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INITIAL-BOUNDARY VALUE PROBLEMS FOR THE WAVE EQUATION

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Abstract. In this work we consider an initial-boundary value problem for the one-dimensional wave equation. We prove the uniqueness of the solution and show that the solution coincides with the wave potential.

1. Introduction

In $\Omega = (0,1)$ consider the one-dimensional potential
$$u(x) = \int_0^1 -\frac{1}{2}|x-y|f(y)dy, \tag{1.1}$$
where f is an integrable function in $(0,1)$. The kernel of the potential is a fundamental solution of the second order differential equation
$$-\varepsilon''(x-y) = \delta(x-y), \tag{1.2}$$
where $\varepsilon(x-y) = -\frac{1}{2}|x-y|$ and δ is the Dirac delta function. Hence the potential (1.1) satisfies the equation
$$-u''(x) = f(x), x \in \Omega. \tag{1.3}$$
On the other hand, integrating by part, we obtain
$$\begin{aligned}
u(x) &= \int_0^1 -\frac{1}{2}|x-y|f(y)dy = \int_0^1 \frac{1}{2}|x-y|u''(y)dy \\
&= \int_0^x \frac{1}{2}(x-y)u''(y)dy + \int_x^1 \frac{1}{2}(y-x)u''(y)dy \\
&= u(x) - x\frac{u'(0)+u'(1)}{2} - \frac{-u'(1)+u(0)+u(1)}{2}, \quad \forall x \in (0,1);
\end{aligned}$$
i.e.,
$$x(u'(0)+u'(1)) + (-u'(1)+u(0)+u(1)) = 0, \quad \forall x \in (0,1).$$
Therefore, the self-adjoint boundary conditions for the potential (1.1) are
$$u'(0)+u'(1) = 0, \quad -u'(1)+u(0)+u(1) = 0. \tag{1.4}$$

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Hence if we solve the equation (1.3) with the boundary conditions (1.4), then we find an unique solution of this boundary value problem in the form (1.1).

The simple method finds equivalent boundary value problems of ODE for one dimensional potential integrals. However this task becomes tedious for PDE, and we obtained boundary conditions of the volume potentials for elliptic equations and showed some their applications in works [2, 3, 4]. In particular in [2], by using a new non-local boundary value problem, which is equivalent to the Newton potential, we found explicitly all eigenvalues and eigenfunctions of the Newton potential in the 2-disk and the 3-ball. The aim of this paper is to give an analogy of the boundary value problem (1.3)-(1.4) for the wave potential. Unlike elliptic and parabolic cases, where we obtained non-local boundary conditions for the corresponding volume potentials, and some other nonclassic non-local boundary initial boundary value problems of hyperbolic equations (see, for example, [6, 7]) we get a local initial boundary value problem for the wave potential.

2. Main result and their proof

In the bounded domain $\Omega \equiv \{(x,t) : (0,l)\times(0,T)\}$ we consider the wave potential

$$u(x,t) = \int_\Omega \varepsilon(x-\xi,t-\tau)f(\xi,\tau)d\xi d\tau, \tag{2.1}$$

where $\varepsilon(x-\xi,t-\tau) = \frac{1}{2}\theta(t-\tau-|x-\xi|)$ is a fundamental solution of Cauchy problem for the wave equation; i.e.,

$$\frac{\partial^2\varepsilon(x-\xi,t-\tau)}{\partial t^2} - \frac{\partial^2\varepsilon(x-\xi,t-\tau)}{\partial x^2} = \delta(x-\xi,t-\tau),$$
$$\frac{\partial^2\varepsilon(x-\xi,t-\tau)}{\partial \tau^2} - \frac{\partial^2\varepsilon(x-\xi,t-\tau)}{\partial \xi^2} = \delta(x-\xi,t-\tau),$$
$$\varepsilon(x-\xi,t-\tau)|_{\tau=t} = \frac{\partial\varepsilon(x-\xi,t-\tau)}{\partial t}|_{\tau=t} = \frac{\partial\varepsilon(x-\xi,t-\tau)}{\partial \tau}|_{\tau=t} = 0$$

if $f(x,t) \in L_2(\Omega)$ then $u(x,t) \in W_2^1(\Omega) \cap W_2^1(\partial\Omega)$ and the wave potential (2.1) satisfies to the equation [1]

