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SOLVABILITY OF FRACTIONAL MULTI-POINT BOUNDARY-VALUE PROBLEMS WITH p-LAPLACIAN OPERATOR AT RESONANCE

TENGFEI SHEN, WENBIN LIU, TAIYONG CHEN, XIAOHUI SHEN

Abstract. In this article, we consider the multi-point boundary-value problem for nonlinear fractional differential equations with p-Laplacian operator:

$$D_{0+}^{\beta}\varphi_p(D_{0+}^{\alpha}u(t)) = f(t, u(t), D_{0+}^{\alpha-2}u(t), D_{0+}^{\alpha-1}u(t), D_{0+}^{\alpha}u(t)), \quad t \in (0,1),$$
$$u(0) = u'(0) = D_{0+}^{\alpha}u(0) = 0, \quad D_{0+}^{\alpha-1}u(1) = \sum_{i=1}^{m}\sigma_i D_{0+}^{\alpha-1}u(\eta_i),$$

where $2 < \alpha \le 3$, $0 < \beta \le 1$, $3 < \alpha + \beta \le 4$, $\sum_{i=1}^{m}\sigma_i = 1$, D_{0+}^{α} is the standard Riemann-Liouville fractional derivative. $\varphi_p(s) = |s|^{p-2}s$ is p-Laplacians operator. The existence of solutions for above fractional boundary value problem is obtained by using the extension of Mawhin's continuation theorem due to Ge, which enrich konwn results. An example is given to illustrate the main result.

1. Introduction

In recent years, fractional differential equations play a important role in many fields such as physics, engineering, biology, control theory, etc., see [1, 12, 15, 17, 18]. It has been studied extensively by scholars have obtained many results, see [2, 5, 10, 11, 14, 19, 22].

However, the existence of solutions for fractional boundary value problems at resonance is less studied, see [3, 4, 7, 9, 20, 21]. There are few articles which consider the boundary value problems (BVPs for shorts) at resonance for nonlinear fractional differential equation with p-Laplacian operator. In 2012, Chen, Liu and Hu [6] considered existence of solutions of boundary value problems for a Caputo fractional differential equation with p-Laplacian operator at resonance by coincidence degree theory by Mawhin:

$$\begin{gathered} D_{0+}^{\beta}\varphi_p(D_{0+}^{\alpha}u(t)) = f(t, u(t), D_{0+}^{\alpha}u(t)), \quad t \in (0,1), \\ D_{0+}^{\alpha}u(0) = D_{0+}^{\alpha}u(1) = 0, \end{gathered} \tag{1.1}$$

where $0 < \alpha, \beta < 1$, $1 < \alpha + \beta \le 2$, D_{0+}^{α} is a Caputo fractional derivative, $\varphi_p(s) = |s|^{p-2}s$ is a p-Laplacian operator, $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous.

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In this article, we study fractional multi-point boundary value problem with p-Laplacian operator at resonance by using the extension of Mawhin's continuation theorem due to Ge,

$$
\begin{gathered}
D_{0+}^{\beta}\varphi_p(D_{0+}^{\alpha}u(t)) = f(t,u(t),D_{0+}^{\alpha-2}u(t),D_{0+}^{\alpha-1}u(t),D_{0+}^{\alpha}u(t)), \quad t\in(0,1), \\
u(0)=u'(0)=D_{0+}^{\alpha}u(0)=0, \quad D_{0+}^{\alpha-1}u(1)=\sum_{i=1}^{m}\sigma_i D_{0+}^{\alpha-1}u(\eta_i),
\end{gathered}
\tag{1.2}
$$

where $2<\alpha\le 3$, $0<\beta\le 1$, $3<\alpha+\beta\le 4$, $\eta_i\in(0,1)$, $\sigma_i\in\mathbb{R}$, $\sum_{i=1}^{m}\sigma_i=1$, $1<m, m\in N$, $\varphi_p(s)=|s|^{p-2}s$, $1<p, 1/p+1/q=1$, φ_p is invertible and its inverse operator is φ_q, D_{0+}^{α} is Riemann-Liouville standard fractional derivative, $f:[0,1]\times\mathbb{R}^4\to\mathbb{R}$ is continuous.

There are few articles to investigate fractional multi-point boundary value problem with p-Laplacian operator at resonance. By constructing suitable continuous linear projectors and using the extension of Mawhin's continuation theorem due to Ge, the existence of solutions were obtained. Our paper perfect and generalize some known results.

