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# DISCONTINUOUS ALMOST AUTOMORPHIC FUNCTIONS AND ALMOST AUTOMORPHIC SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH PIECEWISE CONSTANT ARGUMENTS

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Abstract. In this article we introduce a class of discontinuous almost automorphic functions which appears naturally in the study of almost automorphic solutions of differential equations with piecewise constant argument. Their fundamental properties are used to prove the almost automorphicity of bounded solutions of a system of differential equations with piecewise constant argument. Due to the strong discrete character of these equations, the existence of a unique discrete almost automorphic solution of a non-autonomous almost automorphic difference system is obtained, for which conditions of exponential dichotomy and discrete Bi-almost automorphicity are fundamental.

## 1. Introduction

A first order differential equation with piecewise constant argument (DEPCA) is an equation of the type

$$x'(t) = g(t, x(t), x([t])),$$

where $[\cdot]$ is the greatest integer function. The study of DEPCA began in 1983 with the works of Shah and Wiener [25], then in 1984 Cooke and Wiener studied DEPCA with delay [11]. DEPCA are of considerable importance in applications to some biomedical dynamics, physical phenomena (see [2, 7] and some references therein), discretization problems [28, 15], etc; consequently they have had a huge development, [9, 10, 14, 20, 21, 23, 24] (and some references therein) are evidence of these fact. In this way many results about existence, uniqueness, boundedness, periodicity, almost periodicity, pseudo almost periodicity, stability and other properties of the solutions for these equations have been developed (see [2, 13, 16, 20, 21, 32, 33] and some references therein). In 2006 the study of the almost automorphicity of the solution for a DEPCA was considered in [14, 26].

Let $\mathbb{X}, \mathbb{Y}$ be Banach spaces and $BC(\mathbb{Y}; \mathbb{X})$ denote the space of continuous and bounded functions from $\mathbb{Y}$ to $\mathbb{X}$. A function $f \in BC(\mathbb{R}; \mathbb{X})$ is said to be almost automorphic (in the sense of Bochner) if given any sequence $\{s'_n\}_{n\in\mathbb{N}}$ of real numbers, there exist a subsequence $\{s_n\}_{n\in\mathbb{N}} \subseteq \{s'_n\}_{n\in\mathbb{N}}$ and a function $\tilde{f}$, such that the

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pointwise limits

$$\lim_{n\to\infty} f(t+s_n) = \tilde{f}(t), \quad \lim_{n\to\infty} \tilde{f}(t-s_n) = f(t), t \in \mathbb{R} \tag{1.1}$$

hold.

When the previous limits are uniform in all the real line, we say that the function $f$ is almost periodic in the Bochner sense. Following the classical notation we denote by $AP(\mathbb{R};\mathbb{X})$ and $AA(\mathbb{R};\mathbb{X})$ the Banach spaces of almost periodic and almost automorphic functions respectively. For detailed information about these functions we remit to the references [4, 5, 6, 12, 16, 17, 27].

Our interest in this work is to prove the almost automorphicity of the bounded solutions of the DEPCA

$$x'(t) = Ax(t) + Bx([t]) + f(t), \tag{1.2}$$

where $A, B \in M_{p\times p}(\mathbb{R})$ are matrices and $f$ is an almost automorphic function.

The following definition expresses what we understand by solution for the DEPCA (1.2).

**Definition 1.1.** A function $x(t)$ is a solution of a DEPCA (1.2) in the interval $I$, if the following conditions are satisfied:

i) $x(t)$ is continuous in all $I$.
ii) $x(t)$ is differentiable in all $I$, except possibly in the points $n \in I \cap \mathbb{Z}$ where there should be a lateral derivative.
iii) $x(t)$ satisfies the equation in all the open interval $]n, n+1[, n \in \mathbb{Z}$ as well as is satisfied by its right side derivative in each $n \in \mathbb{Z}$.

DEPCA are differential equations of hybrid type; that is, they have the structure of continuous and discrete dynamical systems, more precisely in (1.2) the continuity occurs on intervals of the form $]n, n+1[, n \in \mathbb{Z}$ and the discrete aspect on $\mathbb{Z}$. Due to the continuity of the solution on the whole line for a DEPCA, we get a recursion formula in $\mathbb{Z}$ and thus, we can pass from an interval to its consecutive. The recursion formula appears naturally as solution of a difference equation.

With this objective, we study a general non-autonomous difference equation

$$x(n+1) = D(n)x(n) + h(n), \quad n \in \mathbb{Z}, \tag{1.3}$$

where $D(n) \in M_{p\times p}$ is a discrete almost automorphic matrix and $h$ is a discrete almost automorphic function. To study the equation (1.3) we use conditions of exponential dichotomy and a Bi-almost automorphic Green function [22, 29], obtaining a theorem about the existence of a unique discrete almost automorphic solution for (1.3). In [22, 29], functions with a Bi-property have shown to be very useful. When $D(n)$ is a constant operator on an abstract Banach space, Araya et al. [3] obtained the existence of discrete almost automorphic solutions under some geometric assumptions on the Banach space and spectral conditions on the operator $D$.