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t), \quad (x,t) \in \Omega, \tag{2.2}$$

with the initial conditions

$$u(x,0) = u_t(x,0) = 0, \quad 0 < x < l. \tag{2.3}$$

The wave potential (2.1) is widely used to solve various initial-boundary problems for the wave equation. Here we find the lateral boundary conditions of the wave potential (2.1). Main result of this article reads as follows.

Theorem 2.1. *If $f(x,t) \in L_2(\Omega)$ then the wave potential (2.1) satisfies the lateral boundary conditions*

$$(u_x - u_t)(0,t) = 0, \quad x = 0,\ 0 < t < T; \tag{2.4}$$
$$(u_x + u_t)(l,t) = 0, \quad x = l,\ 0 < t < T\,. \tag{2.5}$$

Conversely, if a function $u(x,t) \in W_2^1(\Omega) \cap W_2^1(\partial\Omega)$ satisfies the equation (2.2), the initial conditions (2.3), and the lateral boundary conditions (2.4)-(2.5), then the function $u(x,t)$ uniquely defines the wave potential (2.1).

Proof. We use techniques from [4]. Consider the one-dimensional wave potential in the bounded domain $\Omega \equiv \{(x,t) : (0,l) \times (0,T)\}$ with the boundary S,

$$u(x,t) = \int_{\Omega} \varepsilon(x-\xi, t-\tau) f(\xi,\tau) d\xi d\tau.$$

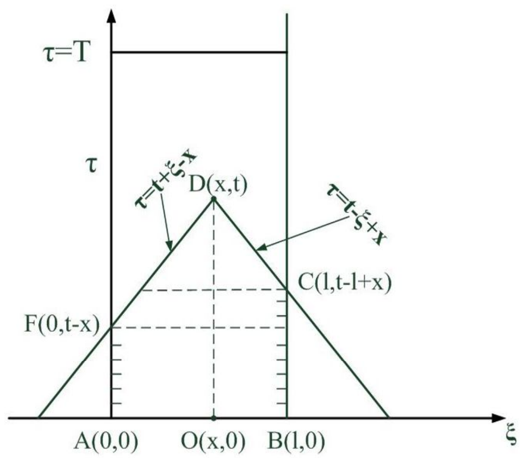

FIGURE 1. Domain of integration

Since $\tau = t - x + \xi$ and $\tau = t + x - \xi$ are characteristics, the integral vanishes outside of characteristic domain. Therefore, we integrate by ABCDF (Figure 1). Assuming that $u(x,t) \in W_2^1(\Omega)$, taking into account properties of the fundamental solution, and integrating by part, we calculate

$$\begin{aligned}
u(x,t) &= \int_{\Omega} \varepsilon(x-\xi, t-\tau) f(\xi,\tau) d\xi d\tau \\
&= \int_{ABCDF} \frac{1}{2}[u_{\tau\tau}(\xi,\tau) - u_{\xi\xi}(\xi,\tau)] d\xi d\tau \\
&= \frac{1}{2}\int_0^x d\xi \int_0^{t+\xi-x} \frac{\partial^2 u(\xi,\tau)}{\partial \tau^2} d\tau + \frac{1}{2}\int_x^l d\xi \int_0^{t-\xi+x} \frac{\partial^2 u(\xi,\tau)}{\partial \tau^2} d\tau \\
&\quad - \frac{1}{2}\int_0^{t-x} d\tau \int_0^l \frac{\partial^2 u(\xi,\tau)}{\partial \xi^2} d\xi - \frac{1}{2}\int_{t-\xi}^{t-l+x} d\tau \int_{\tau-t+x}^l \frac{\partial^2 u(\xi,\tau)}{\partial \xi^2} d\xi \\
&\quad - \frac{1}{2}\int_{t-l+x}^{t} d\tau \int_{\tau-t+x}^{t-\tau+x} \frac{\partial^2 u(\xi,\tau)}{\partial \xi^2} d\xi \\
&= \frac{1}{2}\int_0^x [\frac{\partial u(\xi, t+\xi-x)}{\partial \tau} - \frac{\partial u(\xi,0)}{\partial \tau}] d\xi \\
&\quad + \frac{1}{2}\int_x^l [\frac{\partial u(\xi, t-\xi+x)}{\partial \tau} - \frac{\partial u(\xi,0)}{\partial \tau}] d\xi - \frac{1}{2}\int_0^{t-x} [\frac{\partial u(l,\tau)}{\partial \xi} - \frac{\partial u(0,\tau)}{\partial \xi}] d\tau \\
&\quad - \frac{1}{2}\int_{t-x}^{t-l+x} [\frac{\partial u(l,\tau)}{\partial \xi} - \frac{\partial u(\tau-t+x,\tau)}{\partial \xi}] d\tau - \frac{1}{2}\int_{t-l+x}^{t} [\frac{\partial u(t-\tau+x,\tau)}{\partial \xi}
\end{aligned}$$

