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POSITIVE SOLUTIONS TO A NONLINEAR FRACTIONAL DIRICHLET PROBLEM ON THE HALF-LINE

HABIB MÂAGLI, ABDELWAHEB DHIFLI

Abstract. This concerns the existence of infinitely many positive solutions to the fractional differential equation

$$D^{\alpha}u(x) + f(x, u, D^{\alpha-1}u) = 0, \quad x > 0,$$
$$\lim_{x\to 0^+} u(x) = 0,$$

where $\alpha \in (1, 2]$ and f is a Borel measurable function in $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ satisfying some appropriate conditions.

1. Introduction

Recently, many papers on fractional differential equations have been published. The motivation for those works stems from the fact that fractional equations serve as an excellent tool to describe many phenomena in various fields of science and engineering such as control, porous media, electrochemistry, viscoelasticity, electromagnetic, etc (see [8, 10, 11, 19]). Therefore, the theory of fractional differential equations has been developed very quickly and the investigation for the existence of solutions of fractional differential equations has recently attracted a considerable attention (see [1, 2, 3, 4, 5, 6, 7, 9, 12, 13, 14, 15, 17, 18, 20, 21, 22] and the references therein. For instance, in [18], the first author considered the following nonlinear fractional differential problem in the half-line $\mathbb{R}^+ = (0, \infty)$:

$$\begin{gathered} D^{\alpha}u + f(x, u) = 0, \quad u > 0 \\ \lim_{x\to 0^+} u(x) = 0, \end{gathered} \tag{1.1}$$

where $1 < \alpha \le 2$ and f be a measurable function in $\mathbb{R}^+ \times \mathbb{R}^+$ satisfying an appropriate condition. Then, he established the existence of infinitely many solutions of (1.1).

In this paper, we extend this result to the fractional problem

$$\begin{gathered} D^{\alpha}u + f(x, u, D^{\alpha-1}u) = 0, \quad u > 0 \text{in } \mathbb{R}^+, \\ \lim_{x\to 0^+} u(x) = 0, \end{gathered} \tag{1.2}$$

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where f is a Borel measurable function in $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ satisfying the following assumptions.

(H1) f is continuous with respect to the second and third variable.

(H2) There exist h_1 and h_2 two nonnegative measurable functions on $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ such that

- (i) $|f(x,y,z)| \le h(x,y,z) := yh_1(x,y,z) + zh_2(x,y,z)$ for all $x,y,z \in \mathbb{R}^+$.
- (ii) The function h_j is nondecreasing with respect to the second and the third variables and satisfying $\lim_{(y,z)\to(0,0)} h_j(x,y,z) = 0$ for $j = 1,2$.
- (iii) The integral $\int_0^\infty h(t,\omega_\alpha(t),1)dt$ converges, where $\omega_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}$.

We recall that for a measurable function v, the Riemann-Liouville fractional integral $I^\beta v$ and the Riemann-Liouville derivative $D^\beta v$ of order $\beta > 0$ are defined by

$$I^\beta v(x) = \frac{1}{\Gamma(\beta)} \int_0^x (x-t)^{\beta-1} v(t)dt$$

and

$$D^\beta v(x) = \frac{1}{\Gamma(n-\beta)} \big(\frac{d}{dx}\big)^n \int_0^x (x-t)^{n-\beta-1} v(t)dt = \big(\frac{d}{dx}\big)^n I^{n-\beta} v(x),$$

provided that the right-hand sides are pointwise defined on $\mathbb{R}^+$. Here $n = [\beta]+1$ and $[\beta]$ means the integer part of the number β and Γ is the Euler Gamma function. Moreover, we have the following well-known properties (see [11, 20]).

$$I^\beta I^\gamma v(x) = I^{\beta+\gamma} v(x) \quad \text{for } x \in \mathbb{R}^+,\ v \in L^1_{\rm loc}([0,\infty)),\ \beta+\gamma \ge 1. \tag{1.3}$$

$$D^\beta I^\beta v(x) = v(x), \quad \text{a.e in } \mathbb{R}^+,\ v \in L^1_{\rm loc}([0,\infty)),\ \beta > 0. \tag{1.4}$$

$$D^\beta v(x) = 0 \quad \text{if and only if} \quad v(x) = \sum_{j=1}^n c_j x^{\beta-j}, \tag{1.5}$$

where n is the smallest integer greater than or equal to β and $(c_1, c_2, \dots, c_n) \in \mathbb{R}^n$.

