Electronic Journal of Differential Equations, Vol. 2014 (2014), No. 40, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

SHARP BOUNDS OF THE NUMBER OF ZEROS OF ABELIAN INTEGRALS WITH PARAMETERS

XIANBO SUN, JUNMIN YANG

Abstract. In this article, we study four Abelian integrals over compact level curves of four sixth-degree hyper-elliptic Hamiltonians with parameters. We prove that the sharp bound of the number of zeros for each Abelian integral is 2. The proofs rely mainly on the Chebyshev criterion for Abelian integrals and asymptotic expansions of Abelian integrals.

1. Introduction and main result

The second part of Hilbert's 16th problem and its weak version are two open problems in the qualitative theory of planar differential equations. The first one asks for the maximal number of limit cycles and their distribution for the following planar polynomial differential equation of degree n,

$$\dot{x} = P_n(x,y), \quad \dot{y} = Q_n(x,y). \tag{1.1}$$

A special form of (1.1) is

$$\dot{x} = H_y + \varepsilon p(x,y,\delta), \quad \dot{y} = -H_x + \varepsilon q(x,y,\delta), \tag{1.2}$$

where $H(x,y)$, $p(x,y)$, $q(x,y)$ are polynomials of x and y, and their degrees satisfy $\max\{\deg p, \deg q\} = n$ and $\deg(H) = n+1$, and ε is a positive and sufficiently small parameter. The unperturbed form of (1.2) is

$$\dot{x} = H_y, \quad \dot{y} = -H_x. \tag{1.3}$$

The Hamiltonian function $H(x,y)$ defines at least one family of closed curves L_h which form a period annulus of (1.3) denoted by $\{L_h\}$, where h is energy parameter on an open interval J. Corresponding to system (1.2), the following integral is called Abelian integral or first order Melnikov function,

$$A_n(h) = \oint_{L_h} q(x,y)dx - p(x,y)dy, \quad h \in J, \tag{1.4}$$

which plays an important role in studying the limit cycles of (1.2) (see the Poincaré-Pontryagin Theorem [5]), and finding the upper bound of the maximal number of zeros of $A_n(h)$ is the weak version of the second part of Hilbert's 16th problem (usually called weak Hilbert's 16th problem). Its research advances and the recent

2000 *Mathematics Subject Classification.* 34C05, 34C07, 34C08.
Key words and phrases. Limit cycle; Liénard system; Chebyshev system; bifurcation; heteroclinc loop.

Submitted November 11, 2013. Published February 5, 2014.

popular and efficient methods for special forms of (1.2) can be found in the survey works [17, 18].

Since both problems are difficult, mathematicians try to study special and simpler forms of (1.1) and (1.2). Smale 13th problem restricts Hilbert's 16th problem to the Liénard system

$$\dot{x} = y - f(x), \quad \dot{y} = -x.$$

To study the number of zeros of $A_n(h)$, many mathematicians concentrate on a simpler form of (1.2) as follows

$$\dot{x} = y, \quad \dot{y} = g(x) + \varepsilon f(x)y, \tag{1.5}$$

which is called Liénard system of type (m, n) if $g(x)$ and $f(x)$ are polynomials of degree respectively m and n.

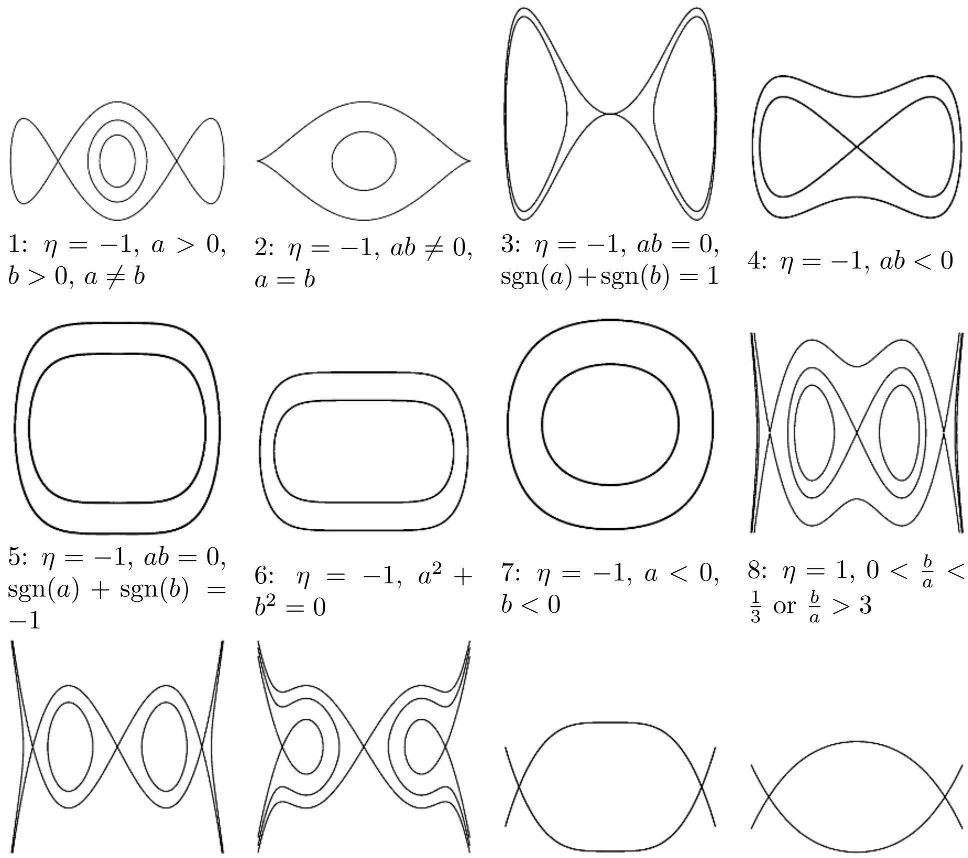
A comprehensive study has been made in [7] for the cases $m + n \leq 4$, except for $(m, n) = (1, 3)$. In all these cases, it has proven that at most one limit cycle can appear and for $(m, n) = (1, 3)$ the same result has been conjectured (see [6]). For type $(3, 2)$, there are several cases according to the portraits of the unperturbed system. Dumortier and Li [8, 9, 10, 11] have made a complete study on these cases and obtained different sharp upper bounds of the number of zeros of Abelian integrals for different cases. Li, Mardešić and Roussarie [19] investigated some Liénard systems of type $(3, 2)$ with symmetry and also obtained the sharp bound. Wang and Xiao [27, 28] investigated some Liénard system of type $(4, 3)$ and proved that 4 is the least upper bound and 3 is the maximum lower bound of the number of the zeros for the corresponding Abelian integral. Some other cases of type $(4, 3)$ are investigated in [4, 25], and the least upper bound and the maximal lower one are obtained. The results of the maximum lower bound for other systems of type $(4, 3)$ can be found in [30, 31, 32].

