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PERSISTENCE AND EXTINCTION IN STOCHASTIC SIRS MODELS WITH GENERAL NONLINEAR INCIDENCE RATE

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Abstract. In this article, a SIRS epidemic model with general nonlinear incidence rate is proposed and investigated. We briefly discuss the global stability of the deterministic system by using Lyapunov function. For the stochastic version, the global existence and positivity of the solution are studied, and the global stability in probability and pth-moment of the system are proved under suitable assumptions on the white noise perturbations. Furthermore, the sufficient conditions for the persistence and extinction of the disease are obtained. Finally, the theoretical results are illustrated by numerical simulations.

1. Introduction

In this article we shall consider the stochastic differential system

$$\begin{aligned} \mathrm{d}S &= (b - \frac{\beta SI}{1+\alpha I^h} - dS + \gamma R)\mathrm{d}t - \sigma \frac{SI}{1+\alpha I^h}\mathrm{d}B(t), \\ \mathrm{d}I &= [\frac{\beta SI}{1+\alpha I^h} - (d+\mu+\eta)I]\mathrm{d}t + \sigma \frac{SI}{1+\alpha I^h}\mathrm{d}B(t), \\ \mathrm{d}R &= [\mu I - (d+\gamma)R]\mathrm{d}t, \end{aligned} \tag{1.1}$$

as a stochastically perturbed system of the ordinary deterministic system

$$\begin{aligned} \dot{S} &= b - \frac{\beta SI}{1+\alpha I^h} - dS + \gamma R, \\ \dot{I} &= \frac{\beta SI}{1+\alpha I^h} - (d+\eta+\mu)I, \\ \dot{R} &= \mu I - (d+\gamma)R, \end{aligned} \tag{1.2}$$

where $S(t) + I(t) + R(t) \equiv N(t)$, denotes the total number of a population at time t; $S(t)$, $I(t)$ and $R(t)$ denote the numbers of the population susceptible to the disease, of the infective members, and of the members who have been removed from the possibility of infection through full immunity, respectively. It is assumed that all newborns are susceptible. The assumptions on system (1.2): b is the recruitment rate of the population; β is the daily contact rate, i.e., the average number of contacts per infective per day. The contact of an infective is an interaction which

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results in infection of the other individual if it is susceptible; d the natural death rates, η the additional disease-caused rate suffered by the infectious individuals and μ is the daily recovery rate of infective individuals; γ is the rate at which recovered individuals lose immunity and return to the susceptible class and α, h are positive parameters. Of course, $b, \beta, d, \eta, \mu, \gamma \in \mathbb{R}_+$.

In the past the classical infectious disease model with bilinear incidence βSI is often used. But in the actual incidence S and I may not be linear relationship. The nonlinear incidence rate $g(I) = \frac{\beta I}{(1+\alpha I)}$ was used by Capasso and Serio [6] in their modeling of cholera. Then, Liu, Levin and Iwasa introduced a more general nonlinear rate $g(I) = \frac{\beta I^q}{(1+\alpha I^h)}(h \geq 1)$ into epidemic models [15], where βI^q measures the infection fore of the disease and $\frac{1}{(1+\alpha I^h)}$ measures the inhibition effect from the behavioral change of the susceptible individuals when their number increases or from the crowding effect of the infective individuals. This incidence rate seems more reasonable than the bilinear incidence rate βSI, because it includes the behavioral change and crowding effect of the infective individuals and prevents the unboundedness of the contact rate by choosing suitable parameters. A variety of nonlinear incidence rates have been used in the literatures [8, 9, 13, 16, 19, 21, 23, 22, 25].

These important and useful studies on deterministic models provide a great insight into the effect of epidemic models. As a matter of fact, the epidemic models are often subject to environmental noise, i.e., due to environmental fluctuation, parameters involved in epidemic models are not absolutely constant, and they may fluctuate around some average values. Based on these factors, more and more people investigated stochastic epidemic system [3, 5, 7, 10, 11, 12, 20, 24, 26].

Taking into account the effect of randomly environment, we incorporate white noise in system (1.2), by replacing the contact rate β in system (1.2) by $\beta + \sigma\dot{B}$, where $\dot{B}$ is a white noise (i.e., $B(t)$ is a Brownian motion) and σ represent the intensity of the white noise. Therefore, system (1.2) can be described by stochastic system (1.1).

This paper is organized as follows: for system (1.2), we firstly consider the global stability of the equilibrium by means of constructing suitable Lyapunov functions. In section 3.1, we prove the existence, uniqueness and positivity of the solution of the stochastic system (1.1). In section 3.2, we show pth-moment exponential stability and almost surely exponential stability of the disease-free equilibrium under certain conditions. In section 3.3, we obtain that stochastic system is stochastically permanent and persistence in mean. In section 3.4, we discuss the stochastic extinction of system (1.1). Finally, we perform some numerical simulations to compare the dynamic behaviors of stochastic system (1.2) and deterministic system (1.1).

2. Global stability of (1.2)

For system (1.2), the basic reproduction number $R_0 = \frac{\beta b}{d(d+\eta+\mu)}$ is the threshold of the system for an epidemic to occur. It is easy, by simple computations, to conclude that system (1.2) has two equilibrium states. If $R_0 \leq 1$, system (1.2) has only a disease-free equilibrium $P^0 = (\frac{b}{d}, 0, 0)$, which is globally asymptotical stable. That is to say, the disease will disappear and the entire population will become susceptible. If $R_0 > 1$, P^0 becomes unstable and there is a unique positive equilibrium

$P^* = (S^*, I^*, R^*)$, which is called the endemic equilibrium and determined by

$$S^* = \frac{b - (d + \eta + \mu)I^*}{d} \quad \text{and} \quad R^* = \frac{\mu}{d} I^*,$$

where

$$\begin{aligned} &\alpha d(d + \mu + \eta)(d + \gamma)(I^*)^h + \beta[d(d + \gamma)(d + \eta) + d\mu]I^* \\ &+ (d + \gamma)[d(d + \mu + \eta) - b\beta] = 0. \end{aligned} \tag{2.1}$$

Through calculation, we can prove the equation (2.1) has only a positive root I^* if and only if $d(d + \mu + \eta) - b\beta < 0$.

The objectives of this section are to prove the global stability of the disease-free equilibrium and endemic equilibrium. It is easy to see that

$$\Gamma = \{(S, I, R) : S \geq 0,\ I \geq 0,\ R \geq 0,\ S + I + R \leq \frac{b}{d}\}$$

is a positive invariant set of system (1.2).

