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BOUNDARY PROBLEMS FOR MIXED PARABOLIC-HYPERBOLIC EQUATIONS WITH TWO LINES OF CHANGING TYPE AND FRACTIONAL DERIVATIVE

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Abstract. In this article, we study a boundary value problem for a parabolic-hyperbolic equation with Caputo fractional derivative. Under certain conditions, we prove its unique solvability using methods of integral equations and Green's functions.

1. Introduction

Gelfand started the study of mixed parabolic-hyperbolic type equations in his work [15]. Later on, Tricomi and Gellerstedt studied main boundary problems, and many authors have continued these studies as seen in the detailed bibliographies of [10, 11]. Son recent works [14, 14, 18, 25] have been devoted to the study of boundary problems for parabolic-hyperbolic equations with two or more lines of changing type. While other works have been devoted to the study of Riemann-Liouville, Caputo, Hadamard, Hadamard-Marchaud and other general fractional operators; see for example [2, 3, 15, 16, 18, 17, 21, 24, 26].

Non-local problems for parabolic-hyperbolic equations with one or two lines of changing type containing the Riemann-Liouville fractional derivative were investigated in [4, 7, 12, 19, 20].

The Caputo fractional derivative is suitable for numerical methods in the study of fractional differential equations [1, 27], and it appears in many mathematical models of real-life processes [9, 16, 22].

In this article we study a boundary value problem for a parabolic-hyperbolic equation with the Caputo fractional derivative, and having two lines where it changes type.

2. Formulation of the problem and main result

Let $0 < \alpha$ be a real number. For a function $\varphi(t)$, given on $(0, \ell)$, $\ell < \infty$ an integral-differential operator in a sense of the Riemann-Liouville starting at 0, is

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defined as follows [21, 24, 26],

$$D_{0t}^{\alpha}\varphi(t)=\begin{cases}\frac{1}{\Gamma(-\alpha)}\int_0^t\frac{\varphi(\tau)d\tau}{(t-\tau)^{\alpha+1}}, & \alpha<0,\\ \varphi(t), & \alpha=0,\\ \frac{d^n}{dt^n}D_{0t}^{\alpha-n}\varphi(t), & n-1<\alpha\le n,\ n\in N.\end{cases}$$

The operator

$${}_CD_{0t}^{\alpha}\varphi(t)=D_{0t}^{\alpha-n}\varphi^{(n)}(t),\quad n-1<\alpha\le n,\ n\in N$$

is called the Caputo fractional differential operator.

The Riemann-Liouville and the Caputo differential operators are related by the equality

$$cD_{0t}^{\alpha}\varphi(t)={}_cD_{0t}^{\alpha}\varphi(t)+\sum_{k=0}^{n}\frac{\varphi^k(0+)}{\Gamma(1+k-\mu)}t^{k-\alpha},\quad t>0. \tag{2.1}$$

Let us consider the equation

$$\frac{\partial^2u}{\partial x^2}-\frac{1-\operatorname{sign}(xy)}{2}\frac{\partial^2u}{\partial y^2}-\frac{1+\operatorname{sign}(xy)}{2}\cdot{}_CD_{0y}^{\alpha}u=f(x,y),\ \alpha\in(0,1). \tag{2.2}$$

This equation for $x>0,y>0$ is the fractional order diffusion equation

$$\frac{\partial^2u(x,y)}{\partial x^2}-{}_CD_{0y}^{\alpha}u(x,y)=f(x,y),$$

which coincides at $\alpha=1$ with the diffusion equation [26]

$$\frac{\partial^2u}{\partial x^2}-\frac{\partial u}{\partial y}=f(x,y).$$

Consider the (2.2) in a finite domain $\Omega\subset\mathbb{R}^2$, bounded for $x>0,\ y>0$ by segments A_0B_0,B_0B of straight lines $y=1,\ x=1$; at $x>0,\ y<0$ by segments AC,BC of characteristics $x+y=0,\ x-y=1$ of the (2.2); at $x<0,\ y>0$ by segments AD,A_0D of characteristics $x+y=0,\ y-x=1$ of the (2.2).