Note that although an almost automorphic solution $x$ of (1.2) is continuous, the function $x([t])$ does not and then it is not almost automorphic. Really, $x([t])$ has friendly properties for our study when in (1.1), $\{s_n\}_{n\in\mathbb{N}}$ are in $\mathbb{Z}$. This class of discontinuous functions, which we call $\mathbb{Z}$-almost automorphic (see definition 2.1), appears inevitably in DEPCA and allow us to study almost automorphic DEPCAs in a correct form (see the notes about Theorem 4.7). This type of problem is present in the study of continuous solutions of DEPCA of diverse kind as periodic or almost periodic type, but it is not sufficiently mentioned in the literature (see [1, 2, 18, 19, 30, 31, 34]). The treatment of almost periodic solutions for a DEPCA was initiated by R. Yuan and H. Jialin [32]. Dads and Lachimi [13] introduced discontinuous almost periodic functions to study the existence of a unique pseudo almost periodic solution in a well posed form to a DEPCA with delay. $\mathbb{Z}$-almost automorphic functions generalize the ones proposed in [13].

Properties derived in Section 2 for $\mathbb{Z}$-almost automorphic functions allow us to simplify the proofs of some important results, some of them known for almost automorphic functions in the literature (see Theorem 4.5 and [26, Lemma 3.3]). We will see that to obtain almost automorphic solutions of DEPCAs is sufficient to consider $\mathbb{Z}$-almost automorphic perturbations. An application of these facts is given by the use of the reduction method in DEPCA (1.2). This paper is organized as follows. In Section 2, we introduce the $\mathbb{Z}$-almost automorphic functions and their basic properties. In Section 3, we introduce the discrete Bi-almost automorphic condition for the Green matrix to study discrete non-autonomous almost automorphic solutions. Finally, in Section 4, we study the almost automorphic solutions of equation (1.2) in several cases.

## 2. $\mathbb{Z}$-almost automorphic functions

In this section we specify the definition of $\mathbb{Z}$-almost automorphic functions with values in $\mathbb{C}^p$ and develop some of their fundamental properties. Let us denote by $B(\mathbb{R};\mathbb{C}^p)$ and $BC(\mathbb{R};\mathbb{C}^p)$ the Banach spaces of respectively bounded and continuous bounded functions from $\mathbb{R}$ to $\mathbb{C}^p$ under the norm of uniform convergence. Now define $BPC(\mathbb{R},\mathbb{C}^p)$ as the space of functions in $B(\mathbb{R};\mathbb{C}^p)$ which are continuous in $\mathbb{R}\backslash\mathbb{Z}$ with finite lateral limits in $\mathbb{Z}$. Note that $BC(\mathbb{R};\mathbb{C}^p) \subseteq BPC(\mathbb{R},\mathbb{C}^p)$.

**Definition 2.1.** A function $f \in BPC(\mathbb{R};\mathbb{C}^p)$ is said to be $\mathbb{Z}$-almost automorphic, if for any sequence of integer numbers $\{s'_n\}_{n\in\mathbb{N}} \subseteq \mathbb{Z}$ there exist a subsequence $\{s_n\}_{n\in\mathbb{N}} \subseteq \{s'_n\}_{n\in\mathbb{N}}$ such that the pointwise limits in (1.1) hold.

When the convergence in Definition 2.1 is uniform, $f$ is called $\mathbb{Z}$-almost periodic. We denote the sets of $\mathbb{Z}$-almost automorphic (periodic) functions by $\mathbb{Z}AA(\mathbb{R};\mathbb{C}^p)$ ($\mathbb{Z}AP(\mathbb{R};\mathbb{C}^p)$). $\mathbb{Z}AA(\mathbb{R};\mathbb{C}^p)$ is an algebra over the field $\mathbb{R}$ or $\mathbb{C}$ and we have respectively $AA(\mathbb{R};\mathbb{C}^p) \subseteq \mathbb{Z}AA(\mathbb{R};\mathbb{C}^p)$ and $AP(\mathbb{R};\mathbb{C}^p) \subseteq \mathbb{Z}AP(\mathbb{R};\mathbb{C}^p)$. Notice that a $\mathbb{Z}$-almost automorphic function is locally integrable.

For functions in $BC(\mathbb{R}\times\mathbb{Y};\mathbb{X})$ we adopt the following notion of almost automorphicity.

**Definition 2.2.** *A function $f \in BC(\mathbb{R}\times\mathbb{Y};\mathbb{X})$ is said to be almost automorphic uniformly in compact subsets of $\mathbb{Y}$, if given any compact set $K \subseteq \mathbb{Y}$ and a sequence $\{s'_n\}_{n\in\mathbb{N}}$ of real numbers, there exists a subsequence $\{s_n\}_{n\in\mathbb{N}} \subseteq \{s'_n\}_{n\in\mathbb{N}}$ and a function $\tilde{f}$, such that for all $x \in K$ and each $t \in \mathbb{R}$ the limits*

$$\lim_{n\to\infty} f(t+s_n,x) = \tilde{f}(t,x), \quad \lim_{n\to\infty} \tilde{f}(t-s_n,x) = f(t,x), \tag{2.1}$$

*hold.*

The vectorial space of almost automorphic functions uniformly in compact subsets is denoted by $AA(\mathbb{R}\times\mathbb{Y};\mathbb{X})$, see [16, 17].