To investigate the problem, we use the condition

$$
\Delta=\frac{1}{\Gamma(\beta+1)^{q-1}(q\beta-\beta+1)}\Big(1-\sum_{i=1}^{m}\sigma_i\eta_i^{q\beta-\beta+1}\Big)\neq 0.
$$

The rest of this article is organized as follows: In Section 2, we give some notations, definitions and Lemmas. In Section 3, basing on the extension of Mawhin's continuation theorem due to Ge, we establish a theorem of existence result for BVP (1.2).

2. Preliminaries

For the convenience of the reader, we present here some basic knowledge and definitions for fractional calculus theory, that can be found in [2, 8, 12].

Let X and Y be two Banach spaces with norms $\|\cdot\|_X$ and $\|u\|_Y$, respectively. A continuous operator

$$
M|_{\operatorname{dom} M\cap X}: X\cap\operatorname{dom} M\to Y
$$

is said to be quasi-linear if

(i) $\operatorname{Im} M := M(X\cap\operatorname{dom} M)$ is a closed subset of Y,
(ii) $\ker M := \{u\in X\cap\operatorname{dom} M : Mu=0\}$ is is linearly homeomorphic to $\mathbb{R}^n$, $n<\infty$.

Let $X_1=\ker M$ and X_2 be the complement space of X_1 in X, then $X=X_1\oplus X_2$. On the other hand, suppose Y_1 is a subspace of Y and Y_2 is the complement space of Y_1 in Y so that $Y=Y_1\oplus Y_2$. Let $P:X\to X_1$ be a projector and $Q:Y\to Y_1$ a semi-projector, and $\Omega\subset X$ an open and bounded set with origin $\theta\in\Omega$. Where θ is the origin of a linear space.

Suppose $N_\lambda:\overline{\Omega}\to Y$, $\lambda\in[0,1]$ is a continuous operator. Denote N_1 by N. Let $\Sigma_\lambda=\{u\in\overline{\Omega}: Mu=N_\lambda u\}$. N_λ is said to be M-compact in $\overline{\Omega}$ if there is a $Y_1\subset Y$ with $\dim Y_1=\dim X_1$ and an operator $R:\overline{\Omega}\times[0,1]\to X$ continuous and compact such that for $\lambda\in[0,1]$,

$$
(I-Q)N_\lambda(\overline{\Omega})\subset\operatorname{Im} M\subset(I-Q)Y, \tag{2.1}
$$

$$
QN_\lambda x=\theta, \lambda\in(0,1)\Leftrightarrow QNx=\theta, \tag{2.2}
$$

$$R(\cdot,\lambda)\mid_{\Sigma_\lambda}=(I-P)\mid_{\Sigma_\lambda} \tag{2.3}$$

and $R(\cdot,0)$ is the zero operator,

$$M[P+R(\cdot,\lambda)]=(I-Q)N_\lambda. \tag{2.4}$$

Lemma 2.1 ([8]). *Let $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ be two Banach spaces, and $\Omega\subset X$ an open and bounded nonempty set. Suppose $M:X\cap\operatorname{dom}M\to Y$ is a quasi-linear operator $N_\lambda:\overline{\Omega}\to Y$, $\lambda\in[0,1]$ is M-compact in $\overline{\Omega}$. In addition, if:*

- (i) *$Mu\neq N_\lambda u$ for all $(u,\lambda)\in(\operatorname{dom}M\cap\partial\Omega)\times(0,1)$,*
- (ii) *$QNu\neq 0$ for all $u\in\partial\Omega\cap\ker M$,*
- (iii) *$\deg(JQN,\ker M\cap\Omega,0)\neq 0$, where $J:\operatorname{Im}Q\to\ker M$ is a homeomorphism with $J(\theta)=\theta$ and $N=N_1$,*

then the equation $Mu=Nu$ has at least one solution in $\operatorname{dom}M\cap\bar{\Omega}$.

Definition 2.2. The Riemann-Liouville fractional integral of order $\alpha>0$ of a function u is given by

$$I_{0^+}^\alpha u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}u(s)ds,$$

provided that the right side integral is pointwise defined on $(0,+\infty)$.

Definition 2.3. The Riemann-Liouville fractional derivative of order $\alpha>0$ of a function u is given by

$$D_{0^+}^\alpha u(t)=\frac{1}{\Gamma(n-\alpha)}(\frac{d}{dt})^n\int_0^t\frac{u(s)}{(t-s)^{\alpha-n+1}}ds,$$

provided that the right side integral is pointwise defined on $(0,+\infty)$.