$$
\begin{aligned}
&- \frac{\partial u(\tau - t + x, \tau)}{\partial \xi}]d\tau \\
&= \frac{1}{2}\int_0^x \frac{\partial u(\xi, t+\xi-x)}{\partial \tau} d\xi + \frac{1}{2}\int_x^l \frac{\partial u(\xi, t-\xi+x)}{\partial \tau} d\xi + \frac{1}{2}\int_0^{t-x} \frac{\partial u(0,\tau)}{\partial \xi} d\tau \\
&- \frac{1}{2}\int_0^{t-l+x} \frac{\partial u(l,\tau)}{\partial \xi} d\tau + \frac{1}{2}\int_{t-x}^{t} \frac{\partial u(\tau - t + x, \tau)}{\partial \xi} d\tau \\
&- \frac{1}{2}\int_{t-l+x}^{t} \frac{\partial u(t-\tau+x, \tau)}{\partial \xi} d\tau, \quad \forall (x,t) \in \Omega.
\end{aligned}
$$

Using the total differential formula, we have

$$
\begin{aligned}
u(x,t) &= \frac{1}{2}\int_0^{t-x} \frac{\partial u(0,\tau)}{\partial \xi} d\tau - \frac{1}{2}\int_0^{t-l+x} \frac{\partial u(l,\tau)}{\partial \xi} d\tau + \frac{1}{2}\int_0^x [\frac{\partial u(\xi, t+\xi-x)}{\partial \tau} \\
&\quad + \frac{\partial u(\xi, t+\xi-x)}{\partial \xi}]d\xi + \frac{1}{2}\int_x^l [\frac{\partial u(\xi, t-\xi+x)}{\partial \tau} - \frac{\partial u(\xi, t-\xi+x)}{\partial \xi}]d\xi \\
&= \frac{1}{2}\int_0^{t-x} \frac{\partial u(0,\tau)}{\partial \xi} d\tau - \frac{1}{2}\int_0^{t-l+x} \frac{\partial u(l,\tau)}{\partial \xi} d\tau \\
&\quad + \frac{1}{2}\int_0^x \frac{du(\xi, t+\xi-x)}{d\xi} d\xi - \frac{1}{2}\int_x^l \frac{du(\xi, t-\xi+x)}{d\xi} d\xi.
\end{aligned}
$$

Thus, we obtain the identity

$$
\begin{aligned}
u(x,t) = u(x,t) + \frac{1}{2}\int_0^{t-x} \frac{\partial u(0,\tau)}{\partial \xi} d\tau - \frac{1}{2}\int_0^{t-l+x} \frac{\partial u(l,\tau)}{\partial \xi} d\tau \\
- \frac{[u(0,t-x) + u(l, t-l+x)]}{2}, \forall (x,t) \in \Omega\,;
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
&\frac{1}{2}\int_0^{t-x} \frac{\partial u(0,\tau)}{\partial \xi} d\tau - \frac{1}{2}\int_0^{t-l+x} \frac{\partial u(l,\tau)}{\partial \xi} d\tau \\
&- \frac{[u(0,t-x) + u(l, t-l+x)]}{2} = 0, \quad \forall (x,t) \in \Omega.
\end{aligned} \tag{2.6}
$$