**Lemma 2.3.** *If $f \in AA(\mathbb{R};\mathbb{C}^p)$ (resp. $AP(\mathbb{R};\mathbb{C}^p)$), then $f([\cdot]) \in \mathbb{Z}AA(\mathbb{R};\mathbb{C}^p)$ (resp. $\mathbb{Z}AP(\mathbb{R};\mathbb{C}^p)$).*

All the next results for $\mathbb{Z}AA(\mathbb{R};\mathbb{C}^p)$ are also valid for $\mathbb{Z}AP(\mathbb{R};\mathbb{C}^p)$.

**Lemma 2.4.** *The space $\mathbb{Z}AA(\mathbb{R};\mathbb{C}^p)$ is a Banach space under the norm of uniform convergence.*

*Proof.* We only need to prove that the space $\mathbb{Z}AA(\mathbb{R};\mathbb{C}^p)$ is closed in the space of bounded functions under the topology of uniform convergence. Let $\{f_n\}_{n\in\mathbb{N}}$ be a uniformly convergent sequence of $\mathbb{Z}$-almost automorphic functions with limit $f$. By definition each function of the sequence is bounded and piecewise continuous with the same points of discontinuities, it is not difficult to see that the limit function $f$ is bounded and piecewise continuous. Given a sequence $\{s'_n\}_{n\in\mathbb{N}} \subseteq \mathbb{Z}$, it only rest to prove the existence of a subsequence $\{s_n\}_{n\in\mathbb{N}} \subseteq \{s'_n\}_{n\in\mathbb{N}}$ and a function $\tilde{f}$, where the pointwise convergence given in (1.1) holds. As in the standard case of the almost automorphic functions the approach follows across the diagonal procedure, see [16, 17]. □

**Lemma 2.5.** *Let $G : \mathbb{C}^p \to \mathbb{C}^p$ be a continuous function and $f \in \mathbb{Z}AA(\mathbb{R};\mathbb{C}^p)$, then $G(f(\cdot)) \in \mathbb{Z}AA(\mathbb{R};\mathbb{C}^p)$.*

**Lemma 2.6.** *Let $f \in AA(\mathbb{R}\times\mathbb{C}^p;\mathbb{C}^p)$ and uniformly continuous on compact subsets of $\mathbb{C}^p$, $\psi \in \mathbb{Z}AA(\mathbb{R};\mathbb{C}^p)$. Then $f(\cdot,\psi(\cdot)) \in \mathbb{Z}AA(\mathbb{R};\mathbb{C}^p)$.*

*Proof.* We have that the range of $\psi \in \mathbb{Z}AA(\mathbb{R};\mathbb{C}^p)$ is relatively compact; that is, $K = \overline{\{\psi(t), t \in \mathbb{R}\}}$ is compact. Let $\{s'_n\}_{n\in\mathbb{N}} \subseteq \mathbb{Z}$ be an arbitrary sequence, then there exist a subsequence $\{s_n\}_{n\in\mathbb{N}} \subseteq \{s'_n\}_{n\in\mathbb{N}}$ and functions $\tilde{f}$ and $\tilde{\psi}$ such that the pointwise limits in (2.1) and

$$\lim_{n\to+\infty} \psi(t+s_n) = \tilde{\psi}(t), \quad \lim_{n\to+\infty} \tilde{\psi}(t-s_n) = \psi(t), t \in \mathbb{R}$$

hold. The equality $\lim_{n\to+\infty} f(t+s_n,\psi(t+s_n)) = \tilde{f}(t,\tilde{\psi}(t))$ follows from

$$\begin{aligned}
&|f(t+s_n,\psi(t+s_n)) - \tilde{f}(t,\tilde{\psi}(t))| \\
&\le |f(t+s_n,\psi(t+s_n)) - f(t+s_n,\tilde{\psi}(t))| + |f(t+s_n,\tilde{\psi}(t)) - \tilde{f}(t,\tilde{\psi}(t))|.
\end{aligned}$$

The proof of $\lim_{n\to+\infty} \tilde{f}(t-s_n,\tilde{\psi}(t-s_n)) = f(t,\psi(t))$ is analogous. □

With analogous arguments we can prove the following Lemma.

**Lemma 2.7.** *Let $f \in AA(\mathbb{R}\times\mathbb{C}^p\times\mathbb{C}^p;\mathbb{C}^p)$ be uniformly continuous on compact subsets of $\mathbb{C}^p\times\mathbb{C}^p$, $\psi \in AA(\mathbb{R};\mathbb{C}^p)$, then $f(\cdot,\psi(\cdot),\psi([\cdot])) \in \mathbb{Z}AA(\mathbb{R};\mathbb{C}^p)$.*

Now we want to give a necessary condition to say when a $\mathbb{Z}$-almost automorphic function is almost automorphic.