Theorem 2.1. *When $R_0 \leq 1$, the disease-free equilibrium P^0 is globally asymptotically stable in Γ.*

Proof. Define a Lyapunov function

$$V(t) = I(t).$$

Then the derivative of V along the positive solution of system (1.2), we obtain

$$\dot{V}|_{(1.2)} = \dot{I} = \frac{\beta SI}{1 + \alpha I^h} - (d + \eta + \mu)I.$$

Notice that $1 + \alpha I^h > 1, S + I + R < \frac{b}{d}$ and $R_0 \leq 1$, from the above, we have that

$$\dot{V}|_{(1.2)} \leq [\frac{\beta b}{d} - (d + \eta + \mu)]I = (d + \eta + \mu)(R_0 - 1)I \leq 0.$$

Thus, the disease-free equilibrium P^0 is globally asymptotically stable. □

Theorem 2.2. *Whenever $R_0 > 1$, the unique endemic equilibrium P^* is globally asymptotically stable in Γ.*

Proof. Through summing the equations of system (1.2), we obtain that the total population size verifies the equation,

$$\dot{N} = b - dN - \eta I. \tag{2.2}$$

It is convenient to choose the variable (N, I, R) instead of (S, I, R). Then, we consider the system

$$\begin{aligned} \dot{N} &= b - dN - \eta I, \\ \dot{I} &= \frac{\beta(N - I - R)I}{1 + \alpha I^h} - (d + \eta + \mu)I, \\ \dot{R} &= \mu I - (d + \gamma)R. \end{aligned} \tag{2.3}$$

So the endemic equilibrium $P^*(S^*, I^*, R^*)$ of system (1.2) corresponds to the endemic equilibrium $\widetilde{P^*}(N^*, I^*, R^*)$ of system (2.3). In order to simplify the expressions, we define

$$f(I) = 1 + \alpha I^h.$$

So system (2.3) becomes

$$\begin{aligned}
\dot{N} &= -d(N-N^*) - \eta(I-I^*),\\
\dot{I} &= \big[\frac{(N-N^*)-(R-R^*)}{f(I^*)} - \frac{(N-I-R)[f(I)-f(I^*)]}{f(I)f(I^*)}\big]\beta I + \frac{(I-I^*)f(I)}{f(I)f(I^*)}\beta I,\\
\dot{R} &= \mu(I-I^*) - (d+\gamma)(R-R^*).
\end{aligned} \tag{2.4}$$

Let us consider the function

$$V(I,R,N) = \frac{1}{\beta}(I - I^* - I^* \ln\frac{I}{I^*}) + \frac{(R-R^*)^2}{2\mu f(I^*)} + \frac{(N-N^*)^2}{2\eta f(I^*)}.$$

Then the derivative of V along the solution of (2.4) is

$$\begin{aligned}
\dot{V}|_{(2.4)} = &-\frac{(d+\gamma)(R-R^*)^2}{\mu f(I^*)} - \frac{d(N-N^*)^2}{\eta f(I^*)} - \frac{(I-I^*)^2}{f(I^*)}\\
&- \frac{(N-I-R)(I-I^*)[f(I)-f(I^*)]}{f(I)f(I^*)}.
\end{aligned}$$

It is clear that $f'(I) > 0$, so $(I-I^*)(f(I)-f(I^*)) > 0$. Obviously, V is positive definite and $\dot{V}$ is negative definite. Therefore the function V is a Lyapunov function for system (2.4) and consequently, by Lyapunov asymptotic stability theorem [17], the equilibrium state P^* is globally asymptotically stable. □

3. Stochastic SIRS model

In this paper, unless otherwise specified, we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all P-null sets). Let $B(t)$ be the Brownian motion defined on this probability space. Denote

$$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i > 0 \text{ for all } 1 \leq i \leq n\}.$$

In general, consider the n-dimensional stochastic differential equation [11]

$$\mathrm{d}x(t) = f(x(t),t)\mathrm{d}t + g(x(t),t)\mathrm{d}B(t), \quad \text{for } t \geq t_0. \tag{3.1}$$

Denote by $C^{2,1}(\mathbb{R}^n \times [t_0,\infty);\mathbb{R}_+)$ the family of all nonnegative functions $V(x,t)$ defined on $\mathbb{R}^n \times [t_0,\infty)$ such that they are continuously twice differentiable in x and once in t. Define the differential operator L associated with (3.1) by

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^n f_i(x,t)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^n [g^T(x,t)g(x,t)]_{ij}\frac{\partial^2}{\partial x_i \partial x_j}.$$

If L acts on a function $V \in C^{2,1}(\mathbb{R}^n \times [t_0,\infty);\mathbb{R}_+)$, then

$$LV(x(t),t) = V_t(x,t) + V_x(x,t)f(x,t) + \frac{1}{2}\operatorname{trace}[g^T(x,t)V_{xx}(x,t)g(x,t)],$$

where $V_t = \frac{\partial V}{\partial t}$, $V_x = (\frac{\partial V}{\partial x_1}, \cdots, \frac{\partial V}{\partial x_n})$ and $V_{xx} = (\frac{\partial^2 V}{\partial x_i x_j})_{n\times n}$. By Itô formula,

$$\mathrm{d}V(x(t),t) = LV(x(t),t)\mathrm{d}t + V_x(x(t))g(x(t),t)\mathrm{d}B(t).$$

3.1. **Positive and global solutions.**

Theorem 3.1. *For any given initial condition $(S(0), I(0), R(0)) \in \Gamma$, there is a unique positive solution $(S(t), I(t), R(t))$ to (1.1) on $t \geq 0$, and the solution will remain in Γ with probability one. Namely, $(S(t), I(t), R(t)) \in \Gamma$ for all $t \geq 0$ almost surely.*

Proof. Let $(S(0), I(0), R(0)) \in \Gamma$. Obviously, since the coefficients of system (1.1) are locally Lipschitz continuous, for any given initial value $(S(0), I(0), R(0)) \in \Gamma$, there is a unique local solution $(S(t), I(t), R(t))$ on $t \in [0, \tau_e)$, where τ_e is the explosion time. First, we show $S(t) + I(t) + R(t) \leq \frac{b}{d}$ for all $t \in [0, \tau_e]$. The total population in system (1.1) verifies the equation

$$\mathrm{d}N(t) = [b - dN - \eta I]\mathrm{d}t \leq [b - dN]\mathrm{d}t. \tag{3.2}$$

Assume $X(t)$ is the solution of differential equation

$$\mathrm{d}X(t) = (b - dX(t))\mathrm{d}t,$$
$$X(0) = N(0),$$

where $N(0) = S(0) + I(0) + R(0)$. By comparison theorem, we obtain

$$N(t) \leq X(t) \leq \frac{b}{d}, \quad t \in [0, \tau_e) \text{ a.s.} \tag{3.3}$$