For the type $(5, 4)$, many works concentrate on the following Liénard systems with symmetry

$$\dot{x} = y, \quad \dot{y} = \eta x(x^2 - a)(x^2 - b) + \varepsilon(\alpha + \beta x^2 + \gamma x^4)y, \tag{1.6}$$

where $\eta = \pm 1$, α, β and γ are real bounded number. Assume the portraits of system $(1.6)_{\varepsilon=0}$ has at least one periodic annulus, there are 12 cases according to the value of a, b and η, see Figure 1.

For case 1, Zhang et al. [34] proved that system (1.6) with $a = 1/2$, $b = 2$ has at most 3 zeros of the corresponding Abelian integral. For case 2, Asheghi and Zangeneh studied (1.6) with $a = b = 1$ and proved that the least upper bound for the number of zeros of the related Abelian integral inside the eye-figure loop is 2 in [1] and both inside and outside the eye-figure loop is 4 in [2]. For case 3, Asheghi and Zangeneh [3] studied (1.6) by taking $a = 0$, $b = 1$ and proved that the corresponded Abelian integral has at most 2 zeros inside the double cuspidal loops. For case 3, Zhao [35] studied system (1.6) with $a = 0$ and $b = 1$ and obtained that 2 is the sharp bound of the number of zeros of Abelian integral associated on the the two bounded period annuluses. For case 8, Xu and Li [29] proved that system (1.6) has at least 5 limit cycles bifurcated from 3 annuluses of the system $(1.6))_{\varepsilon=0}$ with $a = 1/4$, $b = 1$. For case 9, Sun [26] proved there are at most 4 zeros for the corresponding Abelian integral. Later, Zhao [23] proved the sharp bound of number of zeros for the corresponding Abelian integral is 2. For case 10, Qi and Zhao [24] proved that system (1.6) with $a = \frac{21-\sqrt{41}}{20}$ and $b = \frac{21+\sqrt{41}}{20}$ has at most 2 limit cycles bifurcated from each annulus.



1: $\eta = -1$, $a > 0$, $b > 0$, $a \neq b$

2: $\eta = -1$, $ab \neq 0$, $a = b$

3: $\eta = -1$, $ab = 0$, $\text{sgn}(a)+\text{sgn}(b) = 1$

4: $\eta = -1$, $ab < 0$

5: $\eta = -1$, $ab = 0$, $\text{sgn}(a) + \text{sgn}(b) = -1$

6: $\eta = -1$, $a^2 + b^2 = 0$

7: $\eta = -1$, $a < 0$, $b < 0$

8: $\eta = 1$, $0 < \frac{b}{a} < \frac{1}{3}$ or $\frac{b}{a} > 3$

9: $\eta = 1$, $\frac{b}{a} = \frac{1}{3}$ or $\frac{b}{a} = 3$

10: $\eta = 1$, $\frac{1}{3} < \frac{b}{a} < 1$ or $1 < \frac{b}{a} < 3$

11: $\eta = 1$, $ab = 0$, $\text{sgn}(a)+\text{sgn}(b) = 1$

12: $\eta = 1$, $ab < 0$

FIGURE 1. Twelve cases of (1.6) each having at least one annulus surrounding a center

In this article, we study the cases 5, 6, 7 and 12 with some parameters. Without loss of generality we fix $\gamma = 1$ in all cases. For case 12 we take $a = 1$, $b = -\lambda$ without loss of generality. For convenience we assume $\lambda \geq 1$, then system (1.6) becomes

$$\dot{x} = y, \quad \dot{y} = x(x^2 - 1)(x^2 + \lambda) + \varepsilon(\alpha + \beta x^2 + x^4)y, \tag{1.7}$$

with the Hamiltonian function

$$\widetilde{H}(x, y) = \frac{y^2}{2} + \frac{\lambda}{2}x^2 - \frac{\lambda - 1}{4}x^4 - \frac{1}{6}x^6. \tag{1.8}$$

The level sets (i.e. $\widetilde{H}(x, y) = h$) of Hamiltonian function (1.8) are sketched in Figure 2. $\widetilde{H}(x, y) = h$ defines one family of ovals which correspond to a period annulus of system $(1.7)_{\varepsilon=0}$ denoted by $\{\Gamma_h\}$. $H(x, y) = \frac{3\lambda+1}{12}$ defines a 2-polycycles $\Gamma^* = \{(x, y)|H(x, y) = \frac{3\lambda+1}{12}\}$ which consists of two heteroclinic orbits. Γ_0 is an elementary center. The Abelian integral on Γ_h is

$$I(h,\delta) = \oint_{\Gamma_h} (\alpha + \beta x^2 + x^4) y dx \equiv \alpha I_0(h) + \beta I_1(h) + I_2(h), \tag{1.9}$$

for $h \in (0, (3\lambda+1)/12)$, where $\delta = (\alpha, \beta, 1)$, $I_i(h) = \oint_{\Gamma_h} x^{2i} y dx$, $i = 0, 1, 2$.

