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EXISTENCE AND COMPARISON OF SMALLEST EIGENVALUES FOR A FRACTIONAL BOUNDARY-VALUE PROBLEM

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Abstract. The theory of u_0-positive operators with respect to a cone in a Banach space is applied to the fractional linear differential equations

$$D^{\alpha}_{0+}u + \lambda_1 p(t)u = 0 \quad \text{and} \quad D^{\alpha}_{0+}u + \lambda_2 q(t)u = 0,$$

$0 < t < 1$, with each satisfying the boundary conditions $u(0) = u(1) = 0$. The existence of smallest positive eigenvalues is established, and a comparison theorem for smallest positive eigenvalues is obtained.

1. Introduction

We consider the eigenvalue problems

$$D^{\alpha}_{0+}u + \lambda_1 p(t)u = 0, \quad 0 < t < 1, \tag{1.1}$$

$$D^{\alpha}_{0+}u + \lambda_2 q(t)u = 0, \quad 0 < t < 1, \tag{1.2}$$

satisfying the boundary conditions

$$u(0) = u(1) = 0, \tag{1.3}$$

where $1 < \alpha \leq 2$ is a real number, D^{α}_{0+} is the standard Riemann-Liouville derivative, and $p(t)$ and $q(t)$ are continuous nonnegative functions on $[0, 1]$, where neither $p(t)$ nor $q(t)$ vanishes identically on any nondegenerate compact subinterval of $[0, 1]$.

The Krein Rutman theory [14] has been employed extensively to establish the existence of and compare smallest eigenvalues of boundary value problems for differential equations, difference equations, and dynamic equations on time scales. For some examples, see [4, 5, 7, 8, 9, 11, 12, 16, 18] and the references therein. A standard approach to show the existence of smallest eigenvalues is to apply the theory of u_0-positive operators [15]. Operators are defined whose eigenvalues are reciprocals of the eigenvalues of the original boundary value problems. These operators are constructed by using the corresponding Green's function; the u_0-positivity of these operators are obtained by showing the operator maps nonzero elements of a cone into the interior of that cone. Sign properties of the Green's function are employed to map the cone into the cone and higher order derivatives of the Green's

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functions are employed to map elements to the interior of the cone. The theory of u_0-positivity, as developed by Krasnosel'skii [15], gives the existence of largest eigenvalues of the operator, with the corresponding eigenfunction existing in a cone.

In this article, we apply the standard approach, described above, to a boundary value problem for a fractional differential equation. We are not aware of any previous application of u_0-positive operators to fractional differential equations. Fixed point theory is now commonly applied to boundary value problems for fractional equations; see, for example, the bibliography found in [1]. In many of these applications, the common Banach space to employ is $C[0,1]$; this space is not appropriate for applications of u_0-positivity to (1.1), (1.2) or (1.1), (1.3), since the corresponding Green's function, $G(t,s)$, has unbounded slope at $t=0$. The primary contribution of this article then is to consider an appropriate Banach space and cone, with nonempty interior, so that theory of u_0-positive operators does apply. The motivation for the Banach space used here is found in [6, Theorem 2.5] or [17, Theorem 3.1]. The particular approach to construct the Banach space and cone is modeled after [16]. For other work on eigenvalue problems of fractional differential equations, see [2, 10, 19, 20].

In Section 2, we state the preliminary definitions and theorems. In Section 3, we define the appropriate Banach space and establish the existence of and compare smallest eigenvalues of (1.1), (1.2) and (1.1), (1.3).

2. Preliminary definitions and theorems

Definition 2.1. Let $1<\alpha\le 2$. The α-th Riemann-Liouville fractional derivative of the function $u:[0,1]\to\mathbb{R}$, denoted $D_{0+}^{\alpha}u$, is defined as

$$D_{0+}^{\alpha}u(t)=\frac{1}{\Gamma(2-\alpha)}\frac{d^2}{dt^2}\int_0^t (t-s)^{2-\alpha-1}u(s)ds,$$

provided the right-hand side exists.

Definition 2.2. Let $\mathcal{B}$ be a Banach space over $\mathbb{R}$. A closed nonempty subset $\mathcal{P}$ of $\mathcal{B}$ is said to be a cone provided

(i) $\alpha u+\beta v\in\mathcal{P}$, for all $u,v\in\mathcal{P}$ and all $\alpha,\beta\ge 0$, and
(ii) $u\in\mathcal{P}$ and $-u\in\mathcal{P}$ implies $u=0$.