Remark 1.1. Let $1 < \alpha \le 2$. Then a simple calculus, gives for $x \ge 0$,

$$I^{\alpha-1}(1)(x) = \omega_\alpha(x). \tag{1.6}$$

Our main result is the following.

Theorem 1.2. *Assume* (H1) *and* (H2). *Then problem* (1.2) *has infinitely many solutions. More precisely, there exists a number $b > 0$ such that for each $c \in (0,b]$, problem* (1.2) *has a continuous solution u satisfying*

$$u(x) = c\omega_\alpha(x) + \omega_\alpha(x) \int_0^\infty \big(1 - ((1-\frac{t}{x})^+)^{\alpha-1}\big) f(t,u(t),D^{\alpha-1}u(t))dt.$$

and

$$\lim_{x\to\infty} \frac{u(x)}{\omega_\alpha(x)} = \lim_{x\to\infty} D^{\alpha-1}u(x) = c.$$

Note that Theorem 1.2 generalizes a result established by Mâagli and Masmoudi [16] in the case $\alpha = 2$.

In the sequel, for $\lambda \in \mathbb{R}$, we put $\lambda^+ = \max(\lambda,0)$ and we denote by $C([0,\infty])$ the set of continuous functions v on $\mathbb{R}^+$ such that $\lim_{x\to 0^+} v(x)$ and $\lim_{x\to\infty} v(x)$ exist. It is easy to see that $C([0,\infty])$ is a Banach space with the norm $\|v\|_\infty = \sup_{x\ge 0}|v(x)|$. Let

$$E = \{v \in C([0,\infty)) : D^{\alpha-1}(\omega_\alpha v) \in C([0,\infty])\}$$

endowed with the norm $\|v\| = \|D^{\alpha-1}(\omega_\alpha v)\|_\infty$. Then the map

$$\begin{gathered}(E, \|\cdot\|) \to (C([0,\infty]), \|\cdot\|_\infty)\\ v \mapsto D^{\alpha-1}(\omega_\alpha v)\end{gathered}$$

is an isometry. It follows that $(E, \|\cdot\|)$ is a Banach space. Next we quote some results in the following lemmas that will be used later.

Lemma 1.3 ([6]). *Let f be a function in $C([0,\infty))$ such that $f(0) = 0$ and $D^{\alpha-1}f$ belongs to $C([0,\infty))$. Then for $x \geq 0$,*

$$I^{\alpha-1}D^{\alpha-1}f(x) = f(x).$$

Lemma 1.4. *Let m_1, $m_2 \in \mathbb{R}$ such that $m_1 \leq m_2$ and let $v \in C([0,\infty))$ such that $D^{\alpha-1}(\omega_\alpha v) \in C([0,\infty))$ and $m_1 \leq D^{\alpha-1}(\omega_\alpha v)(t) \leq m_2$ for all $t \geq 0$. Then for each $t \geq 0$,*

$$m_1 \leq v(t) \leq m_2.$$

In particular, $\|v\|_\infty \leq \|D^{\alpha-1}(\omega_\alpha v)\|_\infty$ and $E \subset C([0,\infty])$.

Proof. Let $v \in C([0,\infty))$ such that $D^{\alpha-1}(\omega_\alpha v) \in C([0,\infty))$ and

$$m_1 \leq D^{\alpha-1}(\omega_\alpha v) \leq m_2. \tag{1.7}$$

Using Lemma 1.3 and (1.6), we obtain

$$m_1\omega_\alpha \leq I^{\alpha-1}D^{\alpha-1}(\omega_\alpha v) = \omega_\alpha v \leq m_1\omega_\alpha.$$

This implies that for each $t \geq 0$,

$$m_1 \leq v(t) \leq m_2.$$

□

Let $\mathcal{F} = \{v \in E : 0 \leq D^{\alpha-1}(\omega_\alpha v) \leq 1\}$. Then we have the following result.

Lemma 1.5. *Assume* (H2). *Then the family of functions*

$$\Big\{x \mapsto \int_0^x (1 - \frac{t}{x})^{\alpha-1} f(t, \omega_\alpha(t)v(t), D^{\alpha-1}(\omega_\alpha v)(t))dt,\ v \in \mathcal{F}\Big\}$$

is relatively compact in $C([0,\infty])$.