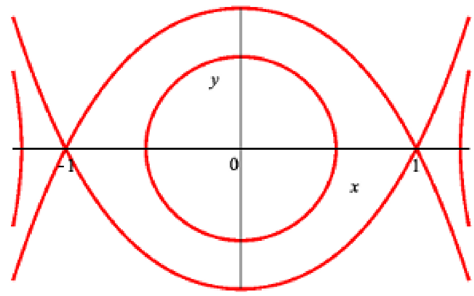


FIGURE 2. The level set of $\widetilde{H}(x,y)$

For case 7, we take $a = -\lambda_1, b = -\lambda_2$, where $\lambda_1, \lambda_2 > 0$, then system (1.6) becomes

$$\dot{x} = y, \quad \dot{y} = -x(x^2+\lambda_1)(x^2+\lambda_2) + \varepsilon(\alpha + \beta x^2 + x^4) y. \tag{1.10}$$

For case 6, we take $a = 0, b = -\lambda_3$, where $\lambda_3 > 0$, then system (1.6) becomes

$$\dot{x} = y, \quad \dot{y} = -x^3(x^2+\lambda_3) + \varepsilon(\alpha + \beta x^2 + x^4) y. \tag{1.11}$$

For case 5, we take $a = b = 0$, then system (1.6) becomes

$$\dot{x} = y, \quad \dot{y} = -x^5 + \varepsilon(\alpha + \beta x^2 + x^4) y. \tag{1.12}$$

The corresponding Abelian integrals of systems (1.10), (1.11), (1.12) are, respectively,

$$\begin{aligned}
I^b(h,\delta) &= \oint_{\Gamma_h^b} (\alpha + \beta x^2 + x^4) y dx \equiv \alpha I_0^b(h) + \beta I_1^b(h) + I_2^b(h),\\
I^c(h,\delta) &= \oint_{\Gamma_h^c} (\alpha + \beta x^2 + x^4) y dx \equiv \alpha I_0^c(h) + \beta I_1^c(h) + I_2^c(h),\\
I^d(h,\delta) &= \oint_{\Gamma_h^d} (\alpha + \beta x^2 + x^4) y dx \equiv \alpha I_0^d(h) + \beta I_1^d(h) + I_2^d(h),
\end{aligned}$$

where $I(h)$, $I^b(h)$ and $I^c(h)$ have parameters λ, λ_1, λ_2, λ_3. Using some algebraic method, some polynomial techniques and expansions of Abelian integrals, the following results are obtained.

Theorem 1.1. *For all α and β, each of $I(h,\delta)$, $I^b(h,\delta)$, $I^c(h,\delta)$ and $I^d(h,\delta)$ has at most 2 zeros, counting the multiplicity. Taking $0 < \alpha \ll -\beta \ll 1$, two zeros of each Abelian integral appear in some small intervals near $h = 0$. Therefore, 2 is the sharp bound.*

By the Poincaré-Pontryagin theorem and Theorem 1.1, each of system (1.7), (1.10), (1.11), (1.12) has at most 2 limit cycles bifurcated from the corresponding period annulus, and there exist some (α, β) and $0 < \varepsilon \ll 1$ such that each system has 2 limit cycles bifurcated from the corresponding period annulus. The rest of the article is organized as follows: in section 2 we will introduce some definitions and the new criteria which are used to determine the number of zeros of the Abelian integrals. In sections 3 and 4, we will prove the main results.

2. Preliminary lemmas and definitions

The method we will introduce proposes some criterion functions defined directly by Hamiltonian and integrands of Abelian integrals, through which the problem whether the basis of the vector space generated by Abelian integrals is a Chebyshev system could be reduced to the problem whether the family of criterion functions form a Chebyshev system, since the latter can be tackled by checking the nonvanishing properties of its Wronskians. For this paper to be self-contained, we list some related definitions and criterions. For more details, [21, 12] is referred.

Definition 2.1. Suppose $f_0, f_1, f_2, \dots, f_{n-1}$ are analytic functions on an real open interval J.

(i) The family of polynomials $\{f_0, f_1, f_2, \dots, f_{n-1}\}$ is called Chebyshev system (T-system for short) provided that any nontrivial linear combination

$$k_0 f_0(x) + k_1 f_1(x) + \dots + k_{n-1} f_{n-1}(x)$$

has at most $n-1$ isolated zeros on J.

(ii) An ordered set of n functions $\{f_0, f_1, f_2, \dots, f_{n-1}\}$ is called complete Chebyshev system (CT-system for short) provided any nontrivial linear combination $k_0 f_0(x) + k_1 f_1(x) + \dots + k_{i-1} f_{i-1}(x)$ has at most $i-1$ zeros for all $i = 1, 2, \dots, n$, moreover it is called extended complete Chebyshev system (ECT-system for short) if the multiplicities of zeros taken into account.

(iii) The continuous Wronskian of $\{f_0, f_1, f_2, \dots, f_{n-1}\}$ at $x \in R$ is

$$\begin{aligned} W[f_0, f_1, f_2, \dots, f_{k-1}] &= \det(f_i^j)_{0 \le i,j \le k-1} \\ &= \begin{vmatrix} f_0(x) & f_1(x) & \dots & f_{k-1} \\ f_0'(x) & f_1'(x) & \dots & f_{k-1}'(x) \\ \dots & \dots & \dots & \dots \\ f_0^{(k-1)}(x) & f_1^{(k-1)}(x) & \dots & f_{k-1}^{(k-1)}(x) \end{vmatrix}, \end{aligned}$$

where $f'(x)$ is the first order derivative of $f(x)$ and $f^{(i)}(x)$ is the ith order derivative of $f(x)$, $i \ge 2$.

The above definitions imply that the function tuple $\{f_0, f_1, \dots, f_{k-1}\}$ is an ECT-system on J, therefore it is a CT-system on J, and then a T-system on J, however the inverse implications are not true at all.

Recall that the authors of [12] studied the number of isolated zero of Abelian integrals using a purely algebraic criteria which is developed from the idea introduced in [20]. Let $H(x,y) = A(x) + \frac{1}{2}y^2$ be an analytic function in some open subset of the plane which has a local minimum at $(0,0)$. Then there exists a punctured neighborhood P of the origin foliated by ovals $L_h : H(x,y) = h$ which correspond to the clockwise closed orbits of (1.3). The set of ovals L_h inside the period annulus, is parameterized by the energy levels $h \in (0, h_1) = J$ for some $h_1 \in (0, +\infty]$. The projection of P on the x-axis is an interval (x_l, x_r) with $x_l < 0 < x_r$. Under the above assumptions it is easy to verify that $xA'(x) > 0$ for all $x \in (x_l, x_r) \setminus \{0\}$,

$A(x)$ has a zero of even multiplicity at $x = 0$ and there exists an analytic involution $z(x)$ such that

$$A(x) = A(z(x))$$

for all $x \in (x_l, x_r)$. It is obvious that $z(x) = -x$ if $A(x)$ is a even function.