Definition 2.3. A cone $\mathcal{P}$ is solid if the interior, $\mathcal{P}^\circ$, of $\mathcal{P}$, is nonempty. A cone $\mathcal{P}$ is reproducing if $\mathcal{B}=\mathcal{P}-\mathcal{P}$; i.e., given $w\in\mathcal{B}$, there exist $u,v\in\mathcal{P}$ such that $w=u-v$.

Remark 2.4. Krasnosel'skii [15] showed that every solid cone is reproducing.

Cones give rise to partial orders on Banach spaces and to partial orders on bounded linear operators on Banach spaces in a natural way.

Definition 2.5. Let $\mathcal{P}$ be a cone in a real Banach space $\mathcal{B}$. If $u,v\in\mathcal{B}$, we say $u\le v$ with respect to $\mathcal{P}$ if $v-u\in\mathcal{P}$. If both $M,N:\mathcal{B}\to\mathcal{B}$ are bounded linear operators, we say $M\le N$ with respect to $\mathcal{P}$ if $Mu\le Nu$ for all $u\in\mathcal{P}$.

Definition 2.6. A bounded linear operator $M:\mathcal{B}\to\mathcal{B}$ is u_0-positive with respect to $\mathcal{P}$ if there exists $u_0\in\mathcal{P}\backslash\{0\}$ such that for each $u\in\mathcal{P}\backslash\{0\}$, there exist $k_1(u)>0$ and $k_2(u)>0$ such that $k_1u_0\le Mu\le k_2u_0$ with respect to $\mathcal{P}$.

The following two results are fundamental to our comparison results and are attributed to Krasnosel'skii [15]. The proof of Theorem 2.7 can be found in Krasnosel'skii's book [15], and the proof of Theorem 2.8 is provided by Keener and Travis [13] as an extension of Krasonel'skii's results.

Theorem 2.7. *Let $\mathcal{B}$ be a real Banach space and let $\mathcal{P} \subset \mathcal{B}$ be a reproducing cone. Let $L : \mathcal{B} \to \mathcal{B}$ be a compact, u_0-positive, linear operator. Then L has an essentially unique eigenvector in $\mathcal{P}$, and the corresponding eigenvalue is simple, positive, and larger than the absolute value of any other eigenvalue.*

Theorem 2.8. *Let $\mathcal{B}$ be a real Banach space and $\mathcal{P} \subset \mathcal{B}$ be a cone. Let both $M, N : \mathcal{B} \to \mathcal{B}$ be bounded, linear operators and assume that at least one of the operators is u_0-positive. If $M \le N$, $Mu_1 \ge \lambda_1 u_1$ for some $u_1 \in \mathcal{P}$ and some $\lambda_1 > 0$, and $Nu_2 \le \lambda_2 u_2$ for some $u_2 \in \mathcal{P}$ and some $\lambda_2 > 0$, then $\lambda_1 \le \lambda_2$. Furthermore, $\lambda_1 = \lambda_2$ implies u_1 is a scalar multiple of u_2.*

3. Comparison of smallest eigenvalues

In [3], Bai and L u showed the Green's function for $-D^{\alpha}_{0+}u(t) = 0$ satisfying (1.3) is

$$G(t,s) = \begin{cases} \frac{[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\ \frac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1. \end{cases} \tag{3.1}$$

Define the Banach Space

$$\mathcal{B} = \{u : u = t^{\alpha-1}v, v \in C^1[0,1], v(1) = 0\},$$

with the norm

$$\|u\| = |v'|_0,$$

where $|v'|_0 = \sup_{t\in[0,1]} |v'(t)|$ denotes the usual supremum norm.

Note that for $v \in C^1[0,1]$, $v(1) = 0$, $0 \le t \le 1$,

$$|v(t)| = |v(t) - v(1)| = \Big| \int_1^t v'(s)ds \Big| \le (1-t)|v'| \le \|u\|.$$

Therefore, $|v|_0 \le \|u\| = |v'|_0$ and

$$|u|_0 = |t^{\alpha-1}v|_0 \le t^{\alpha-1}\|u\|,$$

implies $|u|_0 \le \|u\|$.