**Lemma 2.8.** *Let $f$ be a continuous $\mathbb{Z}$-almost automorphic (periodic) function. If $f$ is uniformly continuous in $\mathbb{R}$, then $f$ is almost automorphic (periodic).*

*Proof.* Let $\{s'_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence of real numbers, then there exists a subsequence $\{s_n\}_{n\in\mathbb{N}} \subseteq \{s'_n\}_{n\in\mathbb{N}}$ of the form $s_n = t_n+\xi_n$ with $\xi_n \in \mathbb{Z}$ and $t_n \in [0,1[$ such that $\lim_{n\to\infty} t_n = t_0 \in [0,1]$. Moreover, $\{\xi_n\}_{n\in\mathbb{N}}$ can be chosen such that the pointwise limits

$$\lim_{n\to\infty} f(t+\xi_n) =: g(t), \quad \lim_{n\to\infty} g(t-\xi_n) = f(t), \quad t \in \mathbb{R} \tag{2.2}$$

hold. As $f$ is uniformly continuous, the function $g$ is too. Let us consider

$$|f(t+t_n+\xi_n) - g(t+t_0)|$$

$$\leq |f(t+t_n+\xi_n)-f(t+t_0+\xi_n)|+|f(t+t_0+\xi_n)-g(t+t_0)|.$$

Let $\epsilon>0$, $\delta=\delta(\epsilon)$ be the parameter in the uniform continuity of $f$. Let $N_0 = N_0(\epsilon)\in\mathbb{N}$ be such that for every $n\geq N_0$, $|t_n-t_0|<\delta$. Then the uniform continuity of $f$ ensures that $|f(t+t_n+\xi_n)-f(t+t_0+\xi_n)|<\frac{\epsilon}{2}$. Moreover, by (2.2) there exists $N_0'=N_0'(t,\epsilon)$ such that if $n\geq N_0'$, then $|f(t+t_0+\xi_n)-g(t+t_0)|<\frac{\epsilon}{2}$. Therefore, given $n\geq M_0=\max\{N_0,N_0'\}$, we have

$$|f(t+s_n)-g(t+t_0)|<\epsilon.$$

Similarly, from the uniform continuity of $g$ and (2.2) we conclude that $|g(t+t_0-s_n)-f(t)|<\epsilon$, for all $n\geq M_0$. Then $f\in AA(\mathbb{R},\mathbb{C}^p)$. □

**Lemma 2.9.** *Let $f\in\mathbb{Z}AA(\mathbb{R};\mathbb{C}^p)$ (resp. $\mathbb{Z}AP(\mathbb{R};\mathbb{C}^p)$). The function $F(t)=\int_0^t f(s)ds$ is bounded if and only if $F(\cdot)$ is almost automorphic (resp. almost periodic).*

*Proof.* The proof of the sufficient condition is immediate. For the necessary condition, since $F$ is uniformly continuous, we need to prove that $F$ is $\mathbb{Z}$-almost automorphic, which follows by the same arguments of [16, Theorem 2.4.4]. □

**Lemma 2.10.** *Let $\Phi:\mathbb{R}\to\mathbb{M}_{p\times p}(\mathbb{R})$ be an absolutely integrable matrix and $A\in M_{p\times p}(\mathbb{R})$ be a constant matrix. The operators*

$$(Lf)(t)=\int_{-\infty}^{\infty}\Phi(t-s)f(s)ds \quad \text{and} \quad (\Upsilon f)(t)=\int_{[t]}^{t}e^{A(t-s)}f(s)ds,$$

*map $\mathbb{Z}AA(\mathbb{R};\mathbb{C}^p)$ into itself.*

*Proof.* We only prove the Lemma for $L$, the proof for $\Upsilon$ is analogous. It is easy to see that the operator $L$ is bounded. Let $\{s_n'\}_{n\in\mathbb{N}}$ be a sequence of integers. Since $f\in\mathbb{Z}AA(\mathbb{R};\mathbb{C}^p)$, there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}\subseteq\{s_n'\}_{n\in\mathbb{N}}$ and a function $\tilde{f}$ such that we have the pointwise limits in (1.1).

Define the function $g(t)=(L\tilde{f})(t)$. Then, by the Lebesgue Convergence Theorem

$$\lim_{n\to+\infty}(Lf)(t+s_n)=\lim_{n\to+\infty}\int_{-\infty}^{\infty}\Phi(t-s)f(s+s_n)ds=g(t).$$

Analogously, the limit $\lim_{n\to\infty}g(t-s_n)=(Lf)(t)$ holds. □

## 3. Almost uutomorphic solutions of difference equations

As it is noted in the literature [13, 26, 32, 33], difference equations are very important in DEPCA studies. In this section, we are interested in obtaining discrete almost automorphic solutions of the system

$$x(n+1)=C(n)x(n)+f(n),\quad n\in\mathbb{Z}, \tag{3.1}$$

where $C(\cdot)\in M_{p\times p}(\mathbb{R})$ is a discrete almost automorphic matrix and $f(\cdot)$ is a discrete almost automorphic function.

**Definition 3.1.** Let $\mathbb{X}$ be a Banach space. A function $f:\mathbb{Z}\to\mathbb{X}$ is called discrete almost automorphic, if for any sequence $\{s_n'\}_{n\in\mathbb{N}}\subseteq\mathbb{Z}$, there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}\subseteq\{s_n'\}_{n\in\mathbb{N}}$, such that the following pointwise limits

$$\lim_{n\to+\infty}f(k+s_n)=:\tilde{f}(k),\ \lim_{n\to+\infty}\tilde{f}(k-s_n)=f(k),\ k\in\mathbb{Z}$$

hold.