Lemma 2.4. *Assume that $u\in C(0,1)\cap L^1(0,1)$ with a fractional derivative of order $\alpha>0$ that belongs to $C(0,1)\cap L^1(0,1)$. Then*

$$I_{0^+}^\alpha D_{0^+}^\alpha u(t)=u(t)+c_1t^{\alpha-1}+c_2t^{\alpha-2}+\dots+c_Nt^{\alpha-N},$$

for some $c_i\in\mathbb{R}$, $i=1,2,\dots,N$, where N is the smallest integer grater than or equal to α.

Lemma 2.5. *Assume $u(t)\in C[0,1]$ and $0\le\beta\le\alpha$, then $D_{0^+}^\beta I_{0^+}^\alpha u(t)=I_{0^+}^{\alpha-\beta}u(t)$. And, for all $\alpha\ge0$, $\beta>-1$, we have*

$$D_{0^+}^\alpha t^\beta=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}t^{\beta-\alpha},$$

giving in particular $D_{0^+}^\alpha t^{\alpha-m}=0$, $m=1,2,\dots,N$, where N is the smallest integer grater than or equal to α.

In this article, we take $X=\{u|u,D_{0^+}^{\alpha-2}u,D_{0^+}^{\alpha-1}u,D_{0^+}^\alpha u\in C[0,1]\}$ with the norm $\|u\|_X=\max\{\|u\|_\infty,\ \|D_{0^+}^{\alpha-2}u\|_\infty\ \|D_{0^+}^{\alpha-1}u\|_\infty,\|D_{0^+}^\alpha u\|_\infty\}$, where $\|u\|_\infty=\max_{t\in[0,1]}|u(t)|$, and $Y=C[0,1]$ with the norm $\|y\|_Y=\|y\|_\infty$. By means of the linear functional analysis theory, it is easy to prove that X and Y are Banach spaces. so, we omit it.

Define the operator $M:\operatorname{dom}M\subset X\to Y$ by

$$Mu=D_{0^+}^\beta\varphi_p(D_{0^+}^\alpha u(t)), \tag{2.5}$$

$$\begin{aligned} \operatorname{dom} M = \Big\{ & u \in X : D_{0^+}^{\beta} \varphi_p(D_{0^+}^{\alpha} u) \in Y,\ u(0) = u'(0) = D_{0^+}^{\alpha} u(0) = 0, \\ & D_{0^+}^{\alpha-1} u(1) = \sum_{i=1}^{m} \sigma_i D_{0^+}^{\alpha-1} u(\eta_i) \Big\}. \end{aligned} \tag{2.6}$$

Define the operator $N_\lambda : X \to Y,\ \lambda \in [0,1]$,

$$N_\lambda u(t) = f(t, u(t), D_{0^+}^{\alpha-2} u(t), D_{0^+}^{\alpha-1} u(t), D_{0^+}^{\alpha} u(t)), t \in [0,1],$$

then (1.2) is equivalent to the operator equation $Mu = Nu$, where $N = N_1$.

3. Main result

In this section, we show existence of solutions for BVP (1.2). Let us make some assumptions which will be used throughout this article.

(H1) There exist nonnegative functions $a, b, c, d, e \in Y$ such that

$$|f(t,u,v,w,z))| \le a(t) + b(t)|u|^{p-1} + c(t)|v|^{p-1} + d(t)|w|^{p-1} + e(t)|z|^{p-1},$$

for all $t \in [0,1]$, $(u,v,w,z) \in \mathbb{R}^4$.

(H2) There exists a constant $A > 0$ such that

$$\begin{aligned} & \int_0^1 \varphi_q\Big(\frac{1}{\Gamma(\beta)} \int_0^s (s-\tau)^{\beta-1} f(\tau,u,v,w,z) d\tau\Big) ds \\ & - \sum_{i=1}^{m} \sigma_i \int_0^{\eta_i} \varphi_q(\frac{1}{\Gamma(\beta)} \int_0^s (s-\tau)^{\beta-1} f(\tau,u,v,w,z) d\tau) ds \neq 0, \end{aligned}$$

for all $t \in [0,1]$, $(u,v,w,z) \in \mathbb{R}^4$, $|v| + |w| > A$.

