{C_0 \int_0^t [a(s)+b(s)]ds\}\\
&\quad + b(t)\|\varphi\|_h + c(t).
\end{aligned}$$

Taking into account that $\|Q\| \le 1$, for any $z \in \mathcal{K}(v)$ we get

$$\begin{aligned}
\|z(t)\| \le \|Bu_0(t)\| &+ C_0\Big(M_\varphi + \|Pv\|_{L^1(J;X)}\Big)[a(t)+b(t)]\\
&\times \exp\{C_0 \int_0^t [a(s)+b(s)]ds\} + b(t)\|\varphi\|_h + c(t).
\end{aligned}$$

This implies that

$$\begin{aligned}
\|z\|_{L^2(J;X)} &\le \|Bu_0\|_{L^2(J;X)}\\
&\quad + C_0\Big(M_\varphi + \sqrt{T}\|P\|\|v\|_{L^2(J;X)}\Big)\|a+b\|_{L^2(J;X)} e^{C_0\|a+b\|_{L^1(J;X)}} \\
&\quad + \|b\|_{L^2(J;X)}\|\varphi\|_h + \|c\|_{L^2(J;X)}.
\end{aligned} \tag{4.7}$$

Thanks to assumption (4.4), (4.7) ensures the existence of a number $R > 0$ such that $\|z\|_{L^2(J;X)} \le R$ provided $\|v\|_{L^2(J;X)} \le R$. That is, $\mathcal{K}(D) \subset D$ with the closed bounded subset D denoted in (4.5). By Lemma 3.3 the set $W \circ P(D)$ is compact and then $K = \mathcal{K}(D)$ is a compact set. Thus we get the desired conclusion. □

Remark 4.6. If the nonlinearity F is uniformly bounded with respect to the second and third arguments; that is

$$\|F(t,\xi,\eta)\| \le c(t), \quad \text{for a.e. } t \in J \text{ and all } \xi \in X,\ \eta \in C([-h,0];X)$$

then condition (4.4) can be relaxed since in this case $a = b = 0$.

Let

$$\mathcal{R}_T(F) = \{x(T;u) : u \in L^2(J;V)\},$$

the set of all terminal state of solutions to the system (1.1)-(1.2). The set $\mathcal{R}_T(F)$ is called the *reachable set* of the control system (1.1)-(1.2). When $F = 0$, the notation $\mathcal{R}_T(0)$ stands for the reachable set of the corresponding linear system.

Definition 4.7. The control system (1.1)-(1.2) is said to be exactly controllable (or controllable) if $\mathcal{R}_T(F) = X$. It is called approximately controllable if $\overline{\mathcal{R}_T(F)} = X$.

It is shown in [18, Lemma 2] that, by hypothesis (B1) one has $\overline{\mathcal{R}_T(0)} = X$. That is, (B1) is a sufficient condition for the approximate controllability of the linear system corresponding to (1.1)-(1.2). One can find in [2, 13] for some other conditions. The following theorem is our main result in this section.

Theorem 4.8. *Under the hypotheses of Theorem 4.5, the control system* (1.1)-(1.2) *is approximately controllable if the corresponding linear system is.*

Proof. We show that $\mathcal{R}_T(0) \subset \mathcal{R}_T(F)$. Let $y_0 \in \mathcal{R}_T(0)$, then there exists $u_0 \in L^2(J;V)$ such that

$$y_0 = S(T)\varphi(0) + \mathcal{S}_T Bu_0.$$

Let v^* be a fixed point of $\mathcal{K}$, then we have

$$QBu_0 = QN_F WPv^* + v^*. \tag{4.8}$$

By the definition of P, it is evident that $Pv^* \in \{v^* + \mathcal{N}\} \cap R[B]$, and then

$$\mathcal{S}_T Pv^* = \mathcal{S}_T v^*. \tag{4.9}$$

On the other hand, Q is the projection from $L^2(J;X)$ into $\mathcal{N}^\perp$, then

$$\mathcal{S}_T Qp = \mathcal{S}_T p, \text{ for all } p \in L^2(J;X). \tag{4.10}$$