Next, we show the solution is global, we have only to prove that $\tau_e = \infty$ a.s. We consider an integer $k_0 > 0$ sufficiently large such that $(S(0), I(0), R(0)) \in [\frac{1}{k_0}, k_0]^3$. For each integer $k > k_0$ we define the stopping time

$$\tau_k = \inf\{t \in [0, \tau_e) : S(t) \notin (\frac{1}{k}, k), I(t) \notin (\frac{1}{k}, k) \text{ or } R(t) \notin (\frac{1}{k}, k)\}, \tag{3.4}$$

where throughout this paper we set $\inf \emptyset = \infty$ (as usual $\emptyset$ denotes the empty set). Obviously, τ_k is increasing as $k \to \infty$. Set $\tau_\infty = \lim_{t\to\infty} \tau_k$, whence $\tau_\infty \leq \tau_e$ a.s. If we can show that $\tau_\infty = \infty$ a.s. then $\tau_e = \infty$ a.s. and $(S(t), I(t), R(t)) \in \Gamma$ a.s. for all $t \geq 0$. So we need only to prove that $\tau_\infty = \infty$ a.s. If this statement is false, there are two constants $\epsilon \in (0, 1)$ and $T > 0$ such that

$$P\{\tau_\infty \leq T\} > \epsilon. \tag{3.5}$$

Hence, there is an integer $k_1 \geq k_0$ such that

$$P\{\tau_k \leq T\} \geq \epsilon, \quad \text{for any } \ k > k_1.$$

Consider the function $V(S(t), I(t), R(t))$ defined for $(S(t), I(t), R(t)) \in \Gamma$ by

$$V(S(t), I(t), R(t)) = -\ln \frac{dS}{b} - \ln \frac{dI}{b} - \ln \frac{dR}{b}.$$

Using Itô formula,

$$\mathrm{d}V = LV\mathrm{d}t + \frac{\sigma(I - S)}{1 + \alpha I^h}\mathrm{d}B(t),$$

where

$$LV = -\frac{b + \gamma R}{S} + d + \frac{\beta I}{1 + \alpha I^h} - \frac{\beta S}{1 + \alpha I^h} + d + \eta + \mu - \frac{\mu I}{R} + d + \gamma + \frac{\sigma^2(I^2 + S^2)}{2(1 + \alpha I^h)^2}.$$

By (3.3), we obtain

$$LV \leq 3d + \eta + \mu + \gamma + \beta\frac{b}{d} + \sigma^2(\frac{b}{d})^2 =: K.$$

Therefore, we obtain

$$\mathrm{d}V \le K\mathrm{d}t + \frac{\sigma(I-S)}{1+\alpha I^h}\mathrm{d}B(t). \tag{3.6}$$

By integrating both sides of (3.6) from 0 to $\tau_k \wedge T$ and then taking the expectation of both sides, it yields

$$E[V(S(\tau_k \wedge T), I(\tau_k \wedge T), R(\tau_k \wedge T))] \le V(S(0), I(0), R(0)) + KT.$$

Let $\Omega_k = \{\tau_k \le T\}$, then $P(\Omega_k) \ge \epsilon$. Note that for every $\omega \in \Omega_k$, there is at least $S(\tau_k, \omega), I(\tau_k, \omega), R(\tau_k, \omega)$ equals k or $\frac{1}{k}$, since

$$-\ln\frac{d}{bk} = -\ln\frac{dk}{b}\cdot\frac{1}{k^2} = -\ln\frac{dk}{b} + 2\ln k \ge -\ln\frac{dk}{b}, \qquad (k > k_0 \ge 1),$$

so

$$V(S(\tau_k,\omega), I(\tau_k,\omega), R(\tau_k,\omega)) \ge -\ln\frac{dk}{b} \wedge -\ln\frac{d}{bk} \ge -\ln\frac{dk}{b}.$$

It then follows that

$$\begin{aligned}
V(S(0), I(0), R(0)) + KT &\ge E[I_{\Omega_k}(\omega)V(S(\tau_k \wedge T), I(\tau_k \wedge T), R(\tau_k \wedge T))] \\
&= E[I_{\Omega_k}(\omega)V(S(\tau_k,\omega), I(\tau_k,\omega), R(\tau_k,\omega))] \\
&\ge E[-I_{\Omega_k}(\omega)\ln\frac{dk}{b}] \\
&= -\ln\frac{dk}{b}E[I_{\Omega_k}(\omega)] \\
&\ge -\epsilon\ln\frac{dk}{b},
\end{aligned}$$

where $I_{\Omega_k}(\omega)$ is the indicator function of $\Omega_k(\omega)$. Letting $k \to \infty$, it leads to the contradiction

$$\infty > V(S(0), I(0), R(0)) + KT = \infty,$$

so we have $\tau_e = \infty$ a.s., which completes the proof. □

3.2. **Behavior of** (1.1) **when** $R_0 < 1$**.** For the deterministic SIRS system (1.2), we prove that $P^0 = (\frac{b}{d}, 0, 0)$ is the disease-free equilibrium and it is globally stable if $R_0 = \frac{b\beta}{d(d+\eta+\mu)} \le 1$. Notice that $P^0 = (\frac{b}{d}, 0, 0)$ is also the disease-free equilibrium for the stochastic system (1.1). In this section, we present the following theorem which gives some conditions for the pth-moment exponential stability of the disease-free equilibrium of stochastic system (1.1) in terms of Lyapunov function [1].

Moment exponential stability.

Lemma 3.2. *Let p, c_1, c_2 and c_3 be positive numbers. Suppose that there exists a function $V(t,x) \in C^{1,2}(\mathbb{R}_+, \mathbb{R}^n)$ such that*

$$c_1|x|^p \le V(t,x) \le c_2|x|^p,$$

and

$$LV(t,x) \le -c_3|x|^p,\ t \ge 0,$$

the equilibrium of system (3.1) *is pth-moment exponentially stable. When $p = 2$, it is usually said to be mean square exponentially stable and the equilibrium $x = 0$ is globally asymptotically stable.*

Lemma 3.3. *Set $p \geq 2$ and $\varepsilon, x, y > 0$. Then*

$$x^{p-1}y \leq \frac{(p-1)\varepsilon}{p}x^p + \frac{1}{p\varepsilon^{p-1}}y^p \quad \text{and} \quad x^{p-2}y^2 \leq \frac{(p-2)\varepsilon}{p}x^p + \frac{2}{p\varepsilon^{\frac{p-2}{2}}}y^p.$$

This lemma can be proved easily by using the elementary inequality

$$u^q v^{1-q} \leq qu + (1-q)v, \quad 0 < q < 1,$$

so we omit its proof.