(H3) There exists a constant $B > 0$ such that

$$\begin{aligned} 0 \neq \Lambda := c\frac{1}{\Delta}\Big(& \int_0^1 \varphi_q(\frac{1}{\Gamma(\beta)} \int_0^s (s-\tau)^{\beta-1} f(\tau, c\tau^{\alpha-1}, c\Gamma(\alpha)\tau, c\Gamma(\alpha), 0) d\tau) ds \\ & - \sum_{i=1}^{m} \sigma_i \int_0^{\eta_i} \varphi_q(\frac{1}{\Gamma(\beta)} \int_0^s (s-\tau)^{\beta-1} f(\tau, c\tau^{\alpha-1}, c\Gamma(\alpha)\tau, c\Gamma(\alpha), 0) d\tau) ds\Big), \end{aligned}$$

for all $|c| > B$, $c \in \mathbb{R}$.

Theorem 3.1. *Let $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ be continuous and the condition* (H1)–(H3) *hold. Then BVP* (1.2) *has at least one solution provided that*

$$\frac{1}{\Gamma(\beta+1)}\Big(\frac{D\|b\|_\infty}{\Gamma(\alpha)^{p-1}} + D\|c\|_\infty + D\|d\|_\infty + \|e\|_\infty\Big) < 1. \tag{3.1}$$

Lemma 3.2. *The operator $M : \operatorname{dom} M \cap X \to Y$ is a quasi-linear, and*

$$\ker M = \{u \in X : u(t) = ct^{\alpha-1},\ \forall t \in [0,1],\ c \in \mathbb{R}\} \tag{3.2}$$

$$\begin{aligned} \operatorname{Im} M = \Big\{ y \in Y : & \int_0^1 \varphi_q(\frac{1}{\Gamma(\beta)} \int_0^s (s-\tau)^{\beta-1} y(\tau) d\tau) ds \\ & - \sum_{i=1}^{m} \sigma_i \int_0^{\eta_i} \varphi_q(\frac{1}{\Gamma(\beta)} \int_0^s (s-\tau)^{\beta-1} y(\tau) d\tau) ds = 0 \Big\} \end{aligned} \tag{3.3}$$

Proof. By Lemma 2.4 and $D_{0+}^{\beta}\varphi_p(D_{0+}^{\alpha}u(t)) = 0$, we have
$$D_{0+}^{\alpha}u(t) = \varphi_q(c_0 t^{\beta-1}).$$
From condition $D_{0+}^{\alpha}u(0) = 0$, we obtain that $c_0 = 0$. Thus,
$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3}.$$
Combined with $u(0) = u'(0) = 0$, we have $c_2 = c_3 = 0$, $u(t) = c_1 t^{\alpha-1}$, $c_1 \in \mathbb{R}$. Thus, (3.2) is satisfied.

If $y \in \operatorname{Im} M$, then there exists a function $u \in \operatorname{dom} M$ such that
$$y(t) = D_{0+}^{\beta}\varphi_p(D_{0+}^{\alpha}u(t)).$$
Then by Lemma 2.4 and boundary value condition, we have
$$\begin{gathered} u(t) = I_{0+}^{\alpha}\varphi_q(I_{0+}^{\beta}y(s)) + c_1 t^{\alpha-1}, \\ D_{0+}^{\alpha-1}u(t) = D_{0+}^{\alpha-1}I_{0+}^{\alpha}\varphi_q(I_{0+}^{\beta}y(s)) + c_1\Gamma(\alpha). \end{gathered}$$
Combing this with $\sum_{i=1}^{m}\sigma_i = 1$, we obtain
$$\begin{aligned} &\int_0^1 \varphi_q(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}y(\tau)d\tau)ds \\ &-\sum_{i=1}^{m}\sigma_i\int_0^{\eta_i}\varphi_q(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}y(\tau)d\tau)ds = 0. \end{aligned}$$

On the other hand, suppose $y \in Y$ and satisfies (3.3). Let $u(t) = I_{0+}^{\alpha}\varphi_q(I_{0+}^{\beta}y(t))$, then $u \in \operatorname{dom} M$ and $Mu(t) = D_{0+}^{\beta}\varphi_p(D_{0+}^{\alpha}u(t)) = y(t)$. so $y \in \operatorname{Im} M$ and $\operatorname{Im} M := M(\operatorname{dom} M)$ is a closed subset of Y. Thus, M is a quasi-linear operator. □