For the number of isolated zeros of nontrivial linear combination of some integrals of special form, the algebraic criterion in [12, Theorem B] can be stated as follows:

Lemma 2.2. *Assume that the function $f_i(x)$ is analytic on the interval (x_l, x_r) for $i = 0, 1, \dots, n-1$, and consider*

$$A_i(h) = \int_{L_h} f_i(x) y^{2s-1} dx, \quad i = 0, 1, \dots, n-1,$$

where for each $h \in (0, h_0)$, L_h is the oval surrounding the origin inside an level curve $\{A(x) + \frac{1}{2}y^2 = h\}$. We define

$$l_i(x) := \frac{f_i(x)}{A'(x)} - \frac{f_i(z(x))}{A'(z(x))}.$$

Then, $\{A_0, A_1, \dots, A_{n-1}\}$ is an ECT-system on $(0, h_1)$ if $\{l_0, l_1, \dots, l_{n-1}\}$ is a CT-system on $(x_l, 0)$ or $(0, x_r)$ and $s > n-2$. And $\{l_0, l_1, \dots, l_{n-1}\}$ is an ECT-system on (x_0, x_r) or (x_l, x_0) if and only if the continuous Wronskian of $\{l_0, l_1, \dots, l_{k-1}\}$ does not vanish for $\forall x \in (0, x_r)$ or for all $z \in (x_l, 0)$ and $k = 1, \dots, n$.

Usually s is not big enough, Lemma 2.2 can not be applied directly. To overcome this problem the next result (see [12, Lemma 4.1]) is useful to increase the power of y in $A_i(h)$.

Lemma 2.3. *Let L_h be an oval inside the level curve $A(x) + \frac{1}{2}(x)y^2 = h$ and consider a function $F(x)$ satisfying $\frac{F(x)}{A'(x)}$ is analytic at $x = 0$. Then, for any $k \in N$,*

$$\oint_{L_h} F(x) y^{k-2} dx = \oint_{L_h} G(x) y^k dx$$

where $G(x) = \frac{1}{k}(\frac{F}{A'})'(x)$.

3. Proof of main result

For briefness we prove only case 12, other cases can be proved similarly. In what follows, we proved that the following generating elements of $I(h, \delta)$,

$$I_i(h) = \int_{\Gamma_h} x^{2i} y dx, \quad i = 0, 1, 2$$

have the Chebyshev property for $h \in (0, \frac{3\lambda+1}{12})$.

By Lemma 2.2, $A(x) = \widetilde{H}(x, 0) = -\frac{3}{2}x^2 + x^4 - \frac{1}{6}x^6$ and $s = 1$, $n = 3$ for system (1.7). The period annulus is foliated by the ovals Γ_h, and the projection of the period annulus on the plan is an open interval $(-1, 1)$. Noting that $xA'(x) > 0$ for all $x \in (-1, 1) \setminus \{0\}$, therefore there exists an analytic involution $z(x)$ such that

$$A(x) = A(z(x)).$$

Our goal is to prove that the vector space generated by Abelian integral $I_i(h)$ has the Chebyshev property for $x \in (0, 1)$ by Lemma 2.2. However, for $s = 1$ and $n = 3$ it does not satisfy the hypothesis $s > n-2$ in Lemma 2.2. Thus the power s of y in the integrand of $I_i(h)$ should be increased such that the condition $s > n-2$ holds.

Lemma 3.1. *For $i = 0, 1, 2$, we have*

$$2hI_i(h) = \int_{\Gamma_h} f_i(x)y^3 dx,$$

where $f_i(x) = \frac{x^{2i}\widetilde{f}_i(x)}{18(x-1)^2(x+1)^2(x^2+\lambda)^2}$ *with*

$$\begin{aligned}\widetilde{f}_i(x) &= 20x^8 + 39x^6\lambda + 21\lambda^2x^4 - 39\lambda^2x^2 + 24\lambda^2 + 4ix^8 - 10ix^6 + 6ix^4 \\ &\quad + 12i\lambda^2 + 10ix^6\lambda - 28ix^4\lambda + 18i\lambda x^2 + 6i\lambda^2x^4 - 18i\lambda^2x^2 \\ &\quad + 21x^4 - 70x^4\lambda + 39\lambda x^2 - 39x^6.\end{aligned}$$

Proof. It is clear that on every periodic orbits $\Gamma_h : \{\widetilde{H}(x, y) = h\}$, $\frac{2A(x)+y^2}{2h} = 1$ holds. Therefore,

$$I_i(h) = \frac{1}{2h}\int_{\Gamma_h}(2A(x) + y^2)x^{2i}ydx = \frac{1}{2h}\int_{\Gamma_h} 2x^{2i}A(x)ydx + \frac{1}{2h}\int_{\Gamma_h} x^{2i}y^3dx, \quad (3.1)$$

for $i = 0, 1, 2$. Noting that the functions $\frac{2x^{2i}A(x)}{A'(x)}$ are analytic on $x = 1$, by Lemma 2.3, we have

$$\int_{\Gamma_h} 2x^{2i}A(x)ydx = \int_{\Gamma_h} G_i(x)y^3dx, \quad (3.2)$$

where

$$G_i(x) = \frac{x^{2i}g_i(x)}{(x-1)^2(x+1)^2(x^2+\lambda)^2}$$

with

$$\begin{aligned}g_i(x) &= 2x^8 + 3x^6\lambda + 3\lambda^2x^4 - 3\lambda^2x^2 + 6\lambda^2 + 4ix^8 - 10ix^6 + 6ix^4 \\ &\quad + 12i\lambda^2 + 10ix^6\lambda - 28ix^4\lambda + 18i\lambda x^2 + 6i\lambda^2x^4 \\ &\quad - 18i\lambda^2x^2 + 3x^4 + 2x^4\lambda + 3\lambda x^2 - 3x^6.\end{aligned}$$

Combine (3.1) and (3.2), so Lemma 3.1 is proved. □

Let

$$\widetilde{I}_i(h) = \int_{\Gamma_h} f_i(x)y^3dx.$$

Then $\{I_0, I_1, I_2\}$ is an ECT-system on $(0, \frac{3\lambda+1}{12})$ if and only if $\{\widetilde{I}_0, \widetilde{I}_1, \widetilde{I}_2\}$ is as well. Since $s = 2$, $n = 3$ and the condition $s > n - 2$ holds, lemma 2.2 can be used to study if $\{\widetilde{I}_0, \widetilde{I}_1, \widetilde{I}_2\}$ is an ECT-system on $(0, \frac{3\lambda+1}{12})$. Thus, setting the criteria functions

$$l_i(x) = (\frac{f_i}{A'})(x) - (\frac{f_i}{A'})(z(x)), \ 0 < x < 1, i = 0, 1, 2, \quad (3.3)$$

where $z(x)$ is the analytic involution $z(x)$ defined by $A(x) = A(z)$. By symmetry of system (1.7), it is obvious $z(x) = -x$.

