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MULTIPLE POSITIVE SOLUTIONS FOR SINGULAR MULTI-POINT BOUNDARY-VALUE PROBLEMS WITH A POSITIVE PARAMETER

CHAN-GYUN KIM, EUN KYOUNG LEE

ABSTRACT. In this article we study the existence, nonexistence, and multiplicity of positive solutions for a singular multi-point boundary value problem with positive parameter. We use the fixed point index theory on a cone and a well-known theorem for the existence of a global continuum of solutions to establish our results.

1. INTRODUCTION

Consider the singular multi-point boundary-value problem

$$(\varphi_p(u'(t)))' + \lambda f(t,u(t)) = 0, \quad t \in (0,1), \tag{1.1}$$

$$u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i), \tag{1.2}$$

where $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, λ a nonnegative real parameter, $\xi_i \in (0,1)$ with $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < 1$, $a_i, b_i \in [0,1)$ with $0 \le \sum_{i=1}^{m-2} a_i < 1$, $0 \le \sum_{i=1}^{m-2} b_i < 1$, and $f \in C((0,1) \times [0,\infty), (0,\infty))$. Here, $f(t,u)$ may be singular at $t = 0$ and/or 1 and satisfies the following conditions:

(F1) for all $M > 0$, there exists $h_M \in \mathcal{A}$ such that $f(t,u) \le h_M(t)$, for all $u \in [0,M]$ and all $t \in (0,1)$, where

$$\mathcal{A} = \{h : \int_0^{1/2} \varphi_p^{-1}\Big(\int_s^{1/2} h(\tau)d\tau\Big)ds + \int_{1/2}^1 \varphi_p^{-1}\Big(\int_{1/2}^s h(\tau)d\tau\Big)ds < \infty\} :$$

(F2) there exists $[\alpha,\beta] \subset (0,1)$ such that $\lim_{u\to\infty} f(t,u)/u^{p-1} = \infty$ uniformly in $[\alpha,\beta]$.

By a positive solution of problem (1.1)-(1.2), we mean a function $u \in C[0,1] \cap C^1(0,1)$ with $\varphi_p(u') \in C^1(0,1)$ that satisfies (1.1)-(1.2) and $u > 0$ in $(0,1)$. Here $\|\cdot\|$ denotes the usual maximum norm in $C[0,1]$.

Motivated by the work of Bitsadze [3, 4], the study of multi-point boundary value problem for linear second-order ordinary differential equations was initially done by

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Il'in and Moiseev [13, 14]. Gupta [11] studied three-point boundary value problems for nonlinear ordinary differential equations. Since then, many researchers have studied nonlinear second-order multi-point boundary value problems under various conditions on the nonlinear term. We refer the reader to [2, 8, 9, 15, 19, 22, 23, 24, 25, 27, 28] and references therein.

Problem (1.1)-(1.2) is a singular boundary value problem since f is allowed to have singularity at $t = 0$ and/or 1. Singular problems have been extensively studied in the literature. For the case of two-point boundary value problems, the results were proved in [1, 5, 6, 12, 18, 21, 26, 29, 30] and for multi-point boundary value problems, the results were proved in [8, 9, 19, 22, 24, 28]. However, there are few results for multi-point boundary value problems having nonlinear term which does not satisfy L^1-Carathéodory condition. Recently, in semi-linear case, Sun et al. [24] studied the following singular three-point boundary-value problem

$$\begin{gathered} y'' + \mu a(t) g_1(t, y) = 0, \quad t \in (0,1) \\ y(0) - \beta y'(0) = 0, \quad y(1) = \alpha y(\eta), \end{gathered} \tag{1.3}$$

where $\mu > 0$ is a parameter, $\beta > 0$, $0 < \eta < 1$, $0 < \alpha\eta < 1$, $(1 - \alpha\eta) + \beta(1 - \alpha) > 0$, $a \in C((0,1),(0,\infty))$ satisfies $0 < \int_0^1 (\beta + s)(1 - s)a(s)ds < \infty$, and $g_1 \in C([0,1] \times (0,\infty),(0,\infty))$ may be singular at $y = 0$. Without any monotone or growth conditions imposed on the nonlinearity g_1, using fixed point index theorem, they obtained not only the existence results of positive solutions to the problem (1.3), but also the explicit interval about positive parameter μ. Kim [19], in p-Laplacian case, presented some sufficient conditions for one or multiple positive solutions to the problem (1.1)-(1.2), where $f(t,u) = h(t)g_2(t,u)$, $h \in \mathcal{A}, g_2 \in C([0,1] \times [0,\infty),[0,\infty))$.

To the authors' knowledge, in the case of p-Laplacian, there is no result about the global structure of positive solutions for parameter $\lambda \in (0,\infty)$ to multi-point boundary-value problems with the nonlinear term admitting stronger singularity than $L^1(0,1)$ at $t = 0$ and/or 1. The following is the main result in this paper.

Theorem 1.1. *Assume that* (F1) *and* (F2) *hold. Assume in addition that* $f(t,u) = h(t)g(t,u)$, *where* $h \in \mathcal{A}$ *and* $g \in C((0,1) \times [0,\infty),(0,\infty))$ *satisfies*

- (A1) *for all* $N > 0$ *and all* $\epsilon > 0$, *there exists* $\delta = \delta(N,\epsilon) > 0$ *such that if* $u, v \in [0,N]$ *and* $|u - v| < \delta$, *then* $|g(t,u) - g(t,v)| < \epsilon$, *for all* $t \in (0,1)$,
- (A2) $\inf\{g(t,u) \mid t \in (0,1),\ u \in [0,\infty)\} > 0$.

Then there exists $\lambda^* > 0$ *such that problem* (1.1)-(1.2) *has at least two positive solutions for* $\lambda \in (0,\lambda^*)$, *at least one positive solution for* $\lambda = \lambda^*$ *and no positive solution for* $\lambda > \lambda^*$.