Theorem 3.4. *Set $p \geq 2$. If $R_0 \leq 1$ and $\frac{1}{2}(p-1)\sigma^2(\frac{b}{d})^2 < (d+\mu+\eta)(1-R_0)$ hold, the disease-free equilibrium P^0 of system (1.1) is pth-moment exponentially stable in Γ.*

Proof. Set $p \geq 2$ and $(S(0), I(0), R(0)) \in \Gamma$, in view of Theorem 3.1 the solution of system (1.1) remains in Γ. Considering the Lyapunov function

$$V = (\frac{b}{d} - S)^p + \frac{1}{p}I^p + R^p,$$

by Itô formula, we obtain

$$\mathrm{d}V = LV\mathrm{d}t + p\sigma(\frac{b}{d} - S)^{p-1}\frac{SI}{1+\alpha I^h}\mathrm{d}B + \frac{p\sigma SI^p}{1+\alpha I^h}\mathrm{d}B,$$

where

$$\begin{aligned} LV = &-pd(\frac{b}{d} - S)^p + p\beta(\frac{b}{d} - S)^{p-1}\frac{SI}{1+\alpha I^h} - p\gamma(\frac{b}{d} - S)^{p-1}R \\ &+ p(p-1)(\frac{b}{d} - S)^{p-2}\frac{\sigma^2 S^2 I^2}{2(1+\alpha I^h)^2} - (d+\eta+\mu)I^p + \frac{\beta SI^p}{1+\alpha I^h} \\ &+ (p-1)\frac{\sigma^2 S^2 I^p}{2(1+\alpha I^h)^2} + p\mu IR^{p-1} - p(d+\gamma)R^p. \end{aligned}$$

In view of Theorem 3.1, we have $\max\{S, I, R\} \leq \frac{b}{d}$, hence

$$\begin{aligned} LV \leq &-pd(\frac{b}{d} - S)^p + \frac{b}{d}p\beta I(\frac{b}{d} - S)^{p-1} \\ &+ \frac{p(p-1)}{2}\sigma^2(\frac{b}{d})^2 I^2(\frac{b}{d} - S)^{p-2} - (d+\eta+\mu)I^p + \frac{b}{d}\beta I^p \\ &+ \frac{(p-1)}{2}\sigma^2(\frac{b}{d})^2 I^p + p\mu IR^{p-1} - p(d+\gamma)R^p. \end{aligned}$$

Simplifying the above, we obtain

$$\begin{aligned} LV \leq &-pd(\frac{b}{d} - S)^p - [p(d+\eta+\mu) - \frac{b}{d}p\beta - \frac{p(p-1)}{2}\sigma^2(\frac{b}{d})^2]I^p - p(d+\gamma)R^p \\ &+ \frac{b}{d}p\beta I(\frac{b}{d} - S)^{p-1} + \frac{p(p-1)}{2}\sigma^2(\frac{b}{d})^2 I^2(\frac{b}{d} - S)^{p-2} + p\mu IR^{p-1}. \end{aligned}$$

Using Lemma 3.3, for any $\varepsilon > 0$, we obtain

$$\begin{aligned} (\frac{b}{d} - S)^{p-1}I &\leq \frac{(p-1)\varepsilon}{p}(\frac{b}{d} - S)^p + \frac{1}{p\varepsilon^{p-1}}I^p, \\ R^{p-1}I &\leq \frac{(p-1)\varepsilon}{p}R^p + \frac{1}{p\varepsilon^{p-1}}I^p, \\ I^2(\frac{b}{d} - S)^{p-2} &\leq \frac{(p-2)\varepsilon}{p}(\frac{b}{d} - S)^p + \frac{2}{p\varepsilon^{\frac{p-1}{2}}}I^p. \end{aligned}$$

Substituting these three inequalities in the above inequality, we obtain

$$\begin{aligned} LV \le & -[pd - (\frac{(p-1)(p-2)}{2}\sigma^2(\frac{b}{d})^2 + \beta\frac{b}{d}(p-1))\varepsilon](\frac{b}{d} - S)^p \\ & - [p(d+\gamma) - \mu(p-1)\varepsilon]R^p - [(d+\eta+\mu)(1-R_0) - \frac{(p-1)}{2}\sigma^2(\frac{b}{d})^2 \\ & - \beta\frac{b}{d}\varepsilon^{1-p} - (p-1)\sigma^2(\frac{b}{d})^2\varepsilon^{\frac{2-p}{p}} - \mu\varepsilon^{1-p}]I^p. \end{aligned}$$

We choose ε sufficiently small such that the coefficients of $(\frac{b}{d} - S)^p$ and R^p be negative, and since $(d+\eta+\mu)(1-R_0) - \frac{(p-1)}{2}\sigma^2(\frac{b}{d})^2 > 0$, the coefficient of I^p must be negative. According to Lemma 3.2, the proof is complete. □

Remark 3.5. From Lemma 3.2, Theorem 3.4 and the case $p = 2$, we get that if the conditions $R_0 < 1$ and $\frac{1}{2}\sigma^2(\frac{b}{d})^2 < (d+\eta+\mu)(1-R_0)$ hold, the disease-free P^0 of system (1.1) is globally asymptotically stable in Γ.

Almost sure exponential stability.

Theorem 3.6. *If $\frac{1}{2}\beta^2 < d\sigma^2$ hold, then the disease-free equilibrium P^0 of system (1.1) is almost sure exponential stable in Γ.*

Proof. The proof is similar to [14]. In view of Theorem 3.1, we define the function

$$V = \ln[(\frac{b}{d} - S) + I + R].$$

With the multi-dimensional Itô formula, we obtain

$$\begin{aligned} \mathrm{d}V = & \frac{1}{\frac{b}{d} - S + I + R}[-b + \frac{2\beta SI}{1+\alpha I^h} + dS - (d+2\gamma)R \\ & - (d+\eta)I - \frac{2\sigma^2 S^2 I^2}{(\frac{b}{d} - S + I + R)^2(1+\alpha I^h)^2}]\mathrm{d}t \\ & + \frac{2\sigma SI}{(\frac{b}{d} - S + I + R)(1+\alpha I^h)}\mathrm{d}B(t). \end{aligned}$$

Set $U = \frac{SI}{(\frac{b}{d} - S + I + R)(1+\alpha I^h)}$, from the above equation, we obtain

$$\begin{aligned} \mathrm{d}V &= [-2\sigma^2U^2 + 2\beta U - \frac{d(\frac{b}{d} - S) + (d+\eta)I + (d+2\gamma)R}{\frac{b}{d} - S + I + R}]\mathrm{d}t + 2\sigma U\mathrm{d}B(t) \\ &\le [-2\sigma^2U^2 + 2\beta U - d]\mathrm{d}t + 2\sigma U\mathrm{d}B(t) \\ &\le \frac{\beta^2 - 2d\sigma^2}{2\sigma^2}\mathrm{d}t + 2\sigma U\mathrm{d}B(t), \end{aligned}$$

namely,

$$\mathrm{d}V \le \frac{\beta^2 - 2d\sigma^2}{2\sigma^2}\mathrm{d}t + 2\sigma U\mathrm{d}B(t). \tag{3.7}$$