Lemma 3.3. *Let $\Omega \subset X$ be an open and bounded set, then N_λ is M-compact in $\overline{\Omega}$.*

Proof. Define the continuous projectors $P : X \to X_1$ and $Q : Y \to Y_1$ by
$$Pu(t) = \frac{1}{\Gamma(\alpha)}D_{0+}^{\alpha-1}u(0)t^{\alpha-1}, \quad t \in [0,1],$$
$$\begin{aligned} Qy(t) = \varphi_p(\frac{1}{\Delta}(&\int_0^1 \varphi_q(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}y(\tau)d\tau)ds \\ &-\sum_{i=1}^{m}\sigma_i\int_0^{\eta_i}\varphi_q(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}y(\tau)d\tau)ds)), t \in [0,1]. \end{aligned}$$
Obviously, $X_1 = \ker M = \operatorname{Im} P$ and $Y_1 = \operatorname{Im} Q$. Thus, we have $\dim Y_1 = \dim X_1 = 1$. For any $y \in Y$, we have
$$\begin{aligned} Q^2y = Q(Qy) = Qy\varphi_p\Big(\frac{1}{\Delta}\Big(&\int_0^1 \varphi_q(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}d\tau)ds \\ &-\sum_{i=1}^{m}\sigma_i\int_0^{\eta_i}\varphi_q\Big(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}d\tau\Big)ds\Big)\Big) = Qy. \end{aligned}$$
Hence, $Q^2 = Q$, Q is a semi-projector. Based on the definition of M and Q, it is easy to see that $\ker Q = \operatorname{Im} M$. Let $\Omega \subset X$ be an open and bounded set with $\theta \in \Omega$. For each $u \in \overline{\Omega}$, we can get $Q[(I-Q)N_\lambda(u)] = 0$. Thus, $(I-Q)N_\lambda(u) \in \operatorname{Im} M = \ker Q$. Taking any $y \in$ ImM and noting $Qy = 0$, we can get $y \in (I-Q)Y$. So (2.1) holds. It is easy to verify (2.2).

Define $R:\overline{\Omega}\times[0,1]\to X_2$ by

$$R(u,\lambda)(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\varphi_q\Big(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}((I-Q)N_\lambda u(\tau))d\tau\Big)ds.$$

By the continuity of f, it is easy to get that $R(u,\lambda)$ is continuous on $\overline{\Omega}\times[0,1]$. Moreover, for all $u\in\overline{\Omega}$, there exists a constant $L>0$ such that $|I^{\beta}_{0^+}(I-Q)N_\lambda u(\tau))|\le L$, so we can easily obtain that $R(\overline{\Omega},\lambda)$, $D^{\alpha-2}_{0^+}R(\overline{\Omega},\lambda)$, $D^{\alpha-1}_{0^+}R(\overline{\Omega},\lambda)$ and $D^{\alpha}_{0^+}R(\overline{\Omega},\lambda)$ are uniformly bounded. By Arzela-Ascoli theorem, we just need to prove that $R:\overline{\Omega}\times[0,1]\to X_2$ is equicontinuous.

For $u\in\overline{\Omega}$, $0<t_1<t_2\le 1$, $2<\alpha\le 3$, $0<\beta\le 1$, $3<\alpha+\beta\le 4$, we have

$$\begin{aligned}
&|R(u,\lambda)(t_2)-R(u,\lambda)(t_1)|\\
&=\frac{1}{\Gamma(\alpha)}|\int_0^{t_2}(t_2-s)^{\alpha-1}\varphi_q(I^{\beta}_{0^+}((I-Q)N_\lambda u(\tau)))ds\\
&\quad-\int_0^{t_1}(t_1-s)^{\alpha-1}\varphi_q(I^{\beta}_{0^+}((I-Q)N_\lambda u(\tau)))ds|\\
&\le\frac{\varphi_q(L)}{\Gamma(\alpha)}(\int_0^{t_1}((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1})ds+\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds)\\
&=\frac{\varphi_q(L)}{\Gamma(\alpha+1)}(t_2^{\alpha}-t_1^{\alpha}),
\end{aligned}$$