The above result is an extension of previous works for two-point boundary-value problems by Choi [5], Wong [26], Dalmasso [6], Ha and Lee [12], Lee [21], Xu and Ma [29], and Kim [18].

The rest of this article is organized as follows. In Section 2, the operator for problem (1.1)-(1.2) is introduced, and well-known facts such as Picone-type identity and Global continuation theorem are presented. In Section 3, the proofs of our results (Theorem 3.4 and Theorem 1.1) and examples for nonlinear term to illustrate our results are given.

2. Preliminaries

First we introduce the operator corresponding to problem (1.1)-(1.2). Throughout this section we assume that (F1) holds. Set

$$\mathcal{K} = \{u \in C[0,1] : u \text{i s a nonnegative concave function on } [0,1],\ u \text{ satisfies (1.2)}\}.$$

Then $\mathcal{K}$ is an ordered cone in $C[0,1]$. For $(\lambda, u) \in [0,\infty) \times \mathcal{K}$, we define $x_{\lambda,u} : [0,1] \to \mathbb{R}$ as $x_{\lambda,u}(t) = x^1_{\lambda,u}(t) - x^2_{\lambda,u}(t)$, where

$$x^1_{\lambda,u}(t) = A^{-1}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi_p^{-1}\Big[\int_s^t \lambda f(\tau, u(\tau))d\tau\Big]ds + \int_0^t \varphi_p^{-1}\Big[\int_s^t \lambda f(\tau, u(\tau))d\tau\Big]ds$$

and

$$x^2_{\lambda,u}(t) = B^{-1}\sum_{i=1}^{m-2} b_i \int_{\xi_i}^1 \varphi_p^{-1}\Big[\int_t^s \lambda f(\tau, u(\tau))d\tau\Big]ds + \int_t^1 \varphi_p^{-1}\Big[\int_t^s \lambda f(\tau, u(\tau))d\tau\Big]ds.$$

Here

$$A = 1 - \sum_{i=1}^{m-2} a_i, \quad B = 1 - \sum_{i=1}^{m-2} b_i.$$

For $\lambda > 0$, $\lim_{t\to 0^+} x_{\lambda,u}(t) < 0$ and $\lim_{t\to 1^-} x_{\lambda,u}(t) > 0$. Indeed we can rewrite $x^1_{\lambda,u}(t)$ as

$$\begin{aligned} &x^1_{\lambda,u}(t)\\ &= A^{-1}\Big(-\sum_{i=1}^{m-2} a_i \int_t^{\xi_i} \varphi_p^{-1}\Big[\int_t^s \lambda f(\tau, u(\tau))d\tau\Big]ds + \int_0^t \varphi_p^{-1}\Big[\int_s^t \lambda f(\tau, u(\tau))d\tau\Big]ds\Big). \end{aligned}$$

By (F1), there exists $h_2 \in \mathcal{A}$ such that

$$0 \le \int_0^t \varphi_p^{-1}\Big[\int_s^t \lambda f(\tau, u(\tau))d\tau\Big]ds \le \int_0^t \varphi_p^{-1}\Big[\int_s^t h_2(\tau)d\tau\Big]ds,$$

and

$$\lim_{t\to 0^+} \int_0^t \varphi_p^{-1}\Big[\int_s^t \lambda f(\tau, u(\tau))d\tau\Big]ds = 0.$$

Clearly $\lim_{t\to 0^+} x^2_{\lambda,u}(t) > 0$, and thus $\lim_{t\to 0^+} x_{\lambda,u}(t) < 0$. In a similar manner we can show $\lim_{t\to 1^-} x_{\lambda,u}(t) > 0$. Since $x_{\lambda,u}$ is continuous and strictly increasing in $(0,1)$, there exists a unique zero $A_{\lambda,u} \in (0,1)$ such that $x_{\lambda,u}(A_{\lambda,u}) = 0$. For $\lambda = 0$, we may take $A_{0,u} = 0$ since $x_{0,u} \equiv 0$. Then, for $(\lambda, u) \in [0,\infty) \times \mathcal{K}$,

$$\begin{aligned} &A^{-1}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi_p^{-1}\Big[\int_s^{A_{\lambda,u}} \lambda f(\tau, u(\tau))d\tau\Big]ds + \int_0^{A_{\lambda,u}} \varphi_p^{-1}\Big[\int_s^{A_{\lambda,u}} \lambda f(\tau, u(\tau))d\tau\Big]ds\\ &= B^{-1}\sum_{i=1}^{m-2} b_i \int_{\xi_i}^1 \varphi_p^{-1}\Big[\int_{A_{\lambda,u}}^s \lambda f(\tau, u(\tau))d\tau\Big]ds + \int_{A_{\lambda,u}}^1 \varphi_p^{-1}\Big[\int_{A_{\lambda,u}}^s \lambda f(\tau, u(\tau))d\tau\Big]ds. \end{aligned}$$

Define $H : [0,\infty) \times \mathcal{K} \to C[0,1]$ as

$$H(\lambda,u)(t) = \begin{cases} A^{-1}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi_p^{-1}\big[\int_s^{A_{\lambda,u}} \lambda f(\tau, u(\tau))d\tau\big]ds \\ + \int_0^t \varphi_p^{-1}\big[\int_s^{A_{\lambda,u}} \lambda f(\tau, u(\tau))d\tau\big]ds, & 0 \le t \le A_{\lambda,u}, \\ B^{-1}\sum_{i=1}^{m-2} b_i \int_{\xi_i}^1 \varphi_p^{-1}\big[\int_{A_{\lambda,u}}^s \lambda f(\tau, u(\tau))d\tau\big]ds \\ + \int_{A_{\lambda,u}}^1 \varphi_p^{-1}\big[\int_{A_{\lambda,u}}^s \lambda f(\tau, u(\tau))d\tau\big]ds, & A_{\lambda,u} \le t \le 1. \end{cases}$$

In view of the definition of $A_{\lambda,u}$, $H(\lambda,u)$ is well-defined, $\|H(\lambda,u)\| = H(\lambda,u)(A_{\lambda,u})$, and $H(\lambda,u)\in\mathcal{K}$ for all $(\lambda,u)\in[0,\infty)\times\mathcal{K}$ (see, e.g., [8, Lemma 2.2]).