Integrating both sides from 0 to t, we have

$$\ln[(\frac{b}{d} - S) + I + R] \le \ln[(\frac{b}{d} - S(0)) + I(0) + R(0)] + \frac{\beta^2 - 2d\sigma^2}{2\sigma^2}t + \int_0^t 2\sigma U\mathrm{d}B(t). \tag{3.8}$$

Let $M(t) = \int_0^t 2\sigma U \mathrm{d}B(t)$. Obviously, $M(t)$ is continuous local martingale and $M(0) = 0$. Furthermore,

$$\limsup_{t\to\infty} \frac{\langle M, M\rangle_t}{t} \le 4\sigma^2(\frac{b}{d})^2 < \infty.$$

By the strong law of large numbers [18, 2], we obtain

$$\lim_{t\to\infty} \frac{M(t)}{t} = 0 \tag{3.9}$$

Under the condition $\frac{1}{2}\beta^2 < d\sigma^2$ and it follows from (3.8) and (3.9) that

$$\limsup_{t\to\infty} \frac{1}{t}\ln[(\frac{b}{d} - S) + I + R] \le \frac{\beta^2 - 2d\sigma^2}{2\sigma^2} < 0$$

This completes the proof. □

Remark 3.7. It is easy to see that if $h = 1$, then Theorems 3.4 and 3.6 become Theorem 4 and Theorem 5 in [14]. For detailed information of the asymptotic behavior, we refer the reader to see [14].

3.3. **Behavior of** (1.1) **when** $R_0 > 1$**.** There is the endemic equilibrium P^* of system (1.2), but not the endemic equilibrium P^* of system (1.1). Because system (1.1) does not have the endemic equilibrium, we wish to find out whether or not the solution goes around P^*.

Asymptotic behavior around the positive equilibrium P^***.** In this section, we will investigate whether or not the solution goes around P^*. The following results give a positive answer.

Theorem 3.8. *If* $2d - \gamma > 0$ *and* $2d - \mu > 0$*, for any positive initial value* $(S(0), I(0), R(0))$*, the solution* $(S(t), I(t), R(t))$ *of system* (1.1) *satisfies*

$$\begin{aligned}&\limsup_{t\to\infty} \frac{1}{t}\int_0^t [(S - S^*)^2 + (I - I^*)^2 + (R - R^*)^2]\mathrm{d}s\\ &\le \frac{(\frac{b}{d})^2\sigma^2(2d + \eta + \mu)(1 + \alpha I^{*h})I^*}{m\beta},\end{aligned}$$

where $m = \min\{2d - \gamma, 2d - \mu\} > 0$.

Proof. Define a C^2-function

$$V(S, I, R) = V_1 + \frac{2(2d + \eta + \mu)(1 + \alpha I^{*h})}{\beta}V_2 + V_3,$$

where

$$V_1 = (S - S^* + I - I^*)^2, \quad V_2 = (I - I^* - I^*\frac{\ln I}{I^*}), V_3 = (R - R^*)^2.$$

Obviously, V_1, V_2 and V_3 are positive definite. By Itô formula, we compute

$$\begin{aligned}\mathrm{d}V_1 &= LV_1\mathrm{d}t,\\ \mathrm{d}V_2 &= LV_2\mathrm{d}t + \frac{\sigma S(I - I^*)}{1 + \alpha I^h}\mathrm{d}B,\\ \mathrm{d}V_3 &= LV_3\mathrm{d}t,\\ \mathrm{d}V &= \mathrm{d}V_1 + \frac{2(2d + \eta + \mu)(1 + \alpha I^{*h})}{\beta}\mathrm{d}V_2 + \mathrm{d}V_3.\end{aligned}$$

In detail,

$$\begin{aligned}
LV_1 &= 2(S-S^*+I-I^*)[b-dS+\gamma R-(d+\eta+\mu)I] \\
&= 2(S-S^*+I-I^*)[dS^*-dS+(d+\eta+\mu)I^*-\gamma R^*+\gamma R-(d+\eta+\mu)I] \\
&= 2(S-S^*+I-I^*)[-d(S-S^*)-(d+\eta+\mu)(I-I^*)+\gamma(R-R^*)] \\
&= -2d(S-S^*)^2-2(d+\eta+\mu)(I-I^*)^2-2(2d+\eta+\mu)(S-S^*)(I-I^*) \\
&\quad +2\gamma(R-R^*)(S-S^*)+2\gamma(R-R^*)(I-I^*)
\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}
LV_2 &= (I-I^*)\beta\Big(\frac{S}{1+\alpha I^h}-\frac{S^*}{1+\alpha {I^*}^h}\Big)+\frac{I^*\sigma^2S^2}{2(1+\alpha I^h)^2} \\
&= (I-I^*)\beta\Big(\frac{S}{1+\alpha I^h}-\frac{S}{1+\alpha {I^*}^h}+\frac{S}{1+\alpha {I^*}^h}-\frac{S^*}{1+\alpha {I^*}^h}\Big)+\frac{I^*\sigma^2S^2}{2(1+\alpha I^h)^2} \\
&= \frac{\beta}{1+\alpha {I^*}^h}(S-S^*)(I-I^*)+\beta\alpha S\frac{(I-I^*)({I^*}^h-I^h)}{(1+\alpha I^h)(1+\alpha {I^*}^h)}+\frac{I^*\sigma^2S^2}{2(1+\alpha I^h)^2} \\
&\le \frac{\beta(S-S^*)(I-I^*)}{1+\alpha {I^*}^h}+\frac{I^*\sigma^2}{2}\Big(\frac{b}{d}\Big)^2.
\end{aligned} \tag{3.11}$$

Next, we calculate

$$\begin{aligned}
LV_3 &= 2(R-R^*)[\mu I-(d+\gamma)R] \\
&= 2(R-R^*)[-(d+\gamma)(R-R^*)+\mu(I-I^*)] \\
&= -2(d+\gamma)(R-R^*)^2+2\mu(R-R^*)(I-I^*).
\end{aligned} \tag{3.12}$$