$$\begin{aligned}
&|D^{\alpha-2}_{0^+}R(u,\lambda)(t_2)-D^{\alpha-2}_{0^+}R(u,\lambda)(t_1)|\\
&=|\int_0^{t_2}(t-s)\varphi_q(I^{\beta}_{0^+}((I-Q)N_\lambda u(\tau)))ds-\int_0^{t_1}(t-s)\varphi_q(I^{\beta}_{0^+}((I-Q)N_\lambda u(\tau)))ds|\\
&\le\varphi_q(L)(\int_0^{t_1}(t_2-s)-(t_1-s)ds+\int_{t_1}^{t_2}(t_2-s)ds)\\
&=\frac{\varphi_q(L)}{2}(t_2^2-t_1^2)
\end{aligned}$$

and

$$\begin{aligned}
&|D^{\alpha-1}_{0^+}R(u,\lambda)(t_2)-D^{\alpha-1}_{0^+}R(u,\lambda)(t_1)|\\
&=|\int_0^{t_2}\varphi_q(I^{\beta}_{0^+}((I-Q)N_\lambda u(\tau)))ds-\int_0^{t_1}\varphi_q(I^{\beta}_{0^+}((I-Q)N_\lambda u(\tau)))ds|\\
&\le\varphi_q(L)(t_2-t_1).
\end{aligned}$$

Since t^{α} is uniformly continuous on $[0,1]$, it follows that $R(\overline{\Omega},\lambda)$, $D^{\alpha-2}_{0^+}R(\overline{\Omega},\lambda)$ and $D^{\alpha-1}_{0^+}R(\overline{\Omega},\lambda)$ are equicontinuous. Similarly, we can get $I^{\beta}_{0^+}((I-Q)N_\lambda u(\tau))\subset C[0,1]$ is equicontinuous, Considering of $\varphi_q(s)$ is uniformly continuous on $[-L,L]$, we have $D^{\alpha}_{0^+}R(\overline{\Omega},\lambda)=\varphi_q(I^{\beta}_{0^+}((I-Q)N_\lambda(\overline{\Omega})))$ is also equicontinuous. So, we can obtain that $R:\overline{\Omega}\times[0,1]\to X_2$ is compact.

For each $u\in\Sigma_\lambda$, we have $D^{\beta}_{0^+}\varphi_p(D^{\alpha}_{0^+}u(t))=N_\lambda(u(t))\in\operatorname{Im}M$. Thus,

$$\begin{aligned}
R(u,\lambda)(t)&=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\varphi_q(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}((I-Q)N_\lambda u(\tau))d\tau)ds\\
&=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\varphi_q(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}D^{\beta}_{0^+}\varphi_p(D^{\alpha}_{0^+}u(\tau))d\tau)ds,
\end{aligned}$$

which together with $u(0) = u'(0) = D_{0^+}^{\alpha} u(0) = 0$ yields

$$R(u,\lambda)(t) = u(t) - \frac{1}{\Gamma(\alpha)} D_{0^+}^{\alpha-1} u(0) t^{\alpha-1} = (I-P)u(t).$$

It is easy to verify that $R(u,0)(t)$ is the zero operator. So (2.3) holds. Besides, for any $u \in \overline{\Omega}$,

$$\begin{aligned}
&M[Pu + R(u,\lambda)](t)\\
&= M[\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi_q(\frac{1}{\Gamma(\beta)} \int_0^s (s-\tau)^{\beta-1}((I-Q)N_\lambda u(\tau)) d\tau) ds\\
&\quad + \frac{1}{\Gamma(\alpha)} D_{0^+}^{\alpha-1} u(0) t^{\alpha-1}\\
&= (I-Q)N_\lambda u(t),
\end{aligned}$$

which implies (2.4). So N_λ is M-compact in $\overline{\Omega}$. $\square$

Lemma 3.4. *Suppose* (H1), (H2) *hold, Then the set*

$$\Omega_1 = \{u \in \operatorname{dom} M \setminus \ker M : Mu = \lambda Nu,\ \lambda \in (0,1)\}$$

is bounded.