We denote the vector space of almost automorphic sequences by $AA(\mathbb{Z},\mathbb{X})$ which becomes a Banach algebra over $\mathbb{R}$ or $\mathbb{C}$ with the norm of uniform convergence (see [3]). In [22, 29], we see the huge importance of the Bi-property of a function $H := H(\cdot,\cdot)$, such as Bi-periodicity, Bi-almost periodicity, Bi-almost automorphicity; i.e., $H$ has simultaneously the property in both variables. This motives the following definition.

**Definition 3.2.** *For $\mathbb{X}$ being a Banach space, a function $H : \mathbb{Z}\times\mathbb{Z} \to \mathbb{X}$ is said to be a discrete Bi-almost automorphic function, if for any sequence $\{s'_n\}_{n\in\mathbb{N}} \subseteq \mathbb{Z}$, there exists a subsequence $\{s_n\}_{n\in\mathbb{N}} \subseteq \{s'_n\}_{n\in\mathbb{N}}$, such that the following pointwise limits*

$$\lim_{n\to+\infty} H(k+s_n, m+s_n) =: \tilde{H}(k,m), \lim_{n\to+\infty} \tilde{H}(k-s_n, m-s_n) = H(k,m), \quad k,m \in \mathbb{Z}$$

*hold.*

Some examples of discrete Bi-almost automorphic functions can be obtained by restriction to the integer numbers of continuous Bi-almost automorphic (periodic) functions in $\mathbb{R}$.

The following definition deals with the discrete version of exponential dichotomy [33]. Suppose that the matrix function $C(n), n \in \mathbb{Z}$, of the equation (3.1) is invertible and consider $Y(n), n \in \mathbb{Z}$, a fundamental matrix solution of the system

$$x(n+1) = C(n)x(n), \quad n \in \mathbb{Z}. \tag{3.2}$$

**Definition 3.3.** *The equation* (3.2) *has an exponential dichotomy with parameters $(\alpha, K, P)$, if there are positive constants $\alpha, K$ and a projection $P$ such that*

$$|G(m,l)| \le Ke^{-\alpha|m-l|}, m,l \in \mathbb{Z},$$

*where $G(m,l)$ is the discrete Green function which takes the explicit form*

$$G(m,l) := \begin{cases} Y(m)PY^{-1}(l), m \ge l \\ -Y(m)(I-P)Y^{-1}(l), \quad m < l. \end{cases}$$

Now, we give conditions to obtain a unique discrete almost automorphic solution of the system (3.1).

**Theorem 3.4.** *Let $f \in AA(\mathbb{Z},\mathbb{C}^p)$. Suppose that the homogeneous part of equation* (3.1) *has an $(\alpha,K,P)$-exponential dichotomy with discrete Bi-almost automorphic Green function $G(\cdot,\cdot)$. Then the unique almost automorphic solution of* (3.1) *takes the form:*

$$x(n) = \sum_{k\in\mathbb{Z}} G(n,k+1)f(k), \ n \in \mathbb{Z} \tag{3.3}$$

*and*

$$|x(n)| \le K(1+e^{-\alpha})(1-e^{-\alpha})^{-1}||f||_\infty, n \in \mathbb{Z}.$$

*Proof.* It is well known that the function given by (3.3) is the unique bounded solution of the discrete equation (3.1) (see [33, Theorem 5.7]). We prove that this solution is discrete almost automorphic. In fact, consider an arbitrary sequence $\{s'_n\}_{n\in\mathbb{N}} \subseteq \mathbb{Z}$. Since $f \in AA(\mathbb{Z},\mathbb{C}^p)$ and $G(\cdot,\cdot)$ is discrete Bi-almost automorphic, there are a subsequence $\{s_n\}_{n\in\mathbb{N}} \subseteq \{s'_n\}_{n\in\mathbb{N}}$ and functions $\tilde{f}(\cdot), \tilde{G}(\cdot,\cdot)$ such that the following pointwise limits

$$\lim_{n\to+\infty} f(m+s_n) =: \tilde{f}(m), \quad \lim_{n\to+\infty} \tilde{f}(m-s_n) = f(m), \quad m \in \mathbb{Z}$$

and

$$\lim_{n\to+\infty} G(m+s_n,l+s_n) =: \tilde{G}(m,l), \quad \lim_{n\to+\infty} \tilde{G}(m-s_n,l-s_n) = G(m,l), \quad m,l\in\mathbb{Z}$$

hold. Note that $|\tilde{G}(m,l)| \le Ke^{-\alpha|m-l|}$, $m,l\in\mathbb{Z}$. Then,

$$\begin{aligned} x(n+s_n) &= \sum_{k\in\mathbb{Z}} G(n+s_n,k+1)f(k) \\ &= \sum_{k\in\mathbb{Z}} G(n+s_n,k+1+s_n)f(k+s_n), \end{aligned}$$

and from the Lebesgue Dominated Convergence Theorem we conclude that

$$\lim_{n\to\infty} x(n+s_n) = \tilde{x}(n),$$

where

$$\tilde{x}(n) = \sum_{k\in\mathbb{Z}} \tilde{G}(n,k+1)\tilde{f}(k).$$

To demonstrate the limit

$$\lim_{n\to\infty} \tilde{x}(n-s_n) = \sum_{k\in\mathbb{Z}} G(n,k+1)f(k) = x(n),$$

we proceed analogously. □

## 4. Almost automorphic solutions for linear DEPCA

Finally, in this section we investigate the almost automorphic solution of the equation (1.2). Before that, we reproduce the following useful result.