Lemma 2.1. *Problem (1.1)-(1.2) has a positive solution u if and only if $H(\lambda,\cdot)$ has a fixed point u in $\mathcal{K}$ for $\lambda>0$.*

Proof. We assume that u is a positive solution of problem (1.1)-(1.2). If $\lambda=0$, $u\equiv 0$ by the facts that $0\le\sum_{i=1}^{m-2}a_i<1$ and $0\le\sum_{i=1}^{m-2}b_i<1$. Thus $\lambda>0$. Since u' is strictly decreasing in $(0,1)$, $u\in\mathcal{K}$. From the fact that u satisfies (BC), $\max\{u(0),u(1)\}<u(\xi_j)$ for some $1\le j\le m-2$, and there exists a unique $A_u\in(0,1)$ such that $u'(A_u)=0$. Integrating (P_λ) from s to A_u, we have

$$u'(s)=\varphi_p^{-1}\Big[\lambda\int_s^{A_u}f(\tau,u(\tau))d\tau\Big]. \tag{2.1}$$

Again integrating (2.1) from 0 to t, we have

$$u(t)=u(0)+\int_0^t\varphi_p^{-1}\Big[\int_s^{A_u}\lambda f(\tau,u(\tau))d\tau\Big]ds,\quad t\in[0,1).$$

Then $u(\xi_i)=u(0)+\int_0^{\xi_i}\varphi_p^{-1}\big[\int_s^{A_u}\lambda f(\tau,u(\tau))d\tau\big]ds$ and

$$\begin{aligned}u(0)&=\sum_{i=1}^{m-2}a_iu(\xi_i)\\&=\sum_{i=1}^{m-2}a_iu(0)+\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}\varphi_p^{-1}\Big[\int_s^{A_u}\lambda f(\tau,u(\tau))d\tau\Big]ds.\end{aligned}$$

Thus

$$u(0)=A^{-1}\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}\varphi_p^{-1}\Big[\int_s^{A_u}\lambda f(\tau,u(\tau))d\tau\Big]ds.$$

Similarly, integrating (2.1) from t to 1,

$$u(t)=u(1)+\int_t^1\varphi_p^{-1}\Big[\int_{A_u}^{s}\lambda f(\tau,u(\tau))d\tau\Big]ds,\ t\in(0,1]$$

and

$$u(1)=B^{-1}\sum_{i=1}^{m-2}b_i\int_{\xi_i}^1\varphi_p^{-1}\Big[\int_{A_u}^{s}\lambda f(\tau,u(\tau))d\tau\Big]ds.$$

Then, by the definition of $A_{\lambda,u}$, $A_u=A_{\lambda,u}$ and consequently $H(\lambda,u)\equiv u$.

Conversely, if we assume that there exists $u\in\mathcal{K}$ such that $H(\lambda,u)=u$ for $\lambda>0$, then one can easily see that u is a positive solution of problem (1.1)-(1.2). □

Lemma 2.2. *Let $M>0$ be given and let $\{(\lambda_n,u_n)\}$ be a sequence in $[0,\infty)\times\mathcal{K}$ with $|\lambda_n|+\|u_n\|\le M$. If $A_{\lambda_n,u_n}\to 0$ (or 1) as $n\to\infty$, then $\lambda_n\to 0$ and $\|H(\lambda_n,u_n)\|\to 0$ as $n\to\infty$.*

Proof. We only prove the case $A_{\lambda_n,u_n}\to 0$ as $n\to\infty$ since the other case can be showed in a similar manner. By the definition of $A_{\lambda,u}$, we can easily know $\lambda_n\to 0$ as $n\to\infty$. By (F1), there exists $h_M\in\mathcal{A}$ such that $f(t,u)\le h_M(t)$, $t\in(0,1)$, $u\in[0,M]$. For sufficiently large n, we have $A_{\lambda_n,u_n}<\xi_1$,

$$0\le H(\lambda_n,u_n)(0)=A^{-1}\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}\varphi_p^{-1}\Big[\int_s^{A_{\lambda_n,u_n}}\lambda_nf(\tau,u_n(\tau))d\tau\Big]ds$$

$$\leq \lambda_n A^{-1} \int_0^{\xi_{m-2}} \varphi_p^{-1}\Big[\int_s^{\xi_{m-2}} h_M(\tau)d\tau\Big]ds,$$

and

$$\|H(\lambda_n,u_n)\| = H(\lambda_n,u_n)(0) + \lambda_n \int_0^{A_{\lambda_n,u_n}} \varphi_p^{-1}\Big[\int_s^{A_{\lambda_n,u_n}} h_M(\tau)d\tau\Big]ds.$$

Thus $\|H(\lambda_n,u_n)\| \to 0$ as $n\to\infty$ since $h_M \in \mathcal{A}$ and $\lambda_n \to 0$ as $n \to \infty$. □

Lemma 2.3. *$H : [0,\infty)\times\mathcal{K} \to \mathcal{K}$ is completely continuous.*

Proof. By Lemma 2.2, Ascoli-Arzelà theorem, and Lebesgue dominated convergence theorem, one can easily show the completely continuity of H (e.g., see [1, 19]). Thus we omit the proof here. □