It follows from (3.10), (3.11) and (3.12) that

$$\begin{aligned}
LV &\le -2d(S-S^*)^2-2(d+\eta+\mu)(I-I^*)^2-2(2d+\eta+\mu)(S-S^*)(I-I^*) \\
&\quad +2\gamma(R-R^*)(S-S^*)+2\gamma(R-R^*)(I-I^*)+\frac{2(2d+\eta+\mu)(1+\alpha {I^*}^h)}{\beta} \\
&\quad \times\Big[\frac{\beta}{1+\alpha {I^*}^h}(S-S^*)(I-I^*)+\frac{I^*\sigma^2S^2}{2}\Big]-2(d+\gamma)(R-R^*)^2 \\
&\quad +2\mu(R-R^*)(I-I^*).
\end{aligned}$$

Since $2ab\le a^2+b^2$, we have

$$\begin{aligned}
2(R-R^*)(I-I^*) &\le (R-R^*)^2+(I-I^*)^2, \\
2(S-S^*)(R-R^*) &\le (S-S^*)^2+(R-R^*)^2.
\end{aligned}$$

The, we have

$$\begin{aligned}
LV &\le -(2d-\gamma)(S-S^*)^2-[2(d+\eta)+\mu-\gamma](I-I^*)^2 \\
&\quad -(2d-\mu)(R-R^*)^2+\frac{I^*\sigma^2(\frac{b}{d})^2(2d+\eta+\mu)(1+\alpha {I^*}^h)}{\beta}.
\end{aligned}$$

Substituting these inequalities into $\mathrm{d}V$, we obtain

$$\begin{aligned}
\mathrm{d}V &\le -m[(S-S^*)^2+(I-I^*)^2+(R-R^*)^2]\mathrm{d}t \\
&\quad +\frac{I^*\sigma^2(\frac{b}{d})^2(2d+\eta+\mu)(1+\alpha {I^*}^h)}{\beta}
\end{aligned}$$

$$+\frac{2(2d+\eta+\mu)(1+\alpha I^{*h})\sigma S(I-I^*)}{\beta(1+\alpha I^h)}\mathrm{d}B(t),$$

where $m=\min\{2d-\gamma,2d-\mu\}>0$. This implies

$$\begin{aligned}V(t)-V(0)&\le\int_0^t LV\mathrm{d}s+M(t)\\&\le\int_0^t\{-m[(S-S^*)^2+(I-I^*)^2+(R-R^*)^2]\\&\quad+\frac{I^*\sigma^2(\frac{b}{d})^2(2d+\eta+\mu)(1+\alpha I^{*h})}{\beta}\}\mathrm{d}s+M(t),\end{aligned}\tag{3.13}$$

where $M(t)$ is a martingale defined by

$$M(t)=\int_0^t\frac{2(2d+\eta+\mu)(1+\alpha I^{*h})\sigma\frac{b}{d}(I-I^*)}{\beta(1+\alpha I^h)}\mathrm{d}B(t).$$

The quadratic variation of this martingale is

$$\begin{aligned}\langle M,M\rangle_t&=\int_0^t\frac{4(2d+\eta+\mu)^2(1+\alpha I^{*h})^2\sigma^2(\frac{b}{d})^2(I-I^*)^2}{\beta^2(1+\alpha I^h)^2}\mathrm{d}s\\&\le\frac{4(2d+\eta+\mu)^2(1+\alpha I^{*h})^2\sigma^2(\frac{b}{d})^2(\frac{b}{d}+I^*)^2}{\beta^2}t.\end{aligned}$$

By the strong law of large numbers for martingales [18, 2], we have $\lim_{t\to\infty}\frac{M(t)}{t}=0$ a.s. Then by (3.13),

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t LV\mathrm{d}s\ge 0.\tag{3.14}$$

Dividing both sides of (3.13) by t, and letting $t\to\infty$, it follows that

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{t}\int_0^t[(S-S^*)^2+(I-I^*)^2+(R-R^*)^2]\\&\le\frac{I^*\sigma^2(\frac{b}{d})^2(2d+\eta+\mu)(1+\alpha I^{*h})}{m\beta}\quad\text{a.s.}\end{aligned}$$

The proof is therefore complete. □

Remark 3.9. The disturbance intensity is relevant to the value of σ. The smaller the value of σ is, the smaller the oscillation is. In other words, if the stochastic perturbations become small, the solution of system (1.1) will be close to the endemic equilibrium P^* of system (1.2).

Stochastic Persistence in Mean. Let us continue to discuss the long time behavior of the stochastic system (1.1). In view of ecology, the bad thing happens when the disease exist. In this section, we will consider another stochastic persistence; that is, stochastic persistence in mean. Now, we present the definition of persistence in mean.

Definition 3.10. System (1.1) is said to be persistent in mean [4], if

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t S(s)\mathrm{d}s>0,\quad\liminf_{t\to\infty}\frac{1}{t}\int_0^t I(s)\mathrm{d}s>0,\quad\liminf_{t\to\infty}\frac{1}{t}\int_0^t R(s)\mathrm{d}s>0,$$

where $(S(t),R(t),I(t))$ is any positive solutions of system (1.1).

Theorem 3.11. *Under the condition*

$$\sigma^2 < \min\Big\{\frac{(S^*)^2md^2\beta}{I^*b^2(2d+\eta+\mu)}, \frac{I^*md^2\beta}{b^2(2d+\eta+\mu)}, \frac{(R^*)^2md^2\beta}{I^*b^2(2d+\eta+\mu)}\Big\},$$

system (1.1) *is persistent in mean.*

Proof. Using Theorem 3.8, we have

$$\begin{aligned}
\limsup_{t\to\infty}\frac{1}{t}\int_0^t (S-S^*)^2\mathrm{d}s &\le \frac{I^*\sigma^2b^2(2d+\eta+\mu)}{md^2\beta},\\
\limsup_{t\to\infty}\frac{1}{t}\int_0^t (I-I^*)^2\mathrm{d}s &\le \frac{I^*\sigma^2b^2(2d+\eta+\mu)}{md^2\beta},\\
\limsup_{t\to\infty}\frac{1}{t}\int_0^t (R-R^*)^2\mathrm{d}s &\le \frac{I^*\sigma^2b^2(2d+\eta+\mu)}{md^2\beta}.
\end{aligned}$$

Notice that

$$2(S^*)^2 - 2S^*S = 2S^*(S^*-S) \le (S^*)^2 + (S-S^*)^2,$$

namely,

$$S \ge \frac{S^*}{2} - \frac{(S-S^*)^2}{2S^*}.$$

Then

$$\begin{aligned}
\liminf_{t\to\infty}\frac{1}{t}\int_0^t S(s)\mathrm{d}s &\ge \frac{S^*}{2} - \limsup_{t\to\infty}\frac{1}{t}\int_0^t\frac{(S-S^*)^2}{2S^*}\mathrm{d}s\\
&\ge \frac{S^*}{2} - \frac{I^*\sigma^2b^2(2d+\eta+\mu)}{2S^*md^2\beta} > 0 \quad \text{a.s.}
\end{aligned}$$