Inserting $z(x) = -x$ in (3.3) gives

$$l_i(x) = -\frac{(x^{2i} + (-x)^{2i})\widetilde{l}_i(x)}{18(x-1)^3(x+1)^3(x^2+\lambda)^3x}$$

with

$$\begin{aligned}\widetilde{l}_i(x) &= 20x^8 + 21\lambda^2x^4 - 39\lambda^2x^2 + 24\lambda^2 + 4ix^8 - 10ix^6 + 6ix^4 \\ &\quad + 12i\lambda^2 + 39x^6\lambda + 10ix^6\lambda - 28ix^4\lambda + 18i\lambda x^2 + 6i\lambda^2x^4\end{aligned}$$

$$- 18i\lambda^2x^2 + 21x^4 - 70x^4\lambda + 39\lambda x^2 - 39x^6.$$

Next, we check that the ordered set of criterion functions $\{l_1(x), l_2(x), l_0(x)\}$ is an ECT-system for $x \in (0,1)$ by verifying the non-vanishing property of continuous Wronskians $W[l_1]$, $W[l_1, l_2]$, $W[l_1, l_2, l_0]$.

Lemma 3.2. *The function tuple $\{l_1(x), l_2(x), l_0(x)\}$ is an ECT-system for $x \in (0,1)$.*

Proof. By the Definition 2.1 (iii) about the continuous Wronskian, with the aid of Maple 13, we have

$$\begin{gathered}
W[l_1(x)] = \frac{-xw_1(x,\lambda)}{9(x-1)^3(x+1)^3(x^2+\lambda)^3},\\
W[l_1(x), l_2(x)] = \frac{2x^3w_2(x,\lambda)}{81(x-1)^5(x+1)^5(x^2+\lambda)^5},\\
W[l_1(x), l_2(x), l_0(x)] = \frac{-16w_3(x,\lambda)}{243(x-1)^7(x+1)^7(x^2+\lambda)^7},
\end{gathered}$$

where

$$\begin{aligned}
w_1(x,\lambda) &= 24x^8 + 27\lambda^2x^4 - 57\lambda^2x^2 + 36\lambda^2 - 49x^6 + 27x^4 + 49x^6\lambda\\
&\quad - 98x^4\lambda + 57\lambda x^2,\\
w_2(x,\lambda) &= 672x^{12} - 2072x^{10} + 2072x^{10}\lambda - 6174x^8\lambda + 2295x^8 + 2295x^8\lambda^2\\
&\quad - 891x^6 + 6993x^6\lambda - 6993\lambda^2x^6 + 891x^6\lambda^3 + 8382\lambda^2x^4 - 2871\lambda^3x^4\\
&\quad - 2871x^4\lambda - 3672\lambda^2x^2 + 3672\lambda^3x^2 - 1728\lambda^3,\\
w_3(x,\lambda) &= 13824\lambda^4 + 6237x^8 + 40392\lambda^2x^4 + 22275\,x^6\lambda + 6237\lambda^4x^8 - 22275\lambda^4x^6\\
&\quad + 40392\lambda^4x^4 - 37152\lambda^4x^2 + 4480x^{16} - 17248x^{14} + 27636x^{12}\\
&\quad - 20979x^{10} - 71280x^8\lambda^3 + 27636x^{12}\lambda^2 - 92073x^{10}\lambda^2 + 17248x^{14}\lambda\\
&\quad - 59304x^{12}\lambda + 20979x^{10}\lambda^3 + 37152\lambda^3x^2 - 71280x^8\lambda - 122265\lambda^2x^6\\
&\quad - 106128\lambda^3x^4 + 92073x^{10}\lambda + 149094x^8\lambda^2 + 122265x^6\lambda^3,
\end{aligned}$$

of degree 8, 12 and 16, respectively.

To check if three Wronskians vanish for $x \in (0,1)$, we only need check if three two-variable polynomials $w_1(x,\lambda)$, $w_2(x,\lambda)$ and $w_3(x,\lambda)$ vanish for $x \in (0,1)$. In order to avoid complicated symbolic computation, such as regular chains with parameter, and real roots isolation, we introduce some transforms.

First, let $\alpha > 0$ and introduce $x = \frac{1}{1+\alpha}$, which satisfies $0 < x < 1$. Then $w_1(x,\lambda)$ becomes $w_1(x,\lambda) = \frac{p_1(\alpha,\lambda)}{(1+\alpha)^8}$, where

$$\begin{aligned}
p_1(\alpha,\lambda) &= 2 + 8\lambda + 6\lambda^2 + 108\alpha^3 + 27\alpha^4 + 113\,\alpha^2 + 54\lambda^2\alpha + 315\lambda^2\alpha^2\\
&\quad + 984\lambda^2\alpha^3 + 1692\lambda^2\alpha^4 + 1674\lambda^2\alpha^5 + 951\lambda^2\alpha^6 + 288\lambda^2\alpha^7 + 36\lambda^2\alpha^8\\
&\quad + 48\lambda\alpha + 316\lambda\alpha^2 + 748\lambda\alpha^3 + 757\lambda\alpha^4 + 342\lambda\alpha^5 + 57\lambda\alpha^6 + 10\alpha,
\end{aligned}$$

which does not vanish on $\{(\alpha,\lambda) | \alpha > 0, \lambda \geq 1\}$ since its coefficients are all positive. Therefore, $W[l_1(x)]$ has not root for $x \in (0,1)$ obviously.

Second, let $\alpha > 0$, $\beta \geq 0$ and introduce

$$x = \frac{1}{1+\alpha}, \quad \lambda = 1 + \beta.$$

Then $w_2(x,\lambda) = -p_2(\alpha,\beta)/(1+\alpha)^{12}$, where $p_2(\alpha,\beta)$ is a polynomial with positive coefficients and has no root on $\{(\alpha,\lambda) : \alpha > 0, \beta \ge 0\}$ (see Appendix A). Hence, $W[l_1(x), l_2(x)]$ has no roots for $x \in (0,1)$.