Next we introduce the generalized Picone identity due to Jaros and Kusano ([16]). Let us consider the following operators:

$$l_p[y] \equiv (\varphi_p(y'))' + q(t)\varphi_p(y),$$
$$L_p[z] \equiv (\varphi_p(z'))' + Q(t)\varphi_p(z).$$

Theorem 2.4 ([20, p 382])**.** *Let $q(t)$ and $Q(t)$ be measurable functions on an interval I. If y and z are any functions such that $y, z, \varphi_p(y')$, $\varphi_p(z')$ are differentiable a.e. on I and $z(t) \neq 0$ for $t \in I$, then the following holds*

$$\begin{aligned}&\frac{d}{dt}\Big\{\frac{|y|^p\varphi_p(z')}{\varphi_p(z)} - y\varphi_p(y')\Big\}\\ &= (q-Q)|y|^p - \big[|y'|^p + (p-1)|\frac{yz'}{z}|^p - p\varphi_p(y)y'\varphi_p\big(\frac{z'}{z}\big)\big] - yl_p[y] + \frac{|y|^p}{\varphi_p(z)}L_p[z].\end{aligned} \tag{2.2}$$

Remark 2.5. By Young's inequality, we have

$$|y'|^p + (p-1)|\frac{yz'}{z}|^p - p\varphi_p(y)y'\varphi_p\big(\frac{z'}{z}\big) \geq 0,$$

and the equality holds if and only if $y' = yz'/z$ in (a,b).

Finally we recall a well-known theorem for the existence of a global continuum of solutions by Leray and Schauder [17].

Theorem 2.6 ([31, Corollary 14.12])**.** *Let X be a Banach space with $X \neq \{0\}$ and let $\mathcal{K}$ be an ordered cone in X. Consider*

$$x = H(\mu, x), \tag{2.3}$$

where $\mu\in[0,\infty)$ and $x \in \mathcal{K}$. If $H : [0,\infty)\times\mathcal{K}\to\mathcal{K}$ is completely continuous and $H(0,x) = 0$ for all $x \in \mathcal{K}$. Then the solution component $\mathcal{C}$ of (2.2) in $[0,\infty)\times\mathcal{K}$ which contains $(0,0)$ is unbounded.

3. Main results

Since $H(0,u) = 0$ and $H(\lambda,0) \neq 0$ if $\lambda \neq 0$, by Lemma 2.3, Theorem 2.6, we obtain the following proposition.

Proposition 3.1. *Assume that (F1) holds. Then there exists an unbounded continuum $\mathcal{C}$ emanating from $(0,0)$ in the closure of the set of positive solutions of problem (1.1)-(1.2) in $[0,\infty)\times\mathcal{K}$.*

To see the shape of $\mathcal{C}$, we need lemmas regarding λ-direction block and *a priori* estimate. Using the generalized Picone identity (Theorem 2.4) and the properties of the p-sine function [7, 32], we obtain the following two lemmas.

Lemma 3.2. *Assume that* (F1) *and* (F2) *hold. Then there exists* $\bar{\lambda} > 0$ *such that if problem* (1.1)-(1.2) *has a positive solution* u_λ*, then* $\lambda \leq \bar{\lambda}$.

Proof. Let u_λ be a positive solution of problem (1.1)-(1.2). Since $f(t,u) > 0$ for all $(t,u) \in (0,1) \times [0,\infty)$, by (F2), there exists $C_1 > 0$ such that

$$f(t,u) > C_1\varphi_p(u) \quad \text{for } u \in [0,\infty),\ t \in [\alpha,\beta]. \tag{3.1}$$

It is easy to check that $w(t) = S_q\left(\pi_q(t-\alpha)/(\beta-\alpha)\right)$, where S_q is the q-sine function and $\frac{1}{p} + \frac{1}{q} = 1$, is a solution of

$$\begin{gathered}(\varphi_p(w'(t)))' + \big(\frac{\pi_q}{\beta-\alpha}\big)^p\varphi_p(w(t)) = 0, \quad t \in (\alpha,\beta),\\ w(\alpha) = w(\beta) = 0.\end{gathered}$$

Taking $y = w$, $z = u_\lambda$, $q(t) = (\pi_q/(\beta-\alpha))^p$ and $Q(t) = \lambda f(t,u_\lambda)/\varphi_p(u_\lambda)$ in (2.2) and integrating (2.2) from α to β, by Remark 2.5,

$$\int_\alpha^\beta \Big(\big(\frac{\pi_q}{\beta-\alpha}\big)^p - \lambda\frac{f(t,u_\lambda)}{\varphi_p(u_\lambda)}\Big)|w|^p dt \geq 0.$$

It follows from (3.1) that

$$\Big(\big(\frac{\pi_q}{\beta-\alpha}\big)^p - \lambda C_1\Big)\int_\alpha^\beta |w|^p dt \geq 0,$$

and thus the proof is complete. □

Lemma 3.3. *Assume that* (F1) *and* (F2) *hold, and let* $J = [D,E]$ *be a compact subset of* $(0,\infty)$*. Then there exists* $M_J > 0$ *such that if* u *is a positive solution of problem* (1.1)-(1.2) *with* $\lambda \in J$*, then* $\|u\| \leq M_J$.