By the same way, we obtain

$$\begin{aligned}
\liminf_{t\to\infty}\frac{1}{t}\int_0^t I(s)\mathrm{d}s &\ge \frac{I^*}{2} - \limsup_{t\to\infty}\frac{1}{t}\int_0^t\frac{(I-I^*)^2}{2I^*}\mathrm{d}s\\
&\ge \frac{I^*}{2} - \frac{\sigma^2b^2(2d+\eta+\mu)}{2md^2\beta} > 0 \quad \text{a.s.}
\end{aligned}$$

and

$$\begin{aligned}
\liminf_{t\to\infty}\frac{1}{t}\int_0^t R(s)\mathrm{d}s &\ge \frac{R^*}{2} - \limsup_{t\to\infty}\frac{1}{t}\int_0^t\frac{(R-R^*)^2}{2I^*}\mathrm{d}s\\
&\ge \frac{R^*}{2} - \frac{I^*\sigma^2b^2(2d+\eta+\mu)}{2R^*md^2\beta} > 0 \quad \text{a.s.}
\end{aligned}$$

The theorem is thus proved. □

3.4. **Extinction.** In the previous sections we have showed that under certain conditions, the original autonomous model (1.2) and the associated stochastic model (1.1) behave similarly in the sense that both have positive solutions which will not explode to infinity in a finite time and, in fact, will be ultimately bounded and permanent. In other words, we show that under certain condition the noise will not spoil these properties. However, we will show in this section that if the noise is sufficiently large, the disease to the associated stochastic system (1.1) become extinct, although the disease to the original system (1.2) may be persistent.

Theorem 3.12. *For any given initial value $(S(0), I(0), R(0)) \in \Gamma$, the solution $(S(t), I(t), R(t))$ of system (1.1) has the property that*

$$\limsup_{t\to\infty} \frac{\ln I(t)}{t} \le -(d+\eta+\mu) + \frac{\beta^2}{4\sigma^2}.$$

Proof. Define $V(I(t)) = \ln I(t)$, by the Itô formula, we have

$$\begin{aligned}
\mathrm{d}V(I(t)) &= \frac{1}{I}\mathrm{d}I(t) - \frac{1}{2I^2(t)}(\mathrm{d}I(t))^2 \\
&= [\frac{\beta S(t)}{1+\alpha I^h(t)} - (d+\eta+\mu) - \frac{\sigma^2 S^2(t)}{(1+\alpha I^h(t))^2}]\mathrm{d}t + \frac{\sigma S(t)}{1+\alpha I^h(t)}\mathrm{d}B(t) \\
&\le [\beta S - \sigma^2 S^2(t) - (d+\eta+\mu)]\mathrm{d}t + \frac{\sigma S(t)}{1+\alpha I^h(t)}\mathrm{d}B(t) \\
&\le [-(d+\eta+\mu) + \frac{\beta^2}{4\sigma^2}]\mathrm{d}t + \frac{\sigma S(t)}{1+\alpha I^h(t)}\mathrm{d}B(t).
\end{aligned}$$

Integrating both sides from 0to t, we have

$$\ln I(t) - \ln I(0) \le [-(d+\eta+\mu) + \frac{\beta^2}{4\sigma^2}]t + M(t), \tag{3.15}$$

where

$$M(t) = \int_0^t \frac{\sigma S(t)}{1+\alpha I^h(t)}\mathrm{d}B(t).$$

Since

$$\limsup_{t\to\infty} \frac{\langle M, M\rangle_t}{t} \le \sigma^2(\frac{b}{d})^2 < \infty,$$

so $\lim_{t\to\infty} \frac{M(t)}{t} = 0$ a.s. Dividing both sides of (3.15) by t, and letting $t \to \infty$ we obtain

$$\limsup_{t\to\infty} \frac{\ln t}{t} \le -[(d+\eta+\mu) - \frac{\beta^2}{4\sigma^2}].$$

The proof is complete. □

Remark 3.13. Obviously, if σ^2 is sufficiently large such that $\sigma^2 > \frac{\beta^2}{4(d+\eta+\mu)}$, then the disease to this stochastic system will become extinct. In other words, the theorem reveals the important fact that the environmental noise may make the disease extinct.

4. Simulations and discussions

In this section we analyze the stochastic behavior of system (1.1) by means of numerical simulations and compare it with the deterministic behavior of system (1.2). One of main aims of this section is to show that stochastic noises play an important role in determining the persistence or extinction of disease. Making use of this numerical simulation method and with the help of Matlab soft-ware, by choosing suitable parameters, we get simulations of system (1.1) and system (1.2) when $h = 2$. The blue lines and the red lines in the figures represent solutions of deterministic system (1.2) and stochastic system (1.1) respectively.

In Figure 1, we choose $S(0) = 0.6$, $I(0) = 0.2$, $R(0) = 0.2$, $b = 0.4$, $\mu = 0.15$, $\eta = 0.15$, $\alpha = 4.0$, $\beta = 0.2$, $\gamma = 0.1$, $d = 0.2$ and $R_0 < 1$. The only difference between conditions of Group(a) and Group(b) is that the values of σ is different.

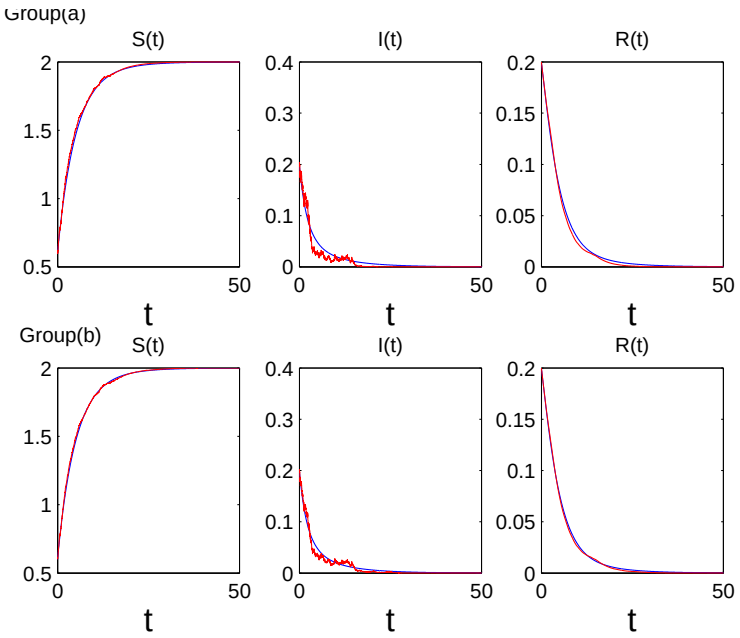

FIGURE 1. Trajectories of stochastic system (1.1) and deterministic system (1.2) with $S(0) = 0.6$, $I(0) = 0.2$, $R(0) = 0.2$, $b = 0.4$, $\mu = 0.15$, $\eta = 0.15$, $\alpha = 4.0$, $\beta = 0.2$, $\gamma = 0.1$ $d = 0.2$; $\sigma = 0.3$ in Group (a) and $\sigma = 0.2$ in Group (b)

In Group(a), we choose $\sigma = 0.3$. At the same time, we choose $\sigma = 0.2$ in Group (b). Figure 1 illustrates the situation where the intensity of noise σ verifies the conditions of the Theorem 3.4. It is observed that disease-free equilibrium state P^0 is stochastically stable.