The parabolic part of the mixed domain Ω will be denoted by Ω_3, and the hyperbolic part by Ω_1, at $x>0$ and by Ω_2 at $x<0$, respectively.

The function $u(x,y)$ is called a regular solution of the (2.2), if it has necessary continuous derivatives participating in the (2.2) and satisfies it in $\Omega_1\cup\Omega_2\cup\Omega_3$.

In the domain Ω we study the following boundary problem.

Problem DS. Find a function $u(x,y)\in C(\overline{\Omega})$, such that:

(1) u is a regular solution of the (2.2) in the domain $\Omega\backslash(AA_0\cup AB)$;

(2) satisfies boundary conditions

$$u(x,y)\big|_{BB_0\cup DC}=0; \tag{2.3}$$

(3) on lines of type changing it satisfies the gluing conditions

$$u(x,-0)=u(x,+0),\ u(-0,y)=u(+0,y), \tag{2.4}$$

$$u_y(x,-0)=l_1(x)\cdot\lim_{y\to+0}y^{1-\alpha}u_y(x,y)+m_1(x)\cdot u(x,0)+n_1(x), \tag{2.5}$$

$$u_x(-0,y)=l_2(y)\cdot u_x(+0,y)+m_2(y)\cdot u(0,y)+n_2(y), \tag{2.6}$$

where $l_i(t)$, $m_i(t)$, $n_i(t)$, $y\in[0,1]$, $i=1,2$ are given functions.

Note that Problem DS generalizes a problem studied in [13].

Theorem 2.1. *Let the following conditions be fulfilled:* $f(x,y) \in C^{\delta}(\overline{\Omega})$, $0<\delta<1$, $l_i(t) \in C^1[0,1]$, $m_i(t), n_i(t) \in C[0,1]$, $l_i(t) \neq 0$, $l_i'(t) - 2l_i(t)\cdot m_i(t) \geq 0$ *for all* $t \in [0,1]$, $l_2(1) > 0$, $i=1,2$. *Then problem DS has a unique regular solution.*

Proof Theorem 2.1. First, we find the main functional relations on AB, AA_0 deduced from the domains Ω_1 and Ω_2. We introduce the following notation

$$u(x,0) = \tau_1(x), \quad , u(0,y) = \tau_2(y), \tag{2.7}$$

$$u_y(x,-0) = \nu_1^-(x), \quad \lim_{y\to+0} y^{1-\alpha} u_y(x,y) = \nu_1^+(x), \tag{2.8}$$

$$u_x(-0,y) = \nu_2^-(y), \quad u_x(+0,y) = \nu_2^+(x). \tag{2.9}$$

The solution to problem DS in Ω_1 can be represented by the D'Alembert's formula

$$u(\xi,\eta) = \frac{1}{2}\Big[\tau_1(\xi) + \tau_1(\eta) - \int_{\xi}^{\eta} \nu_1^-(t)dt\Big] - \int_{\xi}^{\eta} dt \int_{y}^{\eta} f_1(y,\tau)d\tau \tag{2.10}$$

where

$$\xi = x+y, \quad \eta = x-y, \quad f_1(\xi,\eta) = \frac{1}{4} f\big(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\big).$$

Then using condition (2.3) from (2.10) we get the following relation, reduced from the domain Ω_1 to the segment AB,

$$\tau_1'(x) - \nu_1^-(x) = 2\int_0^x f_1(t,x)dt, \quad x \in (0,1).$$

Similarly, from the formula

$$u(\xi,\eta) = \frac{1}{2}\Big[\tau_2(\xi) + \tau_2(\eta) - \int_{\xi}^{\eta} \nu_2^-(t)dt\Big] - \int_{\xi}^{\eta} dt \int_{y}^{\eta} f_2(y,\tau)d\tau \tag{2.11}$$

by (2.3), from the domain Ω_2 one can deduce the following relation between functions $\tau_2(y)$ and $\nu_2^-(y)$:

$$\tau_2'(y) - \nu_2^-(y) = 2\int_0^y f_2(t,y)dt, \quad y \in (0,1), \tag{2.12}$$

where

$$\xi = x+y, \quad \eta = y-x, \quad f_2(\xi,\eta) = -\frac{1}{4} f\big(\frac{\xi-\eta}{2}, \frac{\xi+\eta}{2}\big).$$

First, we prove the uniqueness of the solution of problem DS.