Combining (4.8)-(4.10) yields

$$\mathcal{S}_T Bu_0 = \mathcal{S}_T(f + Pv^*)$$

where $f \in N_F WPv^*$. Therefore,

$$y_0 = S(T)\varphi(0) + \mathcal{S}_T Bu_0 = S(T)\varphi(0) + \mathcal{S}_T(f + Pv^*) = y(T; Pv^*)$$

where y is a solution of (4.1)-(4.2). Since $Pv^* \in R[B]$, there exists a function $u \in L^2(J;V)$ such that $Pv^* = Bu$. Then we have $y(\cdot; Pv^*) = y(\cdot; Bu) = x(\cdot; u)$, where x is a mild solution of the system (1.1)-(1.2). This implies that $\mathcal{R}_T(0) \subset \mathcal{R}_T(F)$ and the proof is complete. □

An example. We end this note with an application to the control system involving a semilinear partial differential equation

$$\frac{\partial y}{\partial t}(t,x) = \frac{\partial^2 y}{\partial x^2}(t,x) + f(t, y(t,x), y(t-h,x)) + Bu(t,x), \ x \in [0,\pi], \quad t \in J, \tag{4.11}$$

$$y(t,0) = y(t,\pi) = 0, \quad t \in J, \tag{4.12}$$

$$y(s,x) = \varphi(s,x), \quad s \in [-h,0], \ x \in [0,\pi], \tag{4.13}$$

where $y, u \in C(J; L^2(0,\pi))$ is the state function and the control function, respectively. The feature in this example is that, the nonlinearity has neither Lipschitz property nor uniform boundedness, in comparison with the existing results in literature.

The operator $A = \frac{d^2}{dx^2}$, with the domain $D(A) = H^2(0,\pi) \cap H_0^1(0,\pi)$, is the infinitesimal generator of a compact semigroup $S(\cdot)$ in $L^2(0,\pi)$. Here $H^2(0,\pi)$ and $H_0^1(0,\pi)$ are usual Sobolev spaces. The expression for $S(\cdot)$ is as follows:

$$(S(t)h)(x) = \sum_{n=1}^{\infty} e^{-n^2 t}\Big(\frac{2}{\pi}\int_0^\pi h(x)\sin nx dx\Big)\sin nx,$$

for $h \in L^2(0,\pi)$. We select B as in [18], that is the intercept operator $B_{\alpha,T}$ is

$$B_{\alpha,T} v(t) = \begin{cases} 0, & 0 \le t < \alpha, \\ v(t), & \alpha \le t \le T, \end{cases}$$

where $v \in L^2(0,T; L^2(0,\pi))$. It is known that $B = B_{\alpha,T}$ satisfies (B1). Then the linear system

$$\frac{\partial y}{\partial t}(t,x) = \frac{\partial^2 y}{\partial x^2}(t,x) + Bu(t,x), \quad x \in [0,\pi], t \in J,$$

$$y(t,0) = y(t,\pi) = 0, \quad t \in J,$$
$$y(0,x) = y_0(x), \quad x \in [0,\pi],$$

is approximately controllable. That is $\overline{\mathcal{R}_T(0)} = L^2(0,\pi)$. As far as nonlinearity f is concerned, we assume that:

(N1) $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous;
(N2) there exist functions $a, b, c \in L^2(J)$ such that
$$|f(t,\xi,\eta)| \le a(t)|\xi| + b(t)|\eta| + c(t),$$
for all $t \in J, \xi, \eta \in \mathbb{R}$.

Applying the abstract results in previous sections, we conclude that the control system (4.11)-(4.13) is approximately controllable in $L^2(0,\pi)$ provided inequality (4.4) holds. Furthermore, the solution set depends upper-semicontinuously in control function u and it is an R_δ-set.

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