**Lemma 4.1.** *Let $f(\cdot)$ be a locally integrable and bounded function. If $x(\cdot)$ is a bounded solution of (1.2), then $x(\cdot)$ is uniformly continuous.*

*Proof.* Since $x(\cdot)$ and $f(\cdot)$ are bounded, there is a constant $M_0 > 0$ such that $\sup_{u\in\mathbb{R}} |Ax(u) + Bx([u]) + f(u)| \le M_0$. Now, as a consequence of the continuity of $x$, we conclude that

$$|x(t) - x(s)| \le \Big| \int_s^t (Ax(u) + Bx([u]) + f(u))du \Big| \le M_0|t-s|.$$

Then, the Lemma holds. □

For a better understanding, we study the equation (1.2) in several cases.

**4.1. B=0.** In this case the equation (1.2) becomes the system of differential equations

$$x'(t) = Ax(t) + f(t), \tag{4.1}$$

which has been well studied when $f \in AA(\mathbb{R};\mathbb{C}^p)$, see [16, 27]. But when $f \in \mathbb{Z}AA(\mathbb{R};\mathbb{C}^p)$ we have the following Massera type extension.

**Theorem 4.2.** *Let $f \in \mathbb{Z}AA(\mathbb{R};\mathbb{C}^p)$. If the eigenvalues of $A$ have non trivial real part, then the equation (4.1) has a unique almost automorphic solution.*

*Proof.* Since the eigenvalues of $A$ have non trivial real part, it is well known that the system $x'(t) = Ax(t)$ has an exponential dichotomy; that is, there are projections $P, Q$ with $P + Q = I$ such that the bounded solution of (4.1) has the form

$$x(t) = \int_{-\infty}^{t} e^{A(t-s)} Pf(s)ds - \int_{t}^{+\infty} e^{A(t-s)} Qf(s)ds.$$

By Lemma 2.10 we can see that this solution is bounded and $\mathbb{Z}$-almost automorphic. By the following Lemma 2.8, we only need to show that this solution is uniformly continuous, but this is a consequence of Lemma 4.1. The conclusion holds. □

For the scalar equation

$$x'(t) = \alpha x(t) + f(t), \tag{4.2}$$

Theorem 4.2 implies the following result.

**Corollary 4.3.** *Let $f \in \mathbb{Z}AA(\mathbb{R};\mathbb{C})$ and, the real part of $\alpha$, $\Re(\alpha) \neq 0$. Then the scalar equation* (4.2) *has a unique almost automorphic solution, given by*

$$x_1(t) = \int_{-\infty}^{t} e^{\alpha(t-s)} f(s)ds, \quad \text{for } \Re(\alpha) < 0,$$
$$x_2(t) = -\int_{t}^{+\infty} e^{\alpha(t-s)} f(s)ds, \quad \text{for } \Re(\alpha) > 0.$$

**Theorem 4.4.** *Let $\alpha$ be a purely imaginary complex number and $f \in \mathbb{Z}AA(\mathbb{R};\mathbb{C})$. If $x(\cdot)$ is a bounded solution of* (4.2) *then $x(\cdot)$ is almost automorphic.*

*Proof.* Let $\alpha = \theta i$, with $\theta \in \mathbb{R}$, then the solution of (4.2) is

$$x(t) = e^{\theta t i} x(0) + \int_{0}^{t} e^{\theta(t-s)i} f(s)ds, \quad t \in \mathbb{R}.$$

Since $x(\cdot)$ is bounded, we have that $\int_0^t e^{i\theta(t-s)} f(s)ds$ is bounded and, by Lemma 2.9, is almost automorphic. Therefore $x(\cdot)$ is almost automorphic. □

**4.2. B ≠0.** By the variation of parameters formula, the solution of DEPCA (1.2), for $t \in [n, n+1[$ and $n \in \mathbb{Z}$, satisfies

$$x(t) = Z(t, [t])x([t]) + H(t, [t]), \tag{4.3}$$

where

$$Z(t, [t]) = e^{A(t-[t])} + \int_{[t]}^{t} e^{A(t-s)} B ds \quad \text{and} \quad H(t, [t]) = \int_{[t]}^{t} e^{A(t-s)} f(s)ds.$$

By continuity of the solution $x$, if $t \to (n+1)^-$ we obtain the difference equation

$$x(n+1) = C(n)x(n) + h(n), n \in \mathbb{Z}, \tag{4.4}$$

where $C(n) = Z(n+1, n)$ and $h(n) = H(n+1, n)$. By Lemma 2.10, $Z$ and $H$ are $\mathbb{Z}$-almost automorphic functions, hence $C(n)$ and $H(n)$ are almost automorphic sequences.

For the existence of the solution $x = x(t)$ of DEPCA (1.2) on all of $\mathbb{R}$, we assume that the matrix $Z(t, [t])$ is invertible for $t \in \mathbb{R}$, see [2, 23, 28]. This hypothesis will be needed in the rest of the section. For example, when $A$ and $B$ are diagonal matrices, we have that

$$Z(t, [t]) = e^{A(t-[t])} + B \int_{[t]}^{t} e^{A(t-s)} ds$$

$$= e^{A(t-[t])}\Big[I + B\int_0^{t-[t]} e^{-Au}du\Big]$$

is invertible if and only if the next assumption holds.