Proof. Suppose on the contrary that there exists a sequence $\{u_n\}$ of positive solutions of problem (1.1)-(1.2) with λ_n instead of λ, and $\{\lambda_n\} \subset J = [D,E]$ and $\|u_n\| \to \infty$ as $n \to \infty$. It follows from the concavity of u_n for all n that

$$u_n(t) \geq \min\{\alpha, 1-\beta\}\|u_n\|, \quad t \in (\alpha,\beta). \tag{3.2}$$

Take $C = 2D^{-1}\left(\pi_q/(\beta-\alpha)\right)^p > 0$. By (F2), there exists $K > 0$ such that $f(t,u) > C\varphi_p(u)$, for $t \in (\alpha,\beta)$, $u > K$. From the assumption, we get $\|u_N\| > (\min\{\alpha, 1-\beta\})^{-1}K$, for sufficiently large N. Therefore, by (3.2), we have

$$f(t,u_N(t)) > C\varphi_p(u_N(t)), \quad t \in (\alpha,\beta).$$

As in the proof of Lemma 3.2, if we take $y(t) = S_q\left(\pi_q(t-\alpha)/(\beta-\alpha)\right)$ and $z = u_N$, by Theorem 2.4 and Remark 2.5,

$$C \leq D^{-1}\big(\frac{\pi_q}{\beta-\alpha}\big)^p.$$

This contradicts the choice of C, and thus the proof is complete. □

Setting $\lambda^* = \sup\{\mu > 0\colon$ for all $\lambda \in (0,\mu)$, there exists at least two positive solutions of problem (1.1)-(1.2), then $\lambda^* > 0$ is well-defined. Indeed by Proposition 3.1, $\mathcal{C}$ emanates from $(0,0)$, and problem (1.1)-(1.2) has a small solution near $(0,0)$ for $\lambda \in (0,s)$ with small $s > 0$. On the other hand, for any $M > 0$, define

$\mathcal{C}_M = \{(\lambda, u) \in \mathcal{C} : \|u\| \geq M\}$ and the projection of $\mathcal{C}_M$ to the λ-axis as Λ_M. Then, by Lemma 3.2 and Lemma 3.3, for large M, $\Lambda_M = (0, a_M]$, where $a_M > 0$ and it is decreasing in M. This implies that, for any interval $(0, s)$ with small $s > 0$, problem (1.1)-(1.2) also has a large solution for $\lambda \in (0, s)$. Thus $\lambda^* > 0$ is well-defined. Moreover it follows from an easy compactness argument that problem (1.1)-(1.2) has at least two positive solution for $\lambda \in (0, \lambda^*)$ and at least one positive solution for $\lambda = \lambda^*$.

The following is the first result in this work.

Theorem 3.4. *Assume that* (F1) *and* (F2) *hold. Then there exists $\lambda_* \geq \lambda^* > 0$ such that problem* (1.1)-(1.2) *has at least two positive solutions for $\lambda \in (0, \lambda^*)$, at least one positive solution for $\lambda \in [\lambda^*, \lambda_*]$, and no positive solution for $\lambda > \lambda_*$.*

Proof. Define $\lambda_* = \sup\{\lambda : \text{problem (1.1)-(1.2) has at least one positive solution}\}$. Then by Lemma 3.2, $\lambda^* \leq \lambda_* < \infty$. We only consider the case $\lambda^* < \lambda_*$, since the proof is done for the case $\lambda^* = \lambda_*$. For $\lambda \in [\lambda^*, \lambda_*)$, there exists $\hat{\lambda} \in [\lambda, \lambda_*)$ such that (1.1)-(1.2) with $\hat{\lambda}$ instead of λ, has a positive solution, say $\hat{u}$. Consider the modified problem

$$\begin{gathered}
(\varphi_p(u'(t)))' + \lambda \bar{f}(t, u(t)) = 0, \quad t \in (0,1), \\
u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i),
\end{gathered} \tag{3.3}$$

where $\bar{f}(t, u) = f(t, \gamma(t, u))$ and $\gamma : (0, 1) \times \mathbb{R} \to \mathbb{R}$ is defined as

$$\gamma(t, u) = \begin{cases} \hat{u}(t), & \text{if } u > \hat{u}(t), \\ u, & \text{if } 0 \leq u \leq \hat{u}(t), \\ 0, & \text{if } u < 0. \end{cases}$$

Then all solutions u of (3.3) are concave and non-trivial. Define $T_\lambda : C[0, 1] \to C[0, 1]$ as

$$T_\lambda(u)(t) = \begin{cases} A^{-1} \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi_p^{-1} \big[\int_s^{\hat{A}} \lambda \bar{f}(\tau, u(\tau)) d\tau \big] ds & \\ + \int_0^t \varphi_p^{-1} \big[\int_s^{\hat{A}} \lambda \bar{f}(\tau, u(\tau)) d\tau \big] ds, & 0 \leq t \leq \hat{A} \\ B^{-1} \sum_{i=1}^{m-2} b_i \int_{\xi_i}^1 \varphi_p^{-1} \big[\int_{\hat{A}}^s \lambda \bar{f}(\tau, u(\tau)) d\tau \big] ds & \\ + \int_{\hat{A}}^1 \varphi_p^{-1} \big[\int_{\hat{A}}^s \lambda \bar{f}(\tau, u(\tau)) d\tau \big] ds, & \hat{A} \leq t \leq 1. \end{cases}$$

where $\hat{A}$ satisfies

$$\begin{aligned}
& A^{-1} \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi_p^{-1} \Big[\int_s^{\hat{A}} \lambda \bar{f}(\tau, u(\tau)) d\tau \Big] ds + \int_0^{\hat{A}} \varphi_p^{-1} \Big[\int_s^{\hat{A}} \lambda \bar{f}(\tau, u(\tau)) d\tau \Big] ds \\
& = B^{-1} \sum_{i=1}^{m-2} b_i \int_{\xi_i}^1 \varphi_p^{-1} \Big[\int_{\hat{A}}^s \lambda \bar{f}(\tau, u(\tau)) d\tau \Big] ds + \int_{\hat{A}}^1 \varphi_p^{-1} \Big[\int_{\hat{A}}^s \lambda \bar{f}(\tau, u(\tau)) d\tau \Big] ds.
\end{aligned}$$

It is easy to check that T_λ is completely continuous on $C[0, 1]$, and u is a solution of (3.3) if and only if $u = T_\lambda u$. It follows from the definition of γ and the continuity of f that there exists $R_1 > 0$ such that $\|T_\lambda u\| < R_1$ for all $u \in C[0, 1]$. Then by Schauder fixed point theorem, there exists $u_\lambda \in C[0, 1]$ such that $T_\lambda u_\lambda = u_\lambda$, and u_λ is a positive solution of (M_λ).