Now we consider the identity (2.6) when $(x,t) \to S$. Taking the limit as $x \to 0$, we have

$$
(u_x - u_t)(0,t) = 0, \quad x = 0,\ 0 < t < T,
$$

and similarly

$$
(u_x + u_t)(l,t) = 0, \quad x = l,\ 0 < t < T,
$$

as $x \to l$. Hence, the one-dimensional wave potential (2.1) satisfies to the lateral boundary conditions (2.4)-(2.5).

Conversely, if a solution of the equation (2.2) satisfies to the initial conditions (2.3) and the lateral boundary conditions (2.4)-(2.5), then it is determined only by the formula (2.1), in the other words it coincides the one-dimensional wave potential (2.1).

Indeed, if a function u_1 satisfies to the equation (2.2), the initial conditions (2.3) and the lateral boundary conditions (2.4)-(2.5), then $u_1 \equiv u$, where u is the wave potential (2.1). If it is not so, then the function $\vartheta(x,t) = u_1(x,t) - u(x,t)$ satisfies

$$
\begin{gathered}
\vartheta_{tt}(x,t) - \vartheta_{xx}(x,t) = 0, \quad (x,t) \in \Omega, \\
\vartheta(x,0) = \vartheta_t(x,0) = 0, \quad 0 < x < 1
\end{gathered}
$$

$$\begin{gathered}(\vartheta_x - \vartheta_t)(0,t) = 0, \quad x = 0,\ 0 < t < T, \\ (\vartheta_x + \vartheta_t)(l,t) = 0, \quad x = l,\ 0 < t < T\,.\end{gathered}$$

Since

$$0 = \int_\Omega \varepsilon(x-\xi, t-\tau) 0 d\xi d\tau = \int_\Omega \varepsilon(x-\xi, t-\tau)[\vartheta_{\tau\tau}(\xi,\tau) - \vartheta_{\xi\xi}(\xi,\tau)] d\xi d\tau,$$

a similar calculation as above shows that

$$\begin{aligned}0 = \int_\Omega \varepsilon(x-\xi, t-\tau)[\vartheta_{\tau\tau}(\xi,\tau) - \vartheta_{\xi\xi}(\xi,\tau)] d\xi d\tau = \vartheta(x,t) + \frac{1}{2}\int_0^{t-x} \frac{\partial\vartheta(0,\tau)}{\partial\xi} d\tau \\ - \frac{1}{2}\int_0^{t-l+x} \frac{\partial\vartheta(l,\tau)}{\partial\xi} d\tau - \frac{[\vartheta(0,t-x) + \vartheta(l,t-l+x)]}{2}, \quad \forall (x,t) \in \Omega.\end{aligned}$$

Denoting,

$$I_\vartheta(x,t) := \frac{1}{2}\int_0^{t-x} \frac{\partial\vartheta(0,\tau)}{\partial\xi} d\tau - \frac{1}{2}\int_0^{t-l+x} \frac{\partial\vartheta(l,\tau)}{\partial\xi} d\tau - \frac{[\vartheta(0,t-x) + \vartheta(l,t-l+x)]}{2}$$

And when $(x,t) \to S$, we obtain

$$\vartheta(x,t)|_{x=0} = -I_\vartheta(x,t)|_{x=0} = 0, \quad \vartheta(x,t)|_{x=l} = -I_\vartheta(x,t)|_{x=l} = 0.$$

Thus, the function $\vartheta(x,t)$ satisfies

$$\begin{gathered}\vartheta_{tt}(x,t) - \vartheta_{xx}(x,t) = 0, \quad (x,t) \in \Omega, \\ \vartheta(x,t) = \vartheta_t(x,t) = 0, \quad 0 < x < l, \\ \vartheta(0,t) = 0, \quad \vartheta(l,t) = 0.\end{gathered} \tag{2.7}$$

Let us define the function

$$E(t) = \int_0^l [(\vartheta_t(x,t))^2 + (\vartheta_x(x,t))^2] dx$$

which we call an energy integral. From the physical point of view, it is a total energy up to a constant, for instance, energy of oscillating string.