Assume that the eigenvalues $\lambda_A$ of $A$ and $\lambda_B$ of $B$ satisfy for $u \in [0,1]$

$$\begin{gathered} \frac{\lambda_B}{\lambda_A}[1 - e^{-u\lambda_A}] \neq -1, \quad \text{if } \lambda_A \neq 0, \\ \lambda_B u \neq -1, \quad \text{if } \lambda_A = 0. \end{gathered} \tag{4.5}$$

As Theorem 4.7 below will show the existence, on all of $\mathbb{R}$, of the solutions of (1.2) also follows from condition (4.5) when matrices $A$ and $B$ are simultaneously triangularizable.

**Theorem 4.5.** *Let $x$ be a bounded solution of* (4.1) *with $f \in \mathbb{Z}AA(\mathbb{R};\mathbb{C}^p)$. Then $x$ is almost automorphic if and only if $x(n)$ in* (4.4) *is discrete almost automorphic.*

*Proof.* If $x$ is an almost automorphic solution then restricting it to $\mathbb{Z}$, $x(n)$ is discrete almost automorphic. For $f \in \mathbb{Z}AA(\mathbb{R};\mathbb{C}^p)$ and $x(n)$ being an almost automorphic sequence, the function $x$ given by (4.3) is well defined. The proof of the almost automorphicity of $x$ will follow at once if we prove its $\mathbb{Z}$-almost automorphicity, by Lemma 4.1.

Let us take an arbitrary sequence $\{s'_n\}_{n\in\mathbb{N}} \subseteq \mathbb{Z}$. Then there are a subsequence $\{s_n\}_{n\in\mathbb{N}} \subseteq \{s'_n\}_{n\in\mathbb{N}}$, functions $\tilde{f}$ and $\nu$ such that the limits in (1.1) and

$$\lim_{n\to+\infty} x(k+s_n) = \nu(k), \quad \lim_{n\to+\infty} \nu(k-s_n) = x(n), \quad k \in \mathbb{Z}$$

hold. Now, consider the limit function

$$y(t) = Z(t,[t])\nu([t]) + \int_{[t]}^t e^{A(t-s)}\tilde{f}(s)ds.$$

Then,

$$|x(t+s_n) - y(t)| \leq |Z(t,[t])||x([t]+s_n) - \nu([t])| + \int_{[t]}^t |e^{A(t-s)}||f(t+s_n) - \tilde{f}(s)|ds,$$

and for each $t \in \mathbb{R}$ we have $\lim_{n\to+\infty} x(t+s_n) = y(t)$. Analogously $\lim_{n\to+\infty} y(t-s_n) = x(t)$. Then, the bounded solution $x$ is $\mathbb{Z}$-almost automorphic. ☐

Note that, without using $\mathbb{Z}AA(\mathbb{R};\mathbb{C}^p)$, to prove directly $x(n) \in AA(\mathbb{Z};\mathbb{C}^p)$ implies $x \in AA(\mathbb{R};\mathbb{C}^p)$ is much more difficult (see [14, Lemma 3] and [26, Lemma 3.3]).

**4.3. A = 0, B ≠0.** Theorem 4.5 includes this important case

$$x'(t) = Bx([t]) + f(t), \tag{4.6}$$

for which the existence condition is reduced to invertibility for $t \in [0,1[$ of $I + tB$. Therefore the following result is obtained.

**Corollary 4.6.** *Let $f \in \mathbb{Z}AA(\mathbb{R};\mathbb{C}^p)$ and $x$ a bounded solution of* (4.6)*. Then, $x$ is almost automorphic if and only if $x(n)$ is discrete almost automorphic.*

**4.4. Reduction Method.** By "simultaneous triangularizations" of matrices $A$ and $B$, we understand that there is an invertible matrix, say $T$, which simultaneously triangularizes both matrices $A$ and $B$. There exist various results to obtain conditions under which simultaneous triangularization holds, see for example the monograph of Heydar Radjavi and Peter Rosenthal [24] and some references therein.

**Theorem 4.7** (Reduction Method). *Consider $f \in \mathbb{Z}AA(\mathbb{R};\mathbb{C}^p)$ and suppose that the matrices $A, B$ of the system* (1.2) *have simultaneous triangularizations and satisfy* (4.5)*. Let $x$ be a bounded solution of* (1.2)*, then $x$ is almost automorphic if and only if $x(n)$, in* (4.4)*, is discrete almost automorphic.*

*Proof.* If $x$ is almost automorphic, then its restriction to $\mathbb{Z}$ is discrete almost automorphic. We will prove that if $x(n)$ is discrete almost automorphic, then $x(\cdot)$ is almost automorphic. In fact, since $A, B$ have a simultaneous triangularization, there is an invertible matrix $T$ such that

$$T^{-1}AT = \bar{A} = \begin{bmatrix} \alpha_1 & a_{12} & a_{13} & \cdots & a_{1p} \\ 0 & \alpha_2 & a_{22} & \cdots & a_{2p} \\ \vdots & & & & \\ 0 & 0 & \cdots & 0 & \alpha_p \end{bmatrix},$$