Proof. For $v \in \mathcal{F}$ and $x > 0$, put

$$Sv(x) = \int_0^x (1 - \frac{t}{x})^{\alpha-1} f(t, \omega_\alpha(t)v(t), D^{\alpha-1}(\omega_\alpha v)(t))dt.$$

By (H2) and Lemma 1.4, we have for $v \in \mathcal{F}$ and $x > 0$,

$$\begin{aligned}|Sv(x)| &\leq \int_0^\infty |f(t, \omega_\alpha(t)v(t), D^{\alpha-1}(\omega_\alpha v)(t))|dt\\ &\leq \int_0^\infty h(t, \omega_\alpha(t)v(t), D^{\alpha-1}(\omega_\alpha v)(t))dt\\ &\leq \int_0^\infty h(t, \omega_\alpha(t), 1)dt < \infty.\end{aligned}$$

Thus the family $S(\mathcal{F})$ is uniformly bounded.

Now, we prove the equicontinuity of $S(\mathcal{F})$ in $[0,\infty]$. Let $x, x' \in \mathbb{R}^+$ and $v \in \mathcal{F}$, then we have

$$|Sv(x) - Sv(x')| \leq \int_0^\infty |((1 - \frac{t}{x})^+)^{\alpha-1} - ((1 - \frac{t}{x'})^+)^{\alpha-1}| h(t, \omega_\alpha(t), 1)dt,$$

$$
\begin{aligned}
|Sv(x)| &\le \int_0^x h(t,\omega_\alpha(t),1)dt, \\
\big|Sv(x) - &\int_0^\infty f(t,\omega_\alpha(t)v(t),D^{\alpha-1}(\omega_\alpha v)(t))dt\big| \\
&\le \int_0^\infty \big(1-((1-\frac{t}{x})^+)^{\alpha-1}\big)h(t,\omega_\alpha(t),1)dt.
\end{aligned}
$$

Using Lebesgue's theorem, we deduce from the above inequalities that $S(\mathcal{F})$ is equicontinuous in $[0,\infty]$. Hence, by Ascoli's theorem, we conclude that $S(\mathcal{F})$ is relatively compact in $C([0,\infty])$. $\square$

2. Proof of Theorem 1.2

In the sequel, we denote

$$g(x,y,z) = \omega_\alpha(x)h_1(x,y,z) + h_2(x,y,z), \quad \text{for } x,y,z \in \mathbb{R}^+.$$

By (H2) and Lebesgue's theorem,

$$\lim_{\beta\to 0}\int_0^\infty g(t,\beta\omega_\alpha(t),\beta)dt = 0.$$

Hence we can fix a number $0<\beta<1$ such that

$$\int_0^\infty g(t,\beta\omega_\alpha(t),\beta)dt \le \frac{1}{3}.$$

Let $b = 2\beta/3$ and $c\in(0,b]$. To apply a fixed point argument, set

$$\Lambda = \{v\in E : \frac{c}{2} \le D^{\alpha-1}(\omega_\alpha v) \le \frac{3c}{2}\}.$$

Then Λ is a nonempty closed bounded and convex set in E. Now, we define the operator T on Λ by

$$Tv(x) = c + \int_0^\infty \Big(1-((1-\frac{t}{x})^+)^{\alpha-1}\Big) f(t,\omega_\alpha(t)v(t),D^{\alpha-1}(\omega_\alpha v)(t))dt, \quad x>0.$$

First, we shall prove that the operator T maps Λ into itself. Let $v\in\Lambda$. Using Lemma 1.5, we deduce that the function Tv is in $C([0,\infty])$. On the other hand, for $x\ge 0$ we have

$$
\begin{aligned}
\omega_\alpha(x)Tv(x) = \omega_\alpha(x)\Big(c + \int_0^\infty f(t,\omega_\alpha(t)v(t),D^{\alpha-1}(\omega_\alpha v)(t))dt\Big) \\
- I^\alpha\big(f(.,\omega_\alpha v, D^{\alpha-1}(\omega_\alpha v))\big)(x).
\end{aligned}
$$