Last, taking $\alpha > 0$ and substituting $x = 1/(1+\alpha)$ into $w_3(x,\lambda)$ yields $w_3(x,\lambda) = \frac{p_3(\alpha,\lambda)}{(1+\alpha)^{16}}$. The polynomial $p_3(\alpha,\lambda)$ has positive coefficients (see Appendix A), and it has no root on $\{(\alpha,\lambda)|\alpha > 0, \lambda \ge 1\}$. Hence, $W[l_1(x), l_2(x), l_0(x)]$ has no root for $x \in (0,1)$. Lemma 3.2 is proved. $\square$

By Lemmas 2.2 and 3.2, $\{\widetilde{I}_1(h), \widetilde{I}_2(h), \widetilde{I}_0(h)\}$ is an ECT-system on $(0, (3\lambda + 1)/12)$, and so $\{I_1, I_2, I_0\}$ is as well. Therefore, $I(h,\delta)$ has at most 2 zeros.

Remark 3.3. With the same methods and techniques, it is not difficult to prove each of $I^b(h,\delta)$, $I^c(h,\delta)$ and $I^d(h,\delta)$ has at most 2 zeros, we omit the proofs here for brevity.

4. Finding zeros in small intervals

Usually, it is difficult to find zeros of $A_n(h)$. One popular method is to detect the expansions of $A_n(h)$ near a center, homoclinic loop and heteroclinic loop of system (1.3), see [14]. When the annulus $\{L_h\}$ of system (1.3) has a homoclinic loop, a heteroclinic loop as the outer boundary, the expansions of $A_n(h)$ near these outer boundaries was studied in [15] and the expression of coefficients are also given. When the inner boundary of $\{L_h\}$ is a center, the expansion of $A_n(h)$ near an elementary center is investigated in [16]. and the expansion of $A_n(h)$ near a nilpotent center is investigated in [33].

By the results of [15, 16, 33], the expansions of $I(h,\delta)$, $I^b(h,\delta)$, $I^c(h,\delta)$ and $I^d(h,\delta)$ near the centers are as follows

$$
\begin{gathered}
I(h,\delta) = b_0(\delta)h + b_1(\delta)h^2 + b_2(\delta)h^3 + h.o.t., \quad h \in (0,\varepsilon_1),\\
I^b(h,\delta) = \widetilde{b}_0(\delta)h + \widetilde{b}_1(\delta)h^2 + \widetilde{b}_2(\delta)h^3 + h.o.t., \quad h \in (0,\varepsilon_2),\\
I^c(h,\delta) = \bar{b}_0(\delta)h^{\frac{3}{4}} + \bar{b}_1(\delta)h^{\frac{5}{4}} + \bar{b}_2(\delta)h^{\frac{7}{4}} + h.o.t., \quad h \in (0,\varepsilon_3),\\
I^d(h,\delta) = \widehat{b}_0(\delta)h^{\frac{4}{6}} + \widehat{b}_1(\delta)h^{\frac{8}{6}} + \widehat{b}_2(\delta)h^{\frac{10}{6}} + h.o.t., \quad h \in (0,\varepsilon_4),
\end{gathered}
$$

where $0 < \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \ll 1$ and

$$
\begin{gathered}
b_0(\delta) = 2\pi\alpha, \quad b_1(\delta) = \frac{3\pi}{4}(\lambda - 1)\alpha + \pi\beta,\\
b_2(\delta) = \frac{5\pi}{96}(21\lambda^2 - 42\lambda + 37)\alpha + \frac{5\pi}{4}(\lambda - 1)\beta + \pi,\\
\widetilde{b}_0(\delta) = 2\pi\alpha, \quad \widetilde{b}_1(\delta) = \frac{3\pi}{4}(\lambda_1 - \lambda_2)\alpha + \pi\beta,\\
\widetilde{b}_2(\delta) = \frac{105\pi}{96}(\lambda_1^2 + 2\lambda_1\lambda_2 + \lambda_2^2 + \frac{80}{105})\alpha + \frac{5\pi}{4}(\lambda_1 + \lambda_2)\beta + \pi,\\
\bar{b}_0(\delta) = \frac{4\pi^{\frac{3}{2}}\sqrt{2}\alpha}{3\Gamma^2(\frac{3}{4})\sqrt[4]{\lambda_3}}, \quad \bar{b}_1(\delta) = \frac{8\sqrt{2}\Gamma^2(\frac{3}{4})(2\lambda_3\beta - \alpha)}{5{\lambda_3}^{\frac{7}{4}}\sqrt{\pi}},\\
\bar{b}_2(\delta) = \frac{2\sqrt{2}\pi^{\frac{3}{2}}(15\alpha - 20\lambda_3\beta + 24\lambda_3^2)}{63\Gamma^2(\frac{3}{4}){\lambda_3}^{\frac{13}{4}}}, \quad \widehat{b}_0(\delta) = \frac{\sqrt{2}6^{\frac{1}{6}}\pi^{\frac{3}{2}}\alpha}{\Gamma(\frac{5}{6})\Gamma(\frac{2}{3})},\\
\widehat{b}_1(\delta) = \frac{2\sqrt{3}\pi\beta}{3}, \quad \widehat{b}_2(\delta) = \frac{3\times 6^{\frac{4}{3}}\Gamma(\frac{5}{6})\Gamma(\frac{2}{3})}{8\sqrt{\pi}},
\end{gathered}
$$

Taking $\alpha = \beta = 0$, then $b_0 = b_1 = \widetilde{b}_0 = \widetilde{b}_1 = \bar{b}_0 = \bar{b}_1 = \widehat{b}_0 = \widehat{b}_1 = 0$ and $b_2 = \widetilde{b}_2 = \bar{b}_2 = \widehat{b}_2 = \pi$. It is easy to find that

$$\det \frac{\partial(b_0, b_1)}{\partial(a_0, a_1)} = \det \frac{\partial(\widetilde{b}_0, \widetilde{b}_1)}{\partial(a_0, a_1)} = \det \frac{\partial(\bar{b}_0, \bar{b}_1)}{\partial(a_0, a_1)} = \det \frac{\partial(\widehat{b}_0, \widehat{b}_1)}{\partial(a_0, a_1)} = 2.$$

Let us take $\alpha \ll -\beta \ll 1$, then $b_0 \ll -b_1 \ll b_2$, $\widetilde{b}_0 \ll -\widetilde{b}_1 \ll \widetilde{b}_2$, $\bar{b}_0 \ll -\bar{b}_1 \ll \bar{b}_2$ and $\widehat{b}_0 \ll -\widehat{b}_1 \ll \widehat{b}_2$, which imply there exist 2 zeros for each Abelian integral $I(h,\delta)$, $I^b(h,\delta)$, $I^c(h,\delta)$ and $I^d(h,\delta)$.