We first claim that $u_\lambda(0) \le \hat{u}(0)$. If the claim is not true, $u_\lambda(0) > \hat{u}(0)$. Put $x(t) = u_\lambda(t) - \hat{u}(t)$. Then

$$0 < x(0) = u_\lambda(0) - \hat{u}(0) = \sum_{i=1}^{m-2} a_i x(\xi_i) \le \sum_{i=1}^{m-2} a_i x(\xi_j) < x(\xi_j),$$

where $x(\xi_j) = \max\{x(\xi_i) | 1 \le i \le m-2\}$. Similarly, $x(1) < x(\xi_j)$. Thus, there exists $\sigma \in (0,1)$ and $a \in [0,\sigma)$ such that $x(\sigma) = \max_{t\in[0,1]} x(t) > 0$, $x'(\sigma) = 0$, $x(a) = 0$, and $x(t) > 0$ for $t \in (a,\sigma]$. Since $\lambda < \hat{\lambda}$, for $t \in (a,\sigma]$, $(\varphi_p(u_\lambda'(t)))' > (\varphi_p(\hat{u}'(t)))'$ and integrating this from t to σ, $u_\lambda'(t) < \hat{u}'(t)$. Again integrating from a to σ, we have $x(\sigma) = u_\lambda(\sigma) - \hat{u}(\sigma) < u_\lambda(a) - \hat{u}(a) = x(a)$. This is a contradiction. Thus the claim is proved. Similarly, we have $u_\lambda(1) \le \hat{u}(1)$. Next we show that $u_\lambda(t) \le \hat{u}(t)$ for $t \in (0,1)$. If it is not true, it follows from $u_\lambda(0) \le \hat{u}(0)$ and $u_\lambda(1) \le \hat{u}(1)$ that there exists an interval $[t_1,t_2] \subset [0,1]$ such that $u_\lambda(t_1) = \hat{u}(t_1)$, $u_\lambda(t_2) = \hat{u}(t_2)$ and $u_\lambda(t) > \hat{u}(t)$ for all $t \in (t_1,t_2)$. Then

$$(\varphi_p(u_\lambda'(t)))' > (\varphi_p(\hat{u}'(t)))', \quad t \in (t_1,t_2) \tag{3.4}$$

and we can choose an interval $[b,c] \subset [t_1,t_2]$ such that $u_\lambda'(b) > \hat{u}'(b)$ and $u_\lambda'(c) < \hat{u}'(c)$. Using (3.4), we can get the contradiction

$$\begin{aligned} 0 &> [\varphi_p(u_\lambda'(c)) - \varphi_p(u_\lambda'(b))] - [\varphi_p(\hat{u}'(c)) - \varphi_p(\hat{u}'(b))] \\ &= \int_b^c \big\{[\varphi_p(u_\lambda'(t))]' - [\varphi_p(\hat{u}'(t))]'\big\} dt > 0. \end{aligned}$$

Therefore, by the definition of γ, u_λ turns out a positive solution of problem (1.1)-(1.2). Furthermore, by Lemma 3.3 and the complete continuity of H, we can show that problem (1.1)-(1.2), with λ_* instead of λ, has a positive solution u_*, and thus the proof is complete. □

Now we consider $f(t,u) = h(t)g(t,u)$ and let u_* be a positive solution of problem (1.1)-(1.2), with λ_* instead of λ.

Lemma 3.5. *Assume that* (F1) *and* (F2) *hold. Assume in addition that g satisfies the conditions* (A1) *and* (A2)*. Then, for all $\lambda \in (0,\lambda_*)$, there exists $\delta_\lambda > 0$ such that $\alpha_\lambda(t) = u_*(t) + \delta_\lambda$ satisfies*

$$(\varphi_p(\alpha_\lambda'(t)))' + \lambda h(t) g(t,\alpha_\lambda(t)) < 0, \quad t \in (0,1). \tag{3.5}$$

Proof. Let λ be fixed in $(0,\lambda_*)$. Put

$$\epsilon = \frac{1}{2}[\lambda_*/\lambda - 1] \inf_{t\in(0,1)} g(t,u_*(t)) > 0.$$

By (A1), there exists $\delta_\lambda > 0$ such that if $u,v \in [0,\|u_*\|+1]$ and $|u-v| < \delta_\lambda$, then $|g(t,u) - g(t,v)| < \epsilon$, $t \in (0,1)$. Put $\alpha_\lambda(t) = u_*(t) + \delta_\lambda$. Then

$$\begin{aligned} (\varphi_p(\alpha_\lambda'(t)))' + \lambda f(t,\alpha_\lambda(t)) &= (\varphi_p(u_*'(t)))' + \lambda f(t,u_*(t)+\delta_\lambda) \\ &= h(t)[-\lambda_* g(t,u_*(t)) + \lambda g(t,u_*(t)+\delta_\lambda)]. \end{aligned}$$

From this, if α_λ does not satisfy (3.5), there exists $t_0 \in (0,1)$ such that

$$-\lambda_* g(t_0,u_*(t_0)) + \lambda g(t_0,u_*(t_0)+\delta_\lambda) \ge 0,$$

and then

$$g(t_0,u_*(t_0)+\delta_\lambda) \ge \frac{\lambda_*}{\lambda} g(t_0,u_*(t_0)).$$