Hence, applying $D^{\alpha-1}$ on both sides of this equality, we conclude by (1.6) and (1.4) that for each $x\ge 0$,

$$D^{\alpha-1}(\omega_\alpha Tv)(x) = c + \int_x^\infty f(t,\omega_\alpha(t)v(t),D^{\alpha-1}(\omega_\alpha v)(t))dt.$$

This implies that $D^{\alpha-1}(\omega_\alpha Tv)$ is in $C([0,\infty])$ and $T\Lambda\subset E$. Furthermore, we have for $v\in\Lambda$ and $x\ge 0$,

$$
\begin{aligned}
|D^{\alpha-1}(\omega_\alpha Tv)(x) - c| &\le \int_0^\infty |f(t,\omega_\alpha(t)v(t),D^{\alpha-1}(\omega_\alpha v)(t))|dt \\
&\le \int_0^\infty h(t,\omega_\alpha(t)v(t),D^{\alpha-1}(\omega_\alpha v)(t))dt
\end{aligned}
$$

$$
\begin{aligned}
&\leq \int_0^\infty h(t, \frac{3c}{2}\omega_\alpha(t), \frac{3c}{2})dt \\
&= \frac{3c}{2}\int_0^\infty g(t, \frac{3c}{2}\omega_\alpha(t), \frac{3c}{2})dt \\
&\leq \frac{3c}{2}\int_0^\infty g(t, \beta\omega_\alpha(t), \beta)dt \leq \frac{c}{2}.
\end{aligned}
$$

It follows that for each $x \geq 0$,

$$\frac{c}{2} \leq D^{\alpha-1}(\omega_\alpha Tv)(x) \leq \frac{3c}{2}.$$

So, we conclude that Λ is invariant under T.

Next, we prove that $T\Lambda$ is relatively compact in $(E, \|\cdot\|)$. For any $v \in \Lambda$ and $x > 0$,

$$\frac{d}{dx}D^{\alpha-1}(\omega_\alpha Tv)(x) = -f(x, \omega_\alpha(x)v(x), D^{\alpha-1}(\omega_\alpha v)(x)) \quad \text{a.e. in } \mathbb{R}^+.$$

Since

$$|\frac{d}{dx}D^{\alpha-1}(\omega_\alpha Tv)(x)| \leq h(x, \omega_\alpha(x)v(x), D^{\alpha-1}(\omega_\alpha v)(x)) \leq h(x, \omega_\alpha(x), 1)$$

and $\int_0^\infty h(x, \omega_\alpha(x), 1)dx < \infty$, it follows that the family $\{D^{\alpha-1}(\omega_\alpha Tv),\, v \in \Lambda\}$ is equicontinuous on $[0, \infty]$. Moreover, $\{D^{\alpha-1}(\omega_\alpha Tv),\, v \in \Lambda\}$ is uniformly bounded. Then from Ascoli's theorem, $\{D^{\alpha-1}(\omega_\alpha v),\, v \in \Lambda\}$ is relatively compact in the space $(C([0, \infty]), \|\cdot\|_\infty)$. This implies that $T\Lambda$ is relatively compact in $(E, \|\cdot\|)$.

Now, we prove the continuity of T in Λ. Let (v_k) be a sequence in Λ such that

$$\|v_k - v\| = \|D^{\alpha-1}(\omega_\alpha v_k) - D^{\alpha-1}(\omega_\alpha v)\|_\infty \to 0 \quad \text{as } k \to \infty.$$

Then by Lemma 1.4, $\|v_k - v\|_\infty \to 0$ as $k \to \infty$ and for any $x \in [0, \infty]$, we have

$$
\begin{aligned}
&|D^{\alpha-1}(\omega_\alpha Tv_k)(x) - D^{\alpha-1}(\omega_\alpha Tv)(x)| \\
&= |\int_x^\infty \big[f(t, \omega_\alpha(t)v_k(t), D^{\alpha-1}(\omega_\alpha v_k)(t)) - f(t, \omega_\alpha(t)v(t), D^{\alpha-1}(\omega_\alpha v)(t))\big]dt| \\
&\leq \int_0^\infty \big|f(t, \omega_\alpha(t)v_k(t), D^{\alpha-1}(\omega_\alpha v_k)(t)) - f(t, \omega_\alpha(t)v(t), D^{\alpha-1}(\omega_\alpha v)(t))\big|dt
\end{aligned}
$$

and

$$|f(t, \omega_\alpha(t)v_k(t), D^{\alpha-1}(\omega_\alpha v_k)(t)) - f(t, \omega_\alpha(t)v(t), D^{\alpha-1}(\omega_\alpha v)(t))| \leq 2h(t, \omega_\alpha(t), 1).$$