4.1. **Conclusion.** The number of zeros of Abelian integral for system (1.6) has been studied for all 12 cases except for case 4. Up to now, the sharp bounds of the numbers of zeros for the corresponding Abelian integrals defined on all period annuluses for one case of system (1.6) are obtained for case 5, 6, 7, 12 and case 9, the sharp bound for other cases are our further research.

Appendix A. This section shows two polynomials with positive coefficients that have no root on $\{(\alpha,\lambda) : \alpha > 0, \lambda \geq 1\}$.

$$\begin{aligned}
&p_2(\alpha,\beta)\\
&= 1728\alpha^{12} + 380160\alpha^9 + 114048\alpha^{10} + 20736\,\alpha^{11} + 238656\alpha^3 + 672144\alpha^4\\
&\quad + 1220736\alpha^5 + 1522752\alpha^6 + 1347456\alpha^7 + 852720\alpha^8 + 49632\alpha^2 + 5952\alpha\\
&\quad + 64\beta + 36\beta^3 + 20736\beta^3\alpha^{11} + 1728\beta^3\alpha^{12} + 3622848\beta\alpha^7 + 2395560\beta\alpha^8\\
&\quad + 3226296\beta^2\alpha^7 + 2235831\beta^2\alpha^8 + 950904\beta^3\alpha^7 + 692991\beta^3\alpha^8 + 1103760\beta\alpha^9\\
&\quad + 338472\beta\alpha^{10} + 1067040\beta^2\alpha^9 + 334800\beta^2\alpha^{10} + 343440\beta^3\alpha^9 + 110376\beta^3\alpha^{10}\\
&\quad + 62208\beta\alpha^{11} + 5184\beta\alpha^{12} + 62208\beta^2\alpha^{11} + 5184\beta^2\alpha^{12} + 11360\beta\alpha + 101296\beta\alpha^2\\
&\quad + 496896\beta\alpha^3 + 1491744\beta\alpha^4 + 96\beta^2 + 7356\beta^2\alpha + 69162\beta^2\alpha^2 + 349356\beta^2\alpha^3\\
&\quad + 1102515\beta^2\alpha^4 + 2910624\beta\alpha^5 + 3875376\beta\alpha^6 + 2293896\beta^2\alpha^5 + 3258564\beta^2\alpha^6\\
&\quad + 1638\beta^3\alpha + 15831\beta^3\alpha^2 + 82476\beta^3\alpha^3 + 271845\beta^3\alpha^4 + 598662\beta^3\alpha^5\\
&\quad + 905049\beta^3\alpha^6.
\end{aligned}$$

$$\begin{aligned}
&p_3(\alpha,\lambda)\\
&= 126 + 1012\lambda + 1026\lambda^4 + 2988\lambda^3 + 2784\lambda^2 + 40236\alpha^3 + 149541\alpha^4 + 223398\alpha^5\\
&\quad + 153657\alpha^6 + 49896\alpha^7 + 6237\alpha^8 + 8519\alpha^2 + 12912\lambda^2\alpha + 123300\lambda^2\alpha^2\\
&\quad + 832788\lambda^2\alpha^3 + 3401511\lambda^2\alpha^4 + 8976510\lambda^2\alpha^5 + 15729117\lambda^2\alpha^6\\
&\quad + 18511416\lambda^2\alpha^7 + 14641209\lambda^2\alpha^8 + 2228\lambda\alpha + 49054\lambda\alpha^2 + 285564\lambda\alpha^3\\
&\quad + 1009941\lambda\alpha^4 + 2174058\lambda\alpha^5 + 2773983\lambda\alpha^6 + 70\alpha + 484704\lambda^2\alpha^{11}\\
&\quad + 40392\lambda^2\alpha^{12} + 222750\lambda\alpha^9 + 22275\lambda\alpha^{10} + 57553848\lambda^3\alpha^7\\
&\quad + 64464741\lambda^3\alpha^8 + 52252794\lambda^3\alpha^9 + 30306969\lambda^3\alpha^{10} + 12249792\lambda^3\alpha^{11}\\
&\quad + 3274704\lambda^3\alpha^{12} + 520128\lambda^3\alpha^{13} + 37152\lambda^3\alpha^{14} + 2102760\lambda\alpha^7 + 931095\lambda\alpha^8\\
&\quad + 7663590\lambda^2\alpha^9 + 2543607\lambda^2\alpha^{10} + 12906\lambda^4\alpha + 116181\lambda^4\alpha^2 + 780624\lambda^4\alpha^3\\
&\quad + 3723408\lambda^4\alpha^4 + 12731364\lambda^4\alpha^5 + 31954230\lambda^4\alpha^6 + 60008256\lambda^4\alpha^7\\
&\quad + 85345326\lambda^4\alpha^8 + 92431746\lambda^4\alpha^9 + 76157037\lambda^4\alpha^{10} + 47344608\lambda^4\alpha^{11}
\end{aligned}$$

$$+ 21819240\lambda^4\alpha^{12} + 7221312\lambda^4\alpha^{13} + 1621728\lambda^4\alpha^{14} + 221184\lambda^4\alpha^{15}$$
$$+ 13824\lambda^4\alpha^{16} + 24876\lambda^3\alpha + 197154\lambda^3\alpha^2 + 1274868\lambda^3\alpha^3 + 5656527\lambda^3\alpha^4$$
$$+ 17269902\lambda^3\alpha^5 + 37205973\lambda^3\alpha^6.$$

Acknowledgements. This work was supported by the National Natural Science Foundations of China (No. 11101118, 11261013), Natural Science Foundation of Hebei Province (A2012205074) and Research project of Guangxi Universities (2013YB216).

References

[1] R. Asheghi, H. R. Z. Zangeneh; *Bifurcations of limit cycles from quintic Hamiltonian systems with an eye-gure loop*, Nonlinear Analysis 68(2008), 2957-2976.

[2] R. Asheghi, H.R.Z. Zangeneh; *Bifurcations of limit cycles from quintic Hamiltonian systems with an eye-figure loop (II).* Nonlinear Analysis 69(2008), 4143-4162.