Define the linear operators

$$Mu(t) = \int_0^1 G(t,s)p(s)u(s)ds \tag{3.2}$$

and

$$Nu(t) = \int_0^1 G(t,s)q(s)u(s)\, ds. \tag{3.3}$$

Theorem 3.1. *The operators $M, N : \mathcal{B} \to \mathcal{B}$ are compact linear operators.*

Proof. We first show $M : \mathcal{B} \to \mathcal{B}$. Let $u \in \mathcal{B}$. So there is a $v \in C^1[0,1]$ such that $u = t^{\alpha-1}v$. Since $v \in C^1[0,1]$ and $p \in C[0,1]$, let $L = |v|_0$ and let $P = |p|_0$. Now

$$Mu(t) = \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)u(s)ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)u(s)ds$$

$$= t^{\alpha-1}\Big(\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds - t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds\Big).$$

Define

$$g(t) = \begin{cases} 0, & t=0, \\ t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds, & 0<t\le 1. \end{cases}$$

First, note $g \in C^1(0,1]$. Now

$$\begin{aligned}
|g(t)| &= \big|t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds\big| \\
&= \big|t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)s^{\alpha-1}v(s)ds\big| \\
&\le PLt^{1-\alpha}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}ds \\
&\le PLt^{1-\alpha}t^{\alpha-1}\int_0^t (t-s)^{\alpha-1}ds \\
&= \frac{PLt^{\alpha}}{\alpha},
\end{aligned}$$

where $\frac{PL}{\alpha} \ge 0$. So $\lim_{t\to 0^+} g(t) = g(0) = 0$ and $g \in C[0,1]$.

Also, for $t>0$,

$$\begin{aligned}
|g'(t)| &= \Big|(1-\alpha)t^{-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds \\
&\quad + (\alpha-1)t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha)}p(s)u(s)ds\Big| \\
&\le \Big|(1-\alpha)t^{-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)s^{\alpha-1}v(s)ds\Big| \\
&\quad + \Big|t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)s^{\alpha-1}v(s)ds\Big| \\
&\le (\alpha-1)PLt^{-\alpha}t^{\alpha-1}\int_0^t (t-s)^{\alpha-1}ds + PLt^{1-\alpha}t^{\alpha-1}\int_0^t (t-s)^{\alpha-2}ds \\
&= \big(\frac{\alpha-1}{\alpha} + \frac{1}{\alpha-1}\big)PLt^{\alpha-1}.
\end{aligned}$$

So, $\lim_{t\to 0^+} g'(t) = 0$. Moreover, using the definition of derivative and L'Hospital's rule,

$$g'(0) = \lim_{t\to 0^+}\frac{g(t)-g(0)}{t} = \lim_{t\to 0^+}\frac{g(t)}{t} = \lim_{t\to 0^+} g'(t) = 0,$$

and so $g' \in C[0,1]$.

Now set

$$\hat{v}(t) = \Big(\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds - t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds\Big).$$

It is an easy calculation to verify that $\hat{v}(1) = 0$. Thus $Mu \in \mathcal{B}$. So $M : \mathcal{B} \to \mathcal{B}$. The proof that $N : \mathcal{B} \to \mathcal{B}$ is similar.

We now show that $M : \mathcal{B} \to \mathcal{B}$ is a compact operator. Let $L > 0$ and consider

$$K = \{u \in \mathcal{B} : \|u\| \le L\}$$

or more appropriately consider

$$\hat{K} = \{v \in C^1[0,1] : v(1) = 0, |v'|_0 \le L\}.$$

Define

$$\hat{M}(v)(t) = \Big(\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)s^{\alpha-1}v(s)ds - t^{1-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)s^{\alpha-1}v(s)ds \Big).$$

To show that M is compact on $\mathcal{B}$ it is sufficient to show that $\{(\hat{M}(v))' : v \in \hat{K}\}$ is uniformly bounded and equicontinuous on $[0,1]$. We provide the details for equicontinuity as the details for uniform boundedness are straightforward.

Assume $|p|_0 = P$ and assume $|v|_0 \le L$. Let $\epsilon > 0$. As in the calculations above for g', $(\hat{M}(v))'(0) = 0$ and

$$|(\hat{M}(v))'(t)| \le \big(\frac{\alpha-1}{\alpha} + \frac{1}{\alpha-1}\big) PLt^{\alpha-1}.$$

Thus, there exists $\delta_1 > 0$ such that if $|t| < \delta_1$ then $|(\hat{M}(v))'(t)| < \frac{\epsilon}{2}$.