Lemma 2.2. *Let the conditions of the theorem be valid. Then problem DS can not have more than one regular solution.*

Proof. Suppose the opposite. Let problem DS has two different regular solutions $u_1(x,y)$, $u_2(x,y)$ and let

$$u(x,y) = u_1(x,y) - u_2(x,y).$$

It is not difficult to see that $u(x,y)$ is a regular solution of the homogeneous problem DS ($f(x,y)=0$, $n_i(t)=0$, $i=1,2$). This is why one only needs to prove that homogeneous problem has only the trivial solution.

Let $u(x,y)$ be a regular solution of the homogeneous problem DS in the domain Ω. Since $uu_{xx} = (u\cdot u_x)_x - u_x^2$, then integrating the identity $u\big(u_{xx} - {}_CD_{0y}^{\alpha} u(x,y)\big) = 0$ along the domain Ω_3, taking (2.3), (2.7)-(2.9) into account, after some evaluations we deduce

$$\iint_{\Omega_3} u \cdot {}_CD_{0y}^{\alpha} u(x,y)\,dx\,dy + \int_0^1 \tau_2(x)\nu_2^+(x)dx + \iint_{\Omega_3} u_x^2\,dx\,dy = 0 \tag{2.13}$$

Consider the integral

$$I_2 = \int_0^1 \tau_2(y)\nu_2^+(y)dy.$$

Taking into consideration the relation

$$\nu_2^+(y) = \frac{1}{l_2(y)}\nu_2^-(y) - \frac{m_2(y)}{l_2(y)}\tau_2(y),$$

which follows from gluing conditions (2.5) and notation (2.9), integral I_2 is rewritten as

$$I_2 = \int_0^1 \frac{1}{l_2(y)}\tau_2(y)\nu_2^-(y)dy - \int_0^1 \frac{m_2(y)}{l_2(y)}\tau_2^2(y)dy.$$

On the other hand from (2.12) it follows that

$$\int_0^1 \frac{1}{l_2(y)}\tau_2(y)\nu_2^-(y)dy = \frac{\tau_2^2(1)}{2l_2(1)} + \frac{1}{2}\int_0^1 \frac{l_2'(y)}{l_2^2(y)}\tau_2^2(y)dy.$$

Hence

$$I_2 = \frac{\tau_2^2(1)}{2l_2(1)} + \frac{1}{2}\int_0^1 \frac{l_2'(y) - l_2(y)m_2(y)}{l_2^2(y)}\tau_2^2(y)dy. \tag{2.14}$$

Using the formula [see [26, p. 53]

$$\lim_{t\to 0} D_{0t}^{\beta-1}\varphi(t) = \Gamma(\beta)\lim_{t\to 0} t^{1-\beta}\varphi(t), t^{1-\beta}\varphi(t) \in C[0,1), 0<\beta<1$$

from the (2.2) passing to the limit at $y \to +0$, taking notations (2.7) and (2.8) into account, we obtain

$$\tau_1''(x) - \Gamma(\alpha)\nu_1^+(x) = 0,\ x \in (0,1). \tag{2.15}$$

Considering condition (2.5), equality (2.15) can be rewritten as

$$\tau_1''(x) - \frac{\Gamma(\alpha)}{l_1(x)}[\nu_1^-(x) - m_1(x)\tau_1(x)] = 0.$$