[3] R. Asheghi, H. R. Z. Zangeneh; *Bifurcations of limit cycles for a quintic Hamiltonian system with a double cuspidal loop.* Comput. Math. Appl. 59(2010), 1409-1418.

[4] A. Atabaigi, R. Z. H. Zangeneh; *Bifurcation of limit cycles in small perturbations of a class of hyper-elliptic Hamiltonian systems of degree 5 with a cusp*, J. Appl. Anal. Comput. 1 (2011), 299-313.

[5] C. Christopher, C. Li; *Limit cycles of differential equations* (Advanced Courses in Mathematics), CRM Barcelona, Birkhäuser Verlag, Basel. (2007).

[6] C. Christopher, N. Lloyd; Small-amplitude limit cycles in Liénard systems, Nonl. Differ. Equat. Appl. 3(1996), 183-190.

[7] F. Dumortier, C. Rousseau; *Cubic Liénard equations with linear damping*, Nonlinearity 3(1990), 1015-1039.

[8] F. Dumortier, C. Li; *Perturbations from an elliptic Hamiltonian of degree four: (I) Saddle loop and two saddle cycle*, J. Differ. Equat. 176 (2001), 114-157.

[9] F. Dumortier, C. Li; *Perturbations from an elliptic Hamiltonian of degree four: (II) Cuspidal loop*, J. Differ. Equat. 175(2001), 209-243.

[10] F. Dumortier, C. Li; *Perturbations from an elliptic Hamiltonian of degree four: (III) Global centre*, J Differ. Equat. 188(2003), 473-511.

[11] F. Dumortier, C. Li; *Perturbations from an elliptic Hamiltonian of degree four: (IV) Figure eight-loop*, J. Differ. Equat. 88(2003), 512-514.

[12] M. Grau, F. Mañosas, J. Villadelprat; *A Chebyshev criterion for Abelian integrals.* Trans. Amer. Math. Soc. 363(2011), 109-129.

[13] M. Han, J. Li; *Lower bounds for the Hilbert number of polynomial systems*, J. Differ. Equat. 252(2012), 3278-3304.

[14] M. Han; *Asymptotic expansions of Melnikov functions and limit cycle bifurcations*, Int. J. Bifur. Chaos 22(2012), 1250296 (30 pages).

[15] M. Han, J. Yang, A. Tarta, Y. Gao; *Limit cycles near homoclinic and heteroclinic loops*, J. Dyn. Differ. Equat. 20(2008), 923-944.

[16] M. Han, J. Yang, P. Yu; *Hopf bifurcations for near-Hamiltonian systems*, Int. J. Bifur. Chaos 19(2009), 4117-4130.

[17] C, Li; *Abelian Integrals and Limit Cycles.* Qual. Theory Dyn. Syst. 11(2012), 111-128.

[18] J. Li; *Hilbert's 16th problem and bifurcations of planar vector fields*, Int. J. Bifur. Chaos. 13(2003), 47-106.

[19] C. Li, P. Mardečić, R. Roussarie; *Perturbations of symmetric elliptic Hamiltonians of degree four*, J. Differ. Equat. 231(2011), 78-91.

[20] C. Li, Z. Zhang; *A criterion for determining the monotonicity of the ratio of two Abilian integral*, J. Differ. Equat. 127(1996), 407-424.

[21] P. Mardešić; *Chebyshev Systems and the Versal Unfolding of the Cusp of Order n*, Travaux en Cours, vol. 57. Hermann, 1998.

[22] F. Mañosas, J. Villadelprat; *Bounding the number of zeros of certain Abelian integrals*, J. Differ. Equat. 251(2011), 1656-1669.

[23] M. Qi, L. Zhao; *Bifurcation of limit cycles for a class of quintic Hamiltonian system with double heteroclinic loops*, J. Beijing Normal University (Natural Science) 48 (2012), 587-591.

[24] M. Qi, L. Zhao; *Bifurcations of Limit Cycles from a Quintic Hamiltonian System with a Figure Double-fish*, Int. J. Bifur. Chaos 23, 1350116 (2013) (15 pages).
[25] X. Sun, J. Su, M. Han; *On the number of zeros of Abelian integral for some Liénard system of type (4,3)*, Chaos, Solitons & Fractals 51(2013), 1-12.
[26] X. Sun; *Bifurcation of limit cycles from a hyper-elliptic Hamiltonian system with a double heteroclinic loops*, Adv. Differ. Equat. 2012, 2012:224
[27] J. Wang, D. Xiao; *On the number of limit cycles in small perturbations of a class of hyper-elliptic Hamiltonian systems with one nilpotent saddle*, J. Differ. Equat. 250 (2011), 2227-2243.
[28] J. Wang; Perturbations of several planar integral system with denerate singularity,[D], Shanghai Jiaotong Univesity(2012).
[29] W. Xu, C. Li; *Limit cycles of some polynomial Liénard systems*, J. Math. Anal. Appl. 389(2012), 367-378.
[30] J. Yang, J. M. Han, V. G. Romanovski; *Limit cycle bifurcations of some Lienard systems*, J. Math. Anal. Appl. 366(2010), 242-255.
[31] J. Yang, M. Han; *Limit cycles bifurcations of some Liénard systems with nilpotent cusp*, Int. J. Bifur. Chaos. 20(2010), 3829-3839.
[32] J. Yang; *On the limit cycles of a kind of Lienard system with a nilpotent center under perturbations*, J. Appl. Anal. Comput. 2 (2012), 325-339.
[33] J. Yang, M. Han; *Computation of expansion coefficients of Melnikov functions near a nilpotent center*, Comput. Math. Appl. 64(2012), 1957-1974.
[34] T. Zhang, M. Tadé, Y. Tian; *On the zeros of the Abelian integrals for a class of Liénard systems*, Physics Letters A 358(2006), 262-274.
[35] L. Zhao; *The perturbations of a class of hyper-elliptic Hamilton systems with a double homoclinic loop through a nilpotent saddle*, Nonlinear Analysis 95 (2014), 374-387.

Xianbo Sun
Department of Applied Mathematics, Guangxi University of Finance and Economics, Nanning, 530003 Guangxi, China
E-mail address: `xianbo01@126.com Tel +86 15977781786`

Junmin Yang
College of Mathematics and Information Science, Hebei Normal University
Shijiazhuang, 050024 Hebei, China
E-mail address: `jmyzhw@sina.com`