On $[\delta_1, 1]$, $\{(\hat{M}(v))' : v \in \hat{K}\}$ is shown to be equicontinuous by showing that $\{(\hat{M}(v))'' : v \in \hat{K}\}$ is uniformly bounded. Now

$$\begin{aligned}(\hat{M}(v))'(t) &= (1-\alpha)t^{-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)u(s)ds \\ &\quad + (\alpha-1)t^{1-\alpha} \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha)} p(s)u(s)ds,\end{aligned}$$

and so

$$\begin{aligned}(\hat{M}(v))''(t) &= -\alpha(1-\alpha)t^{-\alpha-1} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)s^{\alpha-1}v(s)ds \\ &\quad - (\alpha-1)^2 t^{-\alpha} \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha)} p(s)s^{\alpha-1}v(s)ds \\ &\quad - (\alpha-1)^2 t^{-\alpha} \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha)} p(s)s^{\alpha-1}v(s)ds \\ &\quad + (\alpha-1)(\alpha-2)t^{1-\alpha} \int_0^t \frac{(t-s)^{\alpha-3}}{\Gamma(\alpha-1)} p(s)s^{\alpha-1}v(s)ds.\end{aligned}$$

Each of the four terms can be bounded by a constant multiple of $t^{\alpha-2}$.

For the first term, notice

$$\Big| t^{-\alpha-1} \int_0^t (t-s)^{\alpha-1} p(s)s^{\alpha-1}v(s)ds \Big| \le PLt^{-\alpha-1} \Big| \int_0^t (t-s)^{\alpha-1}s^{\alpha-1}ds \Big|.$$

Set $s = rt$. So

$$\begin{aligned}t^{-\alpha-1} \Big| \int_0^t (t-s)^{\alpha-1}s^{\alpha-1}ds \Big| &= t^{-\alpha-1}t^{\alpha-1}t^{\alpha-1}t \Big| \int_0^1 (1-r)^{\alpha-1}r^{\alpha-1}dr \Big| \\ &= t^{\alpha-2}|B(\alpha,\alpha)|,\end{aligned}$$

where B denotes the beta function.

In dealing with the second and third terms, first note

$$\Big| t^{-\alpha} \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha)} p(s)s^{\alpha-1}v(s)ds \Big| \le PLt^{-\alpha} \Big| \int_0^t (t-s)^{\alpha-2}s^{\alpha-1}ds \Big|.$$

Set $s = rt$. Then

$$\begin{aligned} t^{-\alpha}\Big|\int_0^t (t-s)^{\alpha-2}s^{\alpha-1}ds\Big| &= t^{-\alpha}t^{\alpha-2}t^{\alpha-1}t\Big|\int_0^t (1-r)^{\alpha-2}r^{\alpha-1}dr\Big| \\ &= t^{\alpha-2}|B(\alpha,\alpha-1)|. \end{aligned}$$

Notice $B(\alpha, \alpha-1)$ is well-defined since $1 < \alpha \le 2$.

Last, we obtain an analogous estimate for the fourth term. If $\alpha = 2$, this term is zero. If $1 < \alpha < 2$, first integrate by parts to obtain

$$\int_0^t (t-s)^{\alpha-3}s^{\alpha-1}ds = \frac{\alpha-1}{\alpha-2}\int_0^t (t-s)^{\alpha-2}s^{\alpha-2}ds.$$

Thus,

$$\begin{aligned} \Big|t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-3}}{\Gamma(\alpha-1)}p(s)s^{\alpha-1}v(s)ds\Big| &\le PLt^{1-\alpha}\Big|\int_0^t (t-s)^{\alpha-3}s^{\alpha-1}ds\Big| \\ &= PLt^{1-\alpha}\Big|\frac{\alpha-1}{\alpha-2}\int_0^t (t-s)^{\alpha-2}s^{\alpha-2}ds\Big|. \end{aligned}$$

Again, by setting $s = rt$, we obtain

$$\begin{aligned} t^{1-\alpha}\Big|\int_0^t (t-s)^{\alpha-2}s^{\alpha-2}ds\Big| &= t^{1-\alpha}t^{\alpha-2}t^{\alpha-2}t\Big|\int_0^t (1-r)^{\alpha-2}r^{\alpha-2}dr\Big| \\ &= t^{\alpha-2}|B(\alpha-1,\alpha-1)|. \end{aligned}$$