From here we obtain

$$-\int_0^1 [\tau'{}_1(x)]^2dx - \Gamma(\alpha)\int_0^1 \frac{1}{l_1(x)}\tau_1(x)\nu_1^-(x)dx + \Gamma(\alpha)\int_0^1 \frac{m_1(x)}{l_1(x)}\tau_1^2(x)dx = 0. \tag{2.16}$$

On the other hand from (2) it follows that

$$\int_0^1 \frac{1}{l_1(x)}\tau_1(x)\nu_1^-(x)dx = \int_0^1 \frac{l_1'(x)}{2l_2^2(x)}\tau_1^2(x)dx.$$

Then relation (2.16) will have the form

$$\int_0^1 [\tau'{}_1(x)]^2dx + \Gamma(\alpha)\int_0^1 \frac{l_1'(x) - 2l_1(x)m_1(x)}{2l_1^2(x)}\tau_1^2(x)dx = 0.$$

Here considering the conditions of the theorem we have $\tau_1'(x) = 0$. Hence, $\tau_1(x) =$ const. Since, $\tau_1(0) = 0$, it follows that $\tau_1(x) = 0$, $x \in [0,1]$.

Taking (2.14), $\tau_1(x) = 0$ and formula (2.1) into account, (2.13) is rewritten as

$$\iint_{\Omega_3} uD_{RL}^{\alpha}u(x,y)\,dx\,dy + \frac{\tau_2^2(1)}{2l_2(1)} + \frac{1}{2}\int_0^1 \frac{l_2'(x) - 2l_2(x)m_2(x)}{l_2^2(x)}\tau_2^2(x)dx + \iint_{\Omega_3} u_x^2(x,y)\,dx\,dy = 0.$$

According to [24, Theorem 1.7.1], from the last equality we have $u(x,y)\equiv 0$ in $\overline{\Omega}_3$. Further, from formulas (2.10) and (2.11), by virtue of the uniqueness of the solution of the Cauchy problem we have that $u(x,y)\equiv 0$ in $\overline{\Omega}_1\cup\overline{\Omega}_2$. Hence, $u(x,y)\equiv 0$; i.e., $u_1(x,y)\equiv u_2(x,y)$ in $\bar{\Omega}$. The proof is complete □

Now we prove the existence of the solution of problem DS.

From the (2.2) we deduce the following functional relation between functions $\tau_1(x)$ and $\nu_1^+(x)$, on AB

$$\tau_1''(x)-\Gamma(\alpha)\nu_1^+(x)=f(x,0).$$

Defining function $\nu_1^+(x)$. from conditioin (2.5) and (2) and substituting into the above equation we obtain the following problem for the unknown $\tau_1(x)$:

$$\begin{gathered}\tau_1{}''(x)+p(x)\tau_1{}'(x)+q(x)\tau_1(x)=g(x),\\ \tau_1(0)=\tau_1(1)=0,\end{gathered}$$

where

$$p(x)=-\frac{\Gamma(\alpha)}{l_1(x)},\quad q(x)=\frac{m_1(x)}{l_1(x)},\quad g(x)=-\frac{1}{l_1(x)}\big[n_1(x)+2\int_0^x f_1(t,x)dt\big].$$

The uniqueness of the solution of this problem follows from the uniqueness of the solution of problem DS. Note that this solution can be written by Green's function (see [13]). Since, function $\tau_1(x)$ is now known, from (2) we find $\nu_1^-(y)$. Hence, the solution of the problem in Ω_1 is known.

The unknown function $\tau_2(y)$ can be found by the formula of the solution of the first boundary problem for the (2.2) in Ω_3 [6]:

$$\begin{aligned}u(x,y)=&\int_0^y G_\xi(x,y,0,\eta)\tau_2(\eta)d\eta+\int_0^1\tilde{G}(x-\xi,y)\tau_1(\xi)d\xi\\ &-\iint_{\Omega_3}G(x,y,\xi,\eta)f(\xi,\eta)d\xi d\eta,\end{aligned}\tag{2.17}$$

where $G(x,y,\xi,\eta)$ is the Green's function of the first boundary problem for the diffusion equation with the Riemann-Liouville fractional differential operator (see [26, p. 108])