$$T^{-1}BT = \bar{B} = \begin{bmatrix} \beta_1 & b_{12} & b_{13} & \cdots & b_{1p} \\ 0 & \beta_2 & b_{22} & \cdots & b_{2p} \\ \vdots & & & & \\ 0 & 0 & \cdots & 0 & \beta_p \end{bmatrix},$$

where, for $i \in \{1, 2, \cdots, p\}$, $\alpha_i$ and $\beta_i$ are the eigenvalues of $A$ and $B$ respectively. Consider the following change of variables $y(t) = T^{-1}x(t)$, then the boundedness of $x(t)$ is equivalent to the boundedness of $y(t)$, which is a solution of the following new system

$$y'(t) = \bar{A}y(t) + \bar{B}y([t]) + T^{-1}f(t).$$

Observe that, by Lemma 2.5, the sequence $y(n) = T^{-1}x(n) \in AA(\mathbb{Z}, \mathbb{C}^p)$, since $x(n)$ is almost automorphic. Let $T^{-1}f(t) = H(t) = (h_1(t), h_2(t), \cdots, h_p(t))$, then we have the almost automorphic system

$$y'(t) = \bar{A}y(t) + \bar{B}y([t]) + H(t), \tag{4.7}$$

namely,

$$y_1'(t) = \alpha_1 y_1(t) + \sum_{i=2}^{p} a_{1i} y_i(t) + \beta_1 y_1([t]) + \sum_{i=2}^{p} b_{1i} y_i([t]) + h_1(t)$$

$$y_2'(t) = \alpha_2 y_2(t) + \sum_{i=3}^{p} a_{2i} y_i(t) + \beta_2 y_2([t]) + \sum_{i=3}^{p} b_{2i} y_i([t]) + h_2(t)$$

$$\cdots$$

$$y_{p-1}'(t) = \alpha_{p-1} y_{p-1}(t) + a_{p-1p} y_p(t) + \beta_{p-1} y_{p-1}([t]) + b_{p-1p} y_p([t]) + h_{p-1}(t)$$

$$y_p'(t) = \alpha_p y_p(t) + \beta_p y_p([t]) + h_p(t).$$

Now take the $p$ th-equation

$$y_p'(t) = \alpha_p y_p(t) + \beta_p y_p([t]) + h_p(t), \tag{4.8}$$

where the eigenvalues $\alpha_p$ of $A$ and $\beta_p$ of $B$ satisfy (4.5).

Since $AA(\mathbb{R};\mathbb{C}^p) \subseteq \mathbb{Z}AA(\mathbb{R};\mathbb{C}^p)$ and $y_p$ is a bounded solution of (4.8), from Theorem 4.4, $y_p(t)$ is almost automorphic. Consider now the $(p-1)$ th-equation

$$y'_{p-1}(t) = \alpha_{p-1}y_{p-1}(t) + \beta_{p-1}y_{p-1}([t]) + [a_{p-1p}y_p(t) + b_{p-1p}y_p([t]) + h_{p-1}(t)] .$$

By Lemma 2.3, $y_p([t])$ is $\mathbb{Z}$-almost automorphic, then the function

$$z_{p-1}(t) = a_{p-1p}y_p(t) + b_{p-1p}y_p([t]) + h_{p-1}(t)$$

is again $\mathbb{Z}$-almost automorphic. Similarly, we can conclude that $y_{p-1}(t)$ is an almost automorphic solution of the equation

$$y'_{p-1}(t) = \alpha_{p-1}y_{p-1}(t) + \beta_{p-1}y_{p-1}([t]) + z_{p-1}(t), \tag{4.9}$$

since it is a bounded solution. Following this procedure, we obtain the almost automorphic solution $y(t)$ of system (4.7) and thus $x \in AA(\mathbb{R},\mathbb{C}^p)$. $\square$

Note that the discontinuous function $z_{p-1}$ in (4.9) is $\mathbb{Z}$-almost automorphic, although functions $h_{p-1}, h_p \in AA(\mathbb{R},\mathbb{C})$. Then, the presence of $\mathbb{Z}$-almost automorphic terms is proper of DEPCA. The $\mathbb{Z}$-almost automorphic space contains correctly the $\mathbb{Z}$-almost periodic and the interesting $\mathbb{Z}$-periodic situation (which are periodic functions not necessarily continuous), see [8]. Then we conclude.

**Corollary 4.8.** *Let $f \in \mathbb{Z}AP(\mathbb{R},\mathbb{C}^p)$. Then, every bounded solution $x$ of the DEPCA* (1.2) *is almost periodic if and only if $x(n) \in AP(\mathbb{Z},\mathbb{C}^p)$.*

**Corollary 4.9.** *Suppose that $f$ is a $\mathbb{Z}$-$\omega$-periodic function, with $\omega \in \mathbb{Q}$, then*
*(a) If $\omega = p_0 \in \mathbb{Z}$, every bounded solution $x$ of the DEPCA* (1.2) *is $\omega$-periodic if and only if the sequence $x(n), n \in \mathbb{Z}$, is discrete $\omega$-periodic.*
*(b) If $\omega = \frac{p_0}{q_0} \in \mathbb{Q}$ with $p_0, q_0 \in \mathbb{Z}$ relatively primes, then every bounded solution $x$ of the DEPCA* (1.2) *is $q_0\omega$-periodic if and only if the sequence $x(n), n \in \mathbb{Z}$ is discrete $q_0\omega$-periodic.*

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