In Figure 2, we choose $S(0) = 0.6$, $I(0) = 0.2$, $R(0) = 0.2$, $b = 2$, $\mu = 0.15$, $\eta = 0.15$, $\alpha = 4.0$, $\beta = 0.5$, $\gamma = 0.1$, $d = 0.4$ and $R_0 > 1$. The only difference between conditions of Group(a) and Group(b) is that the value of σ is different. In Group (a), we choose $\sigma = 0.02$. At the same time, we choose $\sigma = 0.1$ in Group (b). Figure 2 illustrates that the solution of system (1.1) fluctuates around the solution of system (1.2), which supports the conclusion of Theorem 3.8. From the figure, the fluctuation is getting smaller and smaller when the intensity decreases.

In Figure 3, we choose the same parameters with Figure 2. The only difference between conditions of Figure 2 and Figure 3 is that the values of σ is different. In Figure 3, we choose $\sigma = 0.36$. In view of Theorem 3.12, the system (1.1) will go to extinction. Figure 3 confirms this. By comparing Figure 2 with Figure 3, we can observe that small environmental noise can retain system (1.1) permanent, however sufficiently large environmental noise can make disease to extinct. Theorem 3.12 reveals that a large white noise will force the disease to become extinct while the disease may be persistent under a relatively small white noise.

The results we get and the work of Lahrouz, Omari and Kioach [14] differ in that: in case of $R_0 > 1$. When σ is small enough, the result consist with the deterministic system; that is, the solution converge to the positive equilibrium P^*;

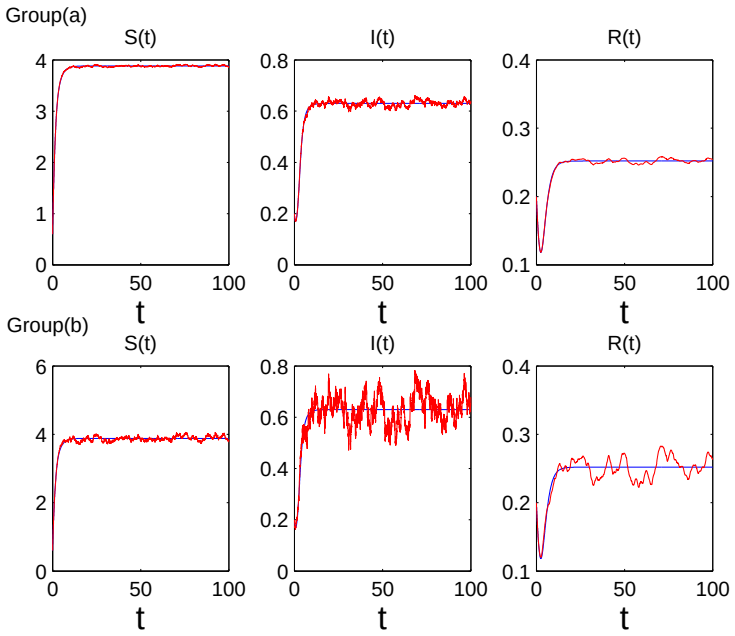

FIGURE 2. Trajectories of stochastic system (1.1) and deterministic system (1.2) with $S(0) = 0.6$, $I(0) = 0.2$, $R(0) = 0.2$, $b = 2$, $\mu = 0.15$, $\eta = 0.15$, $\alpha = 4.0$, $\beta = 0.2$, $\gamma = 0.1$, $d = 0.4$; $\sigma = 0.02$ in Group (a) and $\sigma = 0.1$ in Group (b)

When σ is getting larger, the behavior of system (1.1) become unstable; When σ is getting large enough, the disease to this stochastic system will become extinct. All the above results are new. In the case of $R_0 \leq 1$, we generalize the results of [14]. Evidently, if $h \equiv 1$, Theorem 2.2, Theorem 3.4 and Theorem 3.6 become respectively equal to Theorem 1, Theorem 4 and Theorem 5 in [14].

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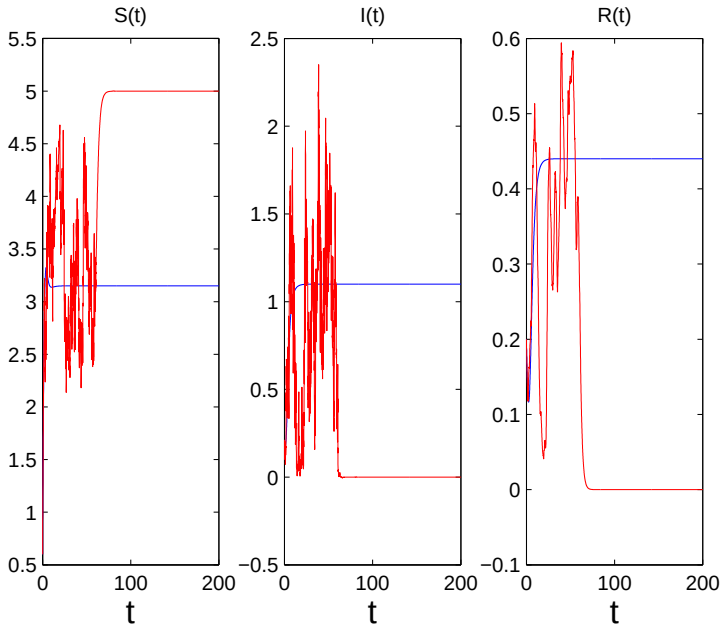

FIGURE 3. Trajectories of stochastic system (1.1) and deterministic system (1.2) with $S(0) = 0.6$, $I(0) = 0.2$, $R(0) = 0.2$, $b = 2$, $\mu = 0.15$, $\eta = 0.15$, $\alpha = 4.0$, $\beta = 0.2$, $\gamma = 0.1$, $d = 0.4$, $\sigma = 0.36$

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