$$G(x,y,\xi,\eta)=\frac{(y-\eta)^{\beta-1}}{2}\sum_{n=-\infty}^{\infty}\Big[e_{1,\beta}^{1,\beta}\Big(-\frac{|x-\xi+2n|}{(y-\eta)^\beta}\Big)-e_{1,\beta}^{1,\beta}\Big(-\frac{|x+\xi+2n|}{(y-\eta)^\beta}\Big)\Big],$$

$$\tilde{G}(x-\xi,y)=\frac{1}{\Gamma(1-\alpha)}\int_0^y\eta^{-\alpha}G(x,y,\xi,\eta)d\eta,\quad \beta=\frac{\alpha}{2},$$

where $e_{1,\beta}^{1,\beta}(z)$ is the Wright's function, which has the form

$$e_{1,\beta}^{1,\delta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{n!\,\Gamma(\delta-\beta n)}.$$

Calculating $u_x(x,y)$ from (2.17) and letting x go to zero, bearing in mind [26, Lemma 2.2.2], we get a relation between functions $\tau_2(y)$ and $\nu_2^+(y)$ on AA_0:

$$\nu_2^+(y)=-\int_0^y K(y-t)\tau_2{}'(t)dt+\Phi_0(y),$$

where

$$K(y-t)=\frac{1}{(y-t)^{\beta}}\Big[\frac{1}{\Gamma(1-\beta)}+2\sum_{n=1}^{\infty}e_{1,\beta}^{1,1-\beta}\Big(-\frac{2n}{(y-t)^{\beta}}\Big)\Big],$$

$$\Phi_0(y)=\lim_{x\to+0}\Big[\int_0^1\tilde{G}_x(x-\xi,y)\tau_1(\xi)d\xi-\iint_{\Omega_3}G_x(x,y,\xi,\eta)f(\xi,\eta)d\xi d\eta\Big].$$

Considering $\nu_2^-(y)=\nu_2^+(x)=\nu_2(x)$, excluding $\nu_2(x)$ from (2.12) and (20), we get the Volterra integral equation of second kind regarding the unknown function ${\tau_2}'(y)$:

$$ {\tau_2}'(y)+\int_0^y K(y-t){\tau_2}'(t)dt=\Phi(y), \tag{2.18}$$

where

$$\Phi(y)=\Phi_0(y)+2\int_0^y f_2(t,y)dt.$$

Since a solution of the integral equation depends on the kernel $K(y-t)$, we shall study it in detail. The function $K(y-t)$ we represent as a sum of two kernels

$$K(y-t)=K_1(y-t)+K_2(y-t),$$

where

$$K_1(y-t)=-\frac{(y-t)^{-\beta}}{\Gamma(1-\beta)},\quad K_2(y-t)=-\frac{2}{(y-t)^{\beta}}\sum_{n=1}^{\infty}e_{1,\beta}^{1,1-\beta}\big(-\frac{2n}{(y-t)^{\beta}}\big).$$

Note that $K_1(y-t)$ is a kernel with weak singularity. The kernel $K_2(y-t)$ represented as a series of Wright's type functions. From [26, (2.2.5), (2.2.24)] it follows that the kernel $K_2(y-t)$ has also weak singularity. Therefore, equation (2.18) is a Volterra equation of second kind with weak singularity.

Using formulas [26, 2.2.19, 2.2.25, 2.3.8, 3.3.3], it is not difficult to show that the right hand side of (2.18) is a continuous function. Then, from the theory of integral equations follows that (2.18) has a unique continuous solution (see, for example [23]).

After finding the functions $\tau_1(x)$, $\tau_2(x)$, $\nu_1(x)$ and $\nu_2(y)$, the solution of problem DS in the domain Ω_3 will be found by formula (2.17), and in the domains Ω_1, Ω_2 as a solution of the Cauchy problem by the formulas (2.10) and (2.11), respectively.

The proof of Theorem 2.1 is complete.

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