So, by (H1) and Lebesgue's theorem,

$$\|Tv_k - Tv\| = \|D^{\alpha-1}(\omega_\alpha Tv_k) - D^{\alpha-1}(\omega_\alpha Tv)\|_\infty \to 0 \quad \text{as } k \to \infty.$$

It follows by the Schauder fixed point theorem that there exists $v \in \Lambda$ such that $Tv = v$. That is,

$$v(x) = c + \int_0^\infty (1 - ((1 - \frac{t}{x})^+)^{\alpha-1})f(t, \omega_\alpha(t)v(t), D^{\alpha-1}(\omega_\alpha v)(t))dt, \quad \text{for } x > 0.$$

We put $u(x) = \omega_\alpha(x)v(x)$. Then for any $x > 0$, we have

$$u(x) = c\omega_\alpha(x) + \omega_\alpha(x)\int_0^\infty (1 - ((1 - \frac{t}{x})^+)^{\alpha-1})f(t, \omega_\alpha(t)v(t), D^{\alpha-1}(\omega_\alpha v)(t))dt.$$

Moreover, for $x > 0$, we have

$$\begin{gathered}
\frac{c}{2}\omega_\alpha(x) \le u(x) \le \frac{3c}{2}\omega_\alpha(x), \\
\lim_{x\to\infty} \frac{u(x)}{\omega_\alpha(x)} = \lim_{x\to\infty} D^{\alpha-1}u(x) = c.
\end{gathered} \tag{2.1}$$

It remains to show that u is a solution of problem (1.2). Indeed, applying D^α on both sides of (2.1) we obtain by (1.5) and (1.4), that

$$D^\alpha u(x) = -f(x, u, D^{\alpha-1}u), \quad \text{a.e. in } \mathbb{R}^+.$$

This completes the proof.

Example 2.1. Let $p, q \ge 0$ such that $\max(p, q) > 1$ and let k be a measurable function satisfying

$$\int_0^\infty t^{(\alpha-1)p}|k(t)|dt < \infty.$$

Then there exists a constant $b > 0$ such that for each $c \in (0, b]$, the problem

$$\begin{gathered}
D^\alpha u + k(x)u^p(D^{\alpha-1}u)^q = 0, \quad u > 0 \quad \text{in } \mathbb{R}^+, \\
\lim_{x\to 0^+} u(x) = 0,
\end{gathered}$$

has a continuous solution u in $\mathbb{R}^+$ satisfying $\lim_{x\to 0^+} \frac{u(x)}{\omega_\alpha(x)} = \lim_{x\to\infty} D^{\alpha-1}u(x) = c$.

Example 2.2. Let $p > 1$ and $q > 1$. Let k_1 and k_2 be two measurable functions such that

$$\int_0^\infty |k_1(t)|t^{p(\alpha-1)}dt < \infty, \quad \int_0^\infty |k_2(t)|dt < \infty.$$

Then there exists a constant $b > 0$ such that for each $c \in (0, b]$, the problem

$$\begin{gathered}
D^\alpha u + k_1(x)u^p + k_2(x)(D^{\alpha-1}u)^q = 0, \quad u > 0 \quad \text{in } \mathbb{R}^+, \\
\lim_{x\to 0^+} u(x) = 0
\end{gathered}$$

has a continuous solution u in $\mathbb{R}^+$ satisfying

$$\lim_{x\to 0^+} \frac{u(x)}{\omega_\alpha(x)} = \lim_{x\to\infty} D^{\alpha-1}u(x) = c.$$

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Habib Mâagli
King Abdulaziz University, Rabigh Campus, College of Sciences and Arts, Department of Mathematics, P.O. Box 344, Rabigh 21911, Saudi Arabia
E-mail address: habib.maagli@fst.rnu.tn

Abdelwaheb Dhifli
Département de Mathématiques, Faculté des Sciences de Tunis, Campus Universitaire, 2092 Tunis, Tunisia
E-mail address: dhifli_waheb@yahoo.fr