By the choice of δ_λ,

$$\epsilon \geq \big(\frac{\lambda_*}{\lambda} - 1\big) g(t_0, u_*(t_0)),$$

which contradicts the choice of ϵ. This completes the proof. $\square$

Proof of Theorem 1.1. Suppose on the contrary that $\lambda^* < \lambda_*$. Let λ be fixed with $\lambda^* \leq \lambda < \lambda_*$. Then by showing that (1.1)-(1.2) has at least two positive solutions for $\lambda \in [\lambda^*, \lambda_*)$, we get a contradiction to the definition of λ^*, which completes the proof. By Lemma 3.5, there exists $\delta_\lambda > 0$ such that $\alpha_\lambda(t) = u_*(t) + \delta_\lambda$ satisfies (3.5). Consider the modified problem

$$\begin{gathered} (\varphi_p(u'(t)))' + \lambda h(t) g(t, \gamma_1(t, u(t))) = 0, \\ u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i), \end{gathered} \tag{3.6}$$

where $\gamma_1 : (0,1) \times \mathbb{R} \to [0, \infty)$ is defined as

$$\gamma_1(t,u) = \begin{cases} \alpha_\lambda(t), & \text{if } u > \alpha_\lambda(t), \\ u, & \text{if } 0 \leq u \leq \alpha_\lambda(t), \\ 0, & \text{if } u < 0. \end{cases}$$

Let u be a positive solution of (3.6). Set

$$\Omega = \{u \in C[0,1] | \ -1 < u(t) < \alpha_\lambda(t), \quad t \in [0,1]\}.$$

Then Ω is bounded and open in $C[0,1]$. We claim that if u is a positive solution of (3.6), then $u \in \Omega \cap \mathcal{K}$. Indeed, by the similar argument as in the proof of Theorem 3.4, $0 \leq u(t) \leq \alpha_\lambda(t)$, $t \in [0,1]$ and

$$\begin{aligned} u(0) &= \sum_{i=1}^{m-2} a_i u(\xi_i) \leq \sum_{i=1}^{m-2} a_i \alpha_\lambda(\xi_i) \\ &= \sum_{i=1}^{m-2} a_i (u_*(\xi_i) + \delta_\lambda) < \sum_{i=1}^{m-2} a_i u_*(\xi_i) + \delta_\lambda \\ &= u_*(0) + \delta_\lambda = \alpha_\lambda(0). \end{aligned}$$

Similarly, $\alpha_\lambda(1) > u(1)$. If the claim is not true, then there exists $[t_0, t_1] \subset (0,1)$ with $t_0 \leq t_1$ such that $0 < u(t) = \alpha_\lambda(t)$, $t \in [t_0, t_1]$ and $0 < u(t) < \alpha_\lambda(t)$, $t \in (t_0 - \delta_1, t_1 + \delta_1) \setminus [t_0, t_1]$ for some $\delta_1 > 0$. Since α_λ satisfies (3.5),

$$\max_{t \in [t_0 - \delta_1, t_1 + \delta_1]} \{(\varphi_p(\alpha_\lambda'(t)))' + \lambda h(t) g(t, \alpha_\lambda(t))\} = -\epsilon_1 < 0.$$

By condition (A1), there exists $\delta_2 > 0$ such that if $|u - v| < \delta_2$ and $u, v \in [0, \|\alpha_\lambda\|]$, then

$$|g(t,u) - g(t,v)| < \epsilon_2,$$

where $\epsilon_2 = \epsilon_1 [2\lambda \max_{t \in [t_0 - \delta_1, t_1 + \delta_1]} h(t)]^{-1} > 0$, and then there exists an interval $[a,b] \subset (t_0 - \delta_1, t_1 + \delta_1)$ such that

$$(u - \alpha_\lambda)'(a) > 0, \ (u - \alpha_\lambda)'(b) < 0$$

and

$$-\delta_2 < \gamma(t, u(t)) - \alpha_\lambda(t) = u(t) - \alpha_\lambda(t) \leq 0, \quad t \in [a,b].$$

Consequently,

$$\varphi_p(u'(a)) - \varphi_p(\alpha'_\lambda(a)) > 0, \quad \varphi_p(u'(b)) - \varphi_p(\alpha'_\lambda(b)) < 0,$$
$$g(t, \gamma(t, u(t))) < g(t, \alpha_\lambda(t)) + \epsilon_2, \quad t \in [a, b].$$

Then, by the choice of ϵ_2,

$$\begin{aligned}
0 &> \varphi_p(u'(b)) - \varphi_p(\alpha'_\lambda(b)) - \varphi_p(u'(a)) + \varphi_p(\alpha'_\lambda(a)),\\
&= [\varphi_p(u'(b)) - \varphi_p(u'(a))] - [\varphi_p(\alpha'_\lambda(b)) - \varphi_p(\alpha'_\lambda(a))]\\
&= \int_a^b \{(\varphi_p(u'(t)))' - (\varphi_p(\alpha'_\lambda(t)))'\}\, dt\\
&= \int_a^b \{-\lambda h(t) g(t, \gamma(t, u(t))) - (\varphi_p(\alpha'_\lambda(t)))'\}\, dt\\
&> \int_a^b \{-\lambda h(t)[g(t, \alpha_\lambda(t)) + \epsilon_2] - (\varphi_p(\alpha'_\lambda(t)))'\}\, dt\\
&> \int_a^b (-\lambda h(t)\epsilon_2 - [(\varphi_p(\alpha'_\lambda(t)))' + \lambda h(t) g(t, \alpha_\lambda(t))])\, dt\\
&\geq \int_a^b (-\lambda \epsilon_2 h(t) + \epsilon_1) dt \geq 0.
\end{aligned}$$