It is obvious that the function $E(t)$ is differentiable because of our conditions on the function $\vartheta(x,t)$. Consequently, its derivative is calculated as

$$E'(t) = \int_0^l [2\vartheta_t(x,t)\vartheta_{tt}(x,t) + 2\vartheta_x(x,t)\vartheta_{xt}(x,t)] dx$$

Integrating by parts, one writes the second term in the form

$$E'(t) = \int_0^l [2\vartheta_t(x,t)\{\vartheta_{tt}(x,t) - \vartheta_{xx}(x,t)\}] dx + 2\vartheta_x(x,t)\vartheta_t(x,t)|_0^l.$$

Note that the integrand function is equal to zero identically since $\vartheta(x,t)$ is a solution of the homogeneous wave equation. Then by differentiating with respect to t boundary conditions, we have $\vartheta_t(0,t) \equiv 0 \equiv \vartheta_t(l,t)$. It follows that the non-integral term also vanishes. So, $E'(t) \equiv 0$, or

$$E(t) = \int_0^l [(\vartheta_t(x,t))^2 + (\vartheta_x(x,t))^2] dx \equiv \text{const.}$$

Actually, we just obtain another law of energy conservation in a closed system which is described by the initial-boundary value problem (2.7) where amount of energy is permanent. Obviously,

$$E(t) = E(0) = \int_0^l [(\vartheta_t(x,0))^2 + (\vartheta_x(x,0))^2]dx$$

From the initial conditions we obtain $\vartheta_t(x,0) = \vartheta_x(x,0) = 0, 0 \le x \le l$. Hence

$$E(0) = 0 \;\Rightarrow\; E(t) \equiv 0\,.$$

Because the integrand functions are nonnegative we have $\vartheta_t(x,t) = \vartheta_x(x,t) = 0$. It follows that $\vartheta(x,t) =$const, and from the initial conditions it follows that $\vartheta(x,t) \equiv 0$; i.e., $u_1 \equiv u$, u_1 coincides with the wave potential potential (2.1).

To summarize, the initial-boundary value problem (2.2)-(2.5) has an unique solution and the solution coincides with the wave potential (2.1). This completes the proof. □

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References

[1] V. S. Vladimirov; *Equations of mathematical physics*. Nauka (1991).
[2] T. Sh. Kalmenov, D. Suragan; *To spectral problems for the volume potential*. Doklady Mathematics, 80 (2) (2009), pp. 646-649.
[3] T.Sh. Kalmenov, D. Suragan; *Boundary conditions for the volume potential for the polyharmonic equation*. Differential Equations, 48 (4) (2012), pp. 604-608.
[4] T. Sh. Kalmenov, D. Suragan; *A boundary condition and Spectral Problems for the Newton Potentials*. Operator Theory: Advances and Applications, 216 (2011), pp. 187-210.
[5] V.G. Maz'ya; *Boundary integral equations*. Analysis IV, Itogi Naukii Tekhniki, Sovrem. Probl. Mat. Fund. Naprav., VINITI, Moscow, 27 (1988), pp. 131-228.
[6] L. A. Pulkina; *Nonlocal problem with integral conditions for hyperbolic equations*. Electronic Journal of Differential Equations, 45 (1999), pp. 1-6.
[7] L. A. Pulkina, M. V. Strigun; *Two initial-boundary value problems with nonlineal boundary conditions for one-dimension hyperbolic equation*. Vestnik SamGU. Estestvenno-Nauchnaya Ser., 2 (83) (2011), pp. 46-56.
[8] D. Suragan, N. Tokmagambetov; *On transparent boundary conditions for the high-order heat equation*. Sib. electron. math. rep., 10 (2013), pp. 141-149.

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