Proof. By lemma 2.4, for each $u \in \operatorname{dom} M$, $D_{0^+}^{\alpha-1} u \in C[0,1]$, we have

$$u(t) = I_{0^+}^{\alpha-1} D_{0^+}^{\alpha-1} u(t) + c_1 t^{\alpha-2} + c_2 t^{\alpha-3}.$$

Combining this with $u(0) = u'(0) = 0$, we get $c_1 = c_2 = 0$. Thus,

$$\begin{aligned}
\|u\|_\infty = \|I_{0^+}^{\alpha-1} D_{0^+}^{\alpha-1} u\|_\infty &\le |\frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} ds| \|D_{0^+}^{\alpha-1} u\|_\infty\\
&\le \frac{1}{\Gamma(\alpha)} \|D_{0^+}^{\alpha-1} u\|_\infty.
\end{aligned}$$

Take any $u \in \Omega_1$, then $Nu \in \operatorname{Im} M = \ker Q$. Thus, $QNu = 0$ for all $t \in [0,1]$. It follows from (H2) that there exists $t_0 \in [0,1]$ such that $|D_{0^+}^{\alpha-2} u(t_0)| + |D_{0^+}^{\alpha-1} u(t_0)| \le A$. Thus

$$\begin{gathered}
D_{0^+}^{\alpha-1} u(t) = D_{0^+}^{\alpha-1} u(t_0) + \int_{t_0}^t D_{0^+}^{\alpha} u(t) dt,\\
D_{0^+}^{\alpha-2} u(t) = D_{0^+}^{\alpha-2} u(t_0) + \int_{t_0}^t D_{0^+}^{\alpha-1} u(t) dt,\\
\|D_{0^+}^{\alpha-1} u\|_\infty \le A + \|D_{0^+}^{\alpha} u\|_\infty,\\
\|D_{0^+}^{\alpha-2} u\|_\infty \le A + \|D_{0^+}^{\alpha-1} u\|_\infty \le 2A + \|D_{0^+}^{\alpha} u\|_\infty,\\
\|u\|_\infty \le \frac{1}{\Gamma(\alpha)} (A + \|D_{0^+}^{\alpha} u\|_\infty).
\end{gathered}$$

Combined with $Mu = \lambda Nu$ and $D_{0^+}^{\alpha} u(0) = 0$, we obtain

$$\varphi_p(D_{0^+}^{\alpha} u(t)) = \lambda I_{0^+}^{\beta} Nu(t).$$

From (H1) and $\lambda \in (0,1)$, we have

$$|\varphi_p(D_{0^+}^{\alpha} u(t))| \le \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} |f(s, u(s), D_{0^+}^{\alpha-2} u(s), D_{0^+}^{\alpha-1} u(s), D_{0^+}^{\alpha} u(s))| ds$$

$$
\begin{aligned}
&\le \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}(a(s)+b(s)|u(s)|^{p-1}+c(s)|D_{0^+}^{\alpha-2}u(s)|^{p-1} \\
&\quad + d(s)|D_{0^+}^{\alpha-1}u(s)|^{p-1}+e(s)|D_{0+}^{\alpha}u(s)|^{p-1})ds \\
&\le \frac{1}{\Gamma(\beta+1)}(\|a\|_\infty+\|b\|_\infty\|u\|_\infty^{p-1}+\|c\|_\infty\|D_{0^+}^{\alpha-2}u\|_\infty^{p-1} \\
&\quad + \|d\|_\infty\|D_{0^+}^{\alpha-1}u\|_\infty^{p-1}+\|e\|_\infty\|D_{0+}^{\alpha}u\|_\infty^{p-1}), \quad \forall t\in[0,1],
\end{aligned}
$$

which together with $|\varphi_p(D_{0+}^{\alpha}u(t))| = |D_{0+}^{\alpha}u(t)|^{p-1}$, and the basic inequality $(|a|+|b|)^p \le C_p(|a|^p+|b|^p)$, where $C_p = 2^{p-1}$ when $p>1$ and where $C_p = 1$ when $0<p\le 1$, $a,b\in\mathbb{R}$ (see [13]). We can get

$$
\begin{aligned}
\|D_{0+}^{\alpha}u\|_\infty^{p-1} &\le \frac{1}{\Gamma(\beta+1)}(\|a\|_\infty+\|b\|_\infty\|u\|_\infty^{p-1}+\|c\|_\infty\|D_{0^+}^{\alpha-2}u\|_\infty^{p-1} \\
&\quad + \|d\|_\infty\|D_{0^+}^{\alpha-1}u\|_\infty^{p-1}+\|e\|_\infty\|D_{0+}^{\alpha}u\|_\infty^{p-1}) \\
&\le \frac{1}{\Gamma(\beta+1)}(\|a\|_\infty+\|b\|_\infty D(\frac{1}{\Gamma(\alpha)^{p-1}}\|D_{0+}^{\alpha}u\|_\infty^{p-1}+\frac{A^{p-1}}{\Gamma(\alpha)^{p-1}}) \\
&\quad + \|c\|_\infty D(2^{p-1}A^{p-1}+\|D_{0+}^{\alpha}u\|_\infty^{p-1})+\|d\|_\infty D(A^{p-1}+\|D_{0+}^{\alpha}u\|_\infty^{p-1}) \\
&\quad + \|e\|_\infty\|D_{0+}^{\alpha}u\|_\infty^{p-1}).
\end{aligned}
$$