Again, $B(\alpha-1, \alpha-1)$ is well-defined since $1 < \alpha < 2$. Therefore, if $\alpha \ne 2$,

$$\begin{aligned} |(\hat{M}(v))''(t)| \le PL\Big[&\frac{\alpha(\alpha-1)|B(\alpha,\alpha)|}{\Gamma(\alpha)} + \frac{2(\alpha-1)^2|B(\alpha,\alpha-1)|}{\Gamma(\alpha)} \\ &+ \frac{(\alpha-1)^2|B(\alpha-1,\alpha-1)|}{\Gamma(\alpha-1)}\Big]t^{\alpha-2}, \end{aligned}$$

and if $\alpha = 2$,

$$|(\hat{M}(v))''(t)| \le \frac{4PL}{3}.$$

So $\{(\hat{M}(v))'' : v \in \hat{K}\}$ is uniformly bounded on $[\delta_1, 1]$.

Since $\{(\hat{M}(v))'' : v \in \hat{K}\}$ is uniformly bounded on $[\delta_1, 1]$, there exists $\delta_2 > 0$ such that if $|t_1 - t_2| < \delta_2$, $t_1, t_2 \in [\delta_1, 1]$, then $|(\hat{M}(v))'(t_1) - (\hat{M}(v))'(t_2)| < \frac{\epsilon}{2}$.

Set $\delta = \min\{\delta_1, \delta_2\}$. If $|t_1 - t_2| < \delta$, $t_1, t_2 \in [0, \delta_1]$, then

$$|(\hat{M}(v))'(t_1) - (\hat{M}(v))'(t_2)| \le |(\hat{M}(v))'(t_1)| + |(\hat{M}(v))'(t_2)| < \epsilon.$$

If $|t_1 - t_2| < \delta$, $t_1, t_2 \in [\delta_1, 1]$, then

$$|(\hat{M}(v))'(t_1) - (\hat{M}(v))'(t_2)| \le \frac{\epsilon}{2} < \epsilon.$$

If $|t_1 - t_2| < \delta$, $0 \le t_1 < \delta_1 \le t_2 \le 1$, then

$$\begin{aligned} &|(\hat{M}(v))'(t_1) - (\hat{M}(v))'(t_2)| \\ &\le |(\hat{M}(v))'(t_1) - (\hat{M}(v))'(\delta_1)| + |(\hat{M}(v))'(\delta_1) - (\hat{M}(v))'(t_2)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Details for the operator N are similar and the proof is complete. $\square$

Define the cone

$$\mathcal{P} = \{u \in \mathcal{B} : u(t) \geq 0 \text{ for } t \in [0,1]\}.$$

Lemma 3.2. *The cone $\mathcal{P}$ is solid in B and hence reproducing.*

Proof. Define

$$\Omega := \{u \in \mathcal{B} \mid u(t) > 0 \text{ for } t \in (0,1), v(0) > 0, v'(1) < 0, \text{ where } u = t^{\alpha-1}v\}. \quad (3.4)$$

We will show $\Omega \subset \mathcal{P}^\circ$. Let $u \in \Omega$. Since $v(0) > 0$, there exists an $\epsilon_1 > 0$ such that $v(0) - \epsilon_1 > 0$. Since $v \in C^1[0,1]$, there exists an $a \in (0,1)$ such that $v(t) > \epsilon_1$ for all $t \in (0,a)$. So $u(t) = t^{\alpha-1}v(t) > \epsilon_1 t^{\alpha-1}$ for all $t \in (0,a)$. Now, since $v'(1) < 0$, there exists an $\epsilon_2 > 0$ such that $v'(1) + \epsilon_2 < 0$. Then, since $v(1) = 0$ and $-v'(1) > \epsilon_2$, there exists a $b \in (a,1)$ such that $v(t) > (1-t)\epsilon_2$ for all $t \in (b,1]$. Thus $u(t) > b^{\alpha-1}(1-t)\epsilon_2$ for all $t \in (b,1]$. Also, since $u(t) > 0$ on $[a,b]$, there exists an $\epsilon_3 > 0$ such that $u(t) - \epsilon_3 > 0$ for all $t \in [a,b]$.