This is a contradiction. Thus the claim is proved. Define

$$Mu(t) = \begin{cases} A^{-1} \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi_p^{-1}\big[\int_s^{A_u} \lambda f(\tau, \gamma_1(\tau, u(\tau))) d\tau\big] ds \\ + \int_0^t \varphi_p^{-1}\big[\int_s^{A_u} \lambda f(\tau, \gamma_1(\tau, u(\tau))) d\tau\big] ds, & 0 \leq t \leq A_u, \\ B^{-1} \sum_{i=1}^{m-2} b_i \int_{\xi_i}^1 \varphi_p^{-1}\big[\int_{A_u}^s \lambda f(\tau, \gamma_1(\tau, u(\tau))) d\tau\big] ds \\ + \int_{A_u}^1 \varphi_p^{-1}\big[\int_{A_u}^s \lambda f(\tau, \gamma_1(\tau, u(\tau))) d\tau\big] ds, & A_u \leq t \leq 1, \end{cases}$$

where A_u is defined as

$$\begin{aligned}
&A^{-1} \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi_p^{-1}\Big[\int_s^{A_u} \lambda f(\tau, \gamma_1(\tau, u(\tau))) d\tau\Big] ds\\
&\quad + \int_0^{A_u} \varphi_p^{-1}\Big[\int_s^{A_u} \lambda f(\tau, \gamma_1(\tau, u(\tau))) d\tau\Big] ds\\
&= B^{-1} \sum_{i=1}^{m-2} b_i \int_{\xi_i}^1 \varphi_p^{-1}\Big[\int_{A_u}^s \lambda f(\tau, \gamma_1(\tau, u(\tau))) d\tau\Big] ds\\
&\quad + \int_{A_u}^1 \varphi_p^{-1}\Big[\int_{A_u}^s \lambda f(\tau, \gamma_1(\tau, u(\tau))) d\tau\Big] ds.
\end{aligned}$$

Then $M : \mathcal{K} \to \mathcal{K}$ is completely continuous, and u is a positive solution of (3.6) if and only if $u = Mu$ on $\mathcal{K}$. By simple calculation, there exists $R_1 > 0$ such that $\|Mu\| < R_1$ for all $u \in \mathcal{K}$ and $\Omega \subset B_{R_1}$. Applying [10, Lemma 2.3.1] with $O = B_{R_1}$,

$$i(M, B_{R_1} \cap \mathcal{K}, \mathcal{K}) = 1.$$

By the above claim and excision property,

$$i(M, \Omega \cap \mathcal{K}, \mathcal{K}) = i(M, B_{R_1} \cap \mathcal{K}, \mathcal{K}) = 1.$$

Since problem (1.1)-(1.2) is equivalent to problem (3.6) on $\Omega \cap \mathcal{K}$, we conclude (1.1)-(1.2) has a positive solution in $\Omega \cap \mathcal{K}$. Assume $H(\lambda, \cdot)$ has no fixed point in

$\partial\Omega\cap\mathcal{K}$, since otherwise the proof is done. Then, $i(H(\lambda,\cdot),\Omega\cap\mathcal{K},\mathcal{K})$ is well-defined, and

$$i(H(\lambda,\cdot),\Omega\cap\mathcal{K},\mathcal{K})=i(M,\Omega\cap\mathcal{K},\mathcal{K})=1 \tag{3.7}$$

since $Mu=H(\lambda,u)$ for $u\in\Omega\cap\mathcal{K}$. By Lemma 3.2, (1.1)-(1.2) with λ_{N_0} instead of λ has no solution in $\mathcal{K}$ for $\lambda_{N_0}>\overline{\lambda}$. Thus, for any open subset $\mathcal{O}$ in X,

$$i(H(\lambda_{N_0},\cdot),\mathcal{O}\cap\mathcal{K},\mathcal{K})=0.$$

By *a priori* estimate (Lemma 3.3) with $I=[\lambda,\lambda_{N_0}]$, there exists $R_2(>R_1)$ such that all possible positive solutions u of (1.1)-(1.2) with μ instead of λ for $\mu\in[\lambda,\lambda_{N_0}]$, satisfy $\|u\|<R_2$.

Define $h:[0,1]\times(\overline{B}_{R_2}\cap\mathcal{K})\to\mathcal{K}$ as

$$h(\tau,u)=H(\tau\lambda_{N_0}+(1-\tau)\lambda,u).$$

Then h is completely continuous on $[0,1]\times\mathcal{K}$, and it satisfies that $h(\tau,u)\neq u$ for all $(\tau,u)\in[0,1]\times(\partial B_{R_2}\cap\mathcal{K})$. By the property of homotopy invariance,

$$i(H(\lambda,\cdot),B_{R_2}\cap\mathcal{K},\mathcal{K})=i(H(\lambda_{N_0},\cdot),B_{R_2}\cap\mathcal{K},\mathcal{K})=0.$$

By (3.7) and the additivity property,

$$i(H(\lambda,\cdot),(B_{R_2}\setminus\overline{\Omega})\cap\mathcal{K},\mathcal{K})=-1.$$

Thus problem (1.1)-(1.2) has another positive solution in $(B_{R_2}\setminus\overline{\Omega})\cap\mathcal{K}$. This completes the proof. $\square$

Finally, we give the examples for the nonlinear term to illustrate our results.

Example 3.6. (1) Put $f_1(t,u)=[t(1-t)]^{-p+1/(u+1)}exp(u)$. Then, it is easily verified that f_1 satisfies the assumptions of Theorem 3.4.

(2) Put $f_2(t,u)=(1-t)^{-\alpha_1}g(t,u)$, where $g(t,u)=c_1t^{-\beta_1}+c_2(u^q+1)$. Then f_2 satisfies the assumptions of Theorem 1.1 if $\alpha_1,\beta_1<p$, $c_1\geq 0$, $c_2>0$, and $q>p-1$.

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Chan-Gyun Kim
Department of Mathematics, College of William and Mary, Williamsburg, Virginia 23187-8795, USA
E-mail address: cgkim75@gmail.com

Eun Kyoung Lee (corresponding author)
Department of Mathematics and Education, Pusan National University, Busan 609-735, Korea
E-mail address: eunkyoung165@gmail.com