where $D = \max\{1, 2^{p-2}\}$. From (3.1), we can see that there exists a constant $M_1>0$ such that

$$
\begin{gathered}
\|D_{0+}^{\alpha}u\|_\infty \le M_1, \quad \|D_{0^+}^{\alpha-1}u\|_\infty \le A+M_1 := M_2, \\
\|D_{0^+}^{\alpha-2}u\|_\infty \le 2A+M_1 := M_3, \quad \|u\|_\infty \le \frac{1}{\Gamma(\alpha)}M_1+\frac{A}{\Gamma(\alpha)} := M_4.
\end{gathered}
$$

Thus

$$
\begin{aligned}
\|u\|_X &= \max\left\{\|u\|_\infty, \|D_{0^+}^{\alpha-2}u\|_\infty, \|D_{0^+}^{\alpha-1}u\|_\infty, \|D_{0+}^{\alpha}u\|_\infty\right\} \\
&\le \max\{M_1,M_2,M_3,M_4\} := M.
\end{aligned}
$$

Therefore, Ω_1 is bounded. □

Lemma 3.5. *Suppose* (H2) *holds, then the set* $\Omega_2 = \{u\in\ker M : Nu\in\operatorname{Im} M\}$ *is bounded.*

Proof. For each $u\in\Omega_2$, we can have that $u(t) = ct^{\alpha-1}$ for all $c\in\mathbb{R}$ and $QNu = 0$. It follow from (H2) that there exists a $t_0\in[0,1]$ such that $|D_{0^+}^{\alpha-1}u(t_0)| + |D_{0^+}^{\alpha-2}u(t_0)| \le A$, which implies $|c| \le \frac{A}{\Gamma(\alpha)(1+t_0)}$. Therefore, Ω_2 is bounded. □

Lemma 3.6. *Suppose* (H3) *holds, then the set*

$$
\Omega_3 = \{u\in\ker M : (-1)^m\lambda J^{-1}u+(1-\lambda)QNu = 0,\ \lambda\in[0,1]\}
$$

is bounded, where $m=1$ *when* $\Lambda<0$ *and* $m=2$ *when* $\Lambda>0$.

Proof. Case 1, suppose $\Lambda<0$, for each $u\in\Omega_3$, we can get that $u(t) = ct^{\alpha-1}$ for all $c\in\mathbb{R}$. We define the isomorphism $J : \operatorname{Im} Q\to\ker M$ by $J(c) = ct^{\alpha-1}, c\in R, t\in$

$[0,1]$. So, we have

$$\begin{aligned}\lambda c = (1-\lambda)\varphi_p\Big(\frac{1}{\Delta}\Big(\int_0^1 \varphi_q(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} f(\tau, c\tau^{\alpha-1}, c\Gamma(\alpha)\tau, c\Gamma(\alpha), 0)d\tau)ds \\ -\sum_{i=1}^m \sigma_i \int_0^{\eta_i} \varphi_q\Big(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} f(\tau, c\tau^{\alpha-1}, c\Gamma(\alpha)\tau, c\Gamma(\alpha), 0)d\tau\Big)ds\Big)\Big).\end{aligned} \tag{3.4}$$

If $\lambda = 0$, then $|c| \le B$ because of the first part of (H3). If $\lambda \in (0,1]$, we can also obtain $|c| \le B$. Otherwise, if $|c| > B$, in view of the first part of (H3), one has

$$\begin{aligned}c(1-\lambda)\varphi_p\Big(\frac{1}{\Delta}\Big(\int_0^1 \varphi_q(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} f(\tau, c\tau^{\alpha-1}, c\Gamma(\alpha)\tau, c\Gamma(\alpha), 0)d\tau)ds \\ -\sum_{i=1}^m \sigma_i \int_0^{\eta_i} \varphi_q\Big(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} f(\tau, c\tau^{\alpha-1}, c\Gamma(\alpha)\tau, c\Gamma(\alpha), 0)d\tau\Big)ds\Big)\Big) \le 0.\end{aligned} \tag{3.