Let $\epsilon = \min\left\{\frac{\epsilon_1}{2}, \frac{b^{\alpha-1}\epsilon_2}{2}, \frac{\epsilon_3}{2}\right\}$. Define $B_\epsilon(u) = \{\hat{u} \in \mathcal{B} : \|u - \hat{u}\| < \epsilon\}$. Let $\hat{u} \in B_\epsilon(u)$. So $\hat{u} = t^{\alpha-1}\hat{v}$, where $\hat{v} \in C^1[0,1]$ with $\hat{v}(1) = 0$. Now

$$|\hat{u}(t) - u(t)| \leq t^{\alpha-1}\|\hat{u} - u\| < \epsilon t^{\alpha-1}.$$

So for $t \in (0,a)$, $\hat{u}(t) > u(t) - t^{\alpha-1}\epsilon > t^{\alpha-1}\epsilon_1 - t^{\alpha-1}\epsilon_1/2 = t^{\alpha-1}\epsilon_1/2$. So $\hat{u}(t) > 0$ for $t \in (0,a)$. By the Mean Value Theorem, for $t \in (b,1)$, $|\hat{u}(t) - u(t)| \leq (1-t)\|\hat{u} - u\| < (1-t)\epsilon$. So for $t \in (b,1)$,

$$\hat{u}(t) > u(t) - (1-t)\epsilon > b^{\alpha-1}(1-t)\epsilon_2 - (1-t)b^{\alpha-1}\epsilon_2/2 = (1-t)b^{\alpha-1}\epsilon_2/2.$$

So for $t \in (b,1)$, $\hat{u}(t) > 0$. Also, $|\hat{u}(t) - u(t)| \leq \|\hat{u} - u\| < \epsilon$. So for $t \in [a,b]$, $\hat{u}(t) > u(t) - \epsilon > \epsilon_3 - \epsilon_3/2 > 0$. So $\hat{u}(t) > 0$ for all $t \in [a,b]$. So $\hat{u} \in \mathcal{P}$ and thus $B_\epsilon(u) \subset \mathcal{P}$. So $\Omega \subset \mathcal{P}^\circ$. □

Lemma 3.3. *The bounded linear operators M and N are u_0-positive with respect to $\mathcal{P}$.*

Proof. First, we show $M : \mathcal{P}\backslash\{0\} \to \Omega \subset \mathcal{P}^\circ$. Let $u \in \mathcal{P}$. So $u(t) \geq 0$. Then since $G(t,s) \geq 0$ on $[0,1] \times [0,1]$ and $p(t) \geq 0$ on $[0,1]$,

$$Mu(x) = \int_0^1 G(t,s)p(s)u(s)ds \geq 0,$$

for $0 \leq t \leq 1$. So $M : \mathcal{P} \to \mathcal{P}$.

Now let $u \in \mathcal{P}\backslash\{0\}$. So there exists a compact interval $[\alpha,\beta] \subset [0,1]$ such that $u(t) > 0$ and $p(t) > 0$ for all $t \in [\alpha,\beta]$. Then, since $G(t,s) > 0$ on $(0,1) \times (0,1)$,

$$\begin{aligned} Mu(t) &= \int_0^1 G(t,s)p(s)u(s)ds \\ &\geq \int_\alpha^\beta G(t,s)p(s)u(s)ds > 0, \end{aligned}$$

for $0 < t < 1$. Now

$$Mu(t) = t^{\alpha-1}\Big(\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds - t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds\Big).$$

Let

$$v(t) = \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds - t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds.$$

So $v(0)=\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds>0$ and

$$v'(1)=-(1-\alpha)\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds-(\alpha-1)\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)}p(s)u(s)ds<0.$$

So $M:\mathcal{P}\backslash\{0\}\to\Omega\subset\mathcal{P}^\circ$.

Now choose any $u_0\in\mathcal{P}\backslash\{0\}$, and let $u\in\mathcal{P}\backslash\{0\}$. So $Mu\in\Omega\subset\mathcal{P}^\circ$. Choose $k_1>0$ sufficiently small and k_2 sufficiently large so that $Mu-k_1u_0\in\mathcal{P}^\circ$ and $u_0-\frac{1}{k_2}Mu\in\mathcal{P}^\circ$. So $k_1u_0\le Mu$ with respect to $\mathcal{P}$ and $Mu\le k_2u_0$ with respect to $\mathcal{P}$. Thus $k_1u_0\le Mu\le k_2u_0$ with respect to $\mathcal{P}$ and so M is u_0-positive with respect to P. Similarly, N is u_0-positive. □

Remark 3.4. Notice that

$$\Lambda u=Mu=\int_0^1 G(t,s)p(s)u(s)ds,$$

if and only if

$$u(t)=\frac{1}{\Lambda}\int_0^1 G(t,s)p(s)u(s)ds,$$

if and only if

$$D_{0+}^\alpha u(t)+\frac{1}{\Lambda}p(t)u(t)=0,\quad 0<t<1,$$

with $u(0)=u(1)=0$.

So the eigenvalues of (1.1),(1.3) are reciprocals of eigenvalues of M, and conversely. Similarly, eigenvalues of (1.2),(1.3) are reciprocals of eigenvalues of N, and conversely.

Theorem 3.5. *Let $\mathcal{B}$, $\mathcal{P}$, M, and N be defined as earlier. Then M (and N) has an eigenvalue that is simple, positive, and larger than the absolute value of any other eigenvalue, with an essentially unique eigenvector that can be chosen to be in $\mathcal{P}^\circ$.*

Proof. Since M is a compact linear operator that is u_0-positive with respect to $\mathcal{P}$, by Theorem 2.7, M has an essentially unique eigenvector, say $u\in\mathcal{P}$, and eigenvalue Λ with the above properties. Since $u\ne 0$, $Mu\in\Omega\subset\mathcal{P}^\circ$ and $u=M\left(\frac{1}{\Lambda}u\right)\in\mathcal{P}^\circ$. □

Theorem 3.6. *Let $\mathcal{B}$, $\mathcal{P}$, M, and N be defined as earlier. Let $p(t)\le q(t)$ on $[0,1]$. Let Λ_1 and Λ_2 be the eigenvalues defined in Theorem 3.5 associated with M and N, respectively, with the essentially unique eigenvectors u_1 and $u_2\in\mathcal{P}^\circ$. Then $\Lambda_1\le\Lambda_2$, and $\Lambda_1=\Lambda_2$ if and only if $p(t)=q(t)$ on $[0,1]$.*

Proof. Let $p(t)\le q(t)$ on $[0,1]$. So for any $u\in\mathcal{P}$ and $t\in[0,1]$,

$$(Nu-Mu)(t)=\int_0^1 G(t,s)(q(s)-p(s))u(s)ds\ge 0.$$

So $Nu-Mu\in\mathcal{P}$ for all $u\in\mathcal{P}$, or $M\le N$ with respect to $\mathcal{P}$. Then by Theorem 2.8, $\Lambda_1\le\Lambda_2$.

If $p(t)=q(t)$, then $\Lambda_1=\Lambda_2$. Now suppose $p(t)\ne q(t)$. So $p(t)<q(t)$ on some subinterval $[\alpha,\beta]\subset[0,1]$. Then $(N-M)u_1\in\Omega\subset\mathcal{P}^\circ$ and so there exists $\epsilon>0$ such that $(N-M)u_1-\epsilon u_1\in\mathcal{P}$. So $\Lambda_1u_1+\epsilon u_1=Mu_1+\epsilon u_1\le Nu_1$, implying $Nu_1\ge(\Lambda_1+\epsilon)u_1$. Since $N\le N$ and $Nu_2=\Lambda_2u_2$, by Theorem 2.8, $\Lambda_1+\epsilon\le\Lambda_2$, or $\Lambda_1<\Lambda_2$. □

By Remark 3.4, the following theorem is an immediate consequence of Theorems 3.5 and 3.6.

Theorem 3.7. *Assume the hypotheses of Theorem 3.6. Then there exists smallest positive eigenvalues* λ_1 *and* λ_2 *of* (1.1),(1.3) *and* (1.2),(1.3)*, respectively, each of which is simple, positive, and less than the absolute value of any other eigenvalue of the corresponding problems. Also, eigenfunctions corresponding to* λ_1 *and* λ_2 *may be chosen to belong to* $\mathcal{P}^\circ$*. Finally,* $\lambda_1 \geq \lambda_2$*, and* $\lambda_1 = \lambda_2$ *if and only if* $p(t) = q(t)$ *for all* $t \in [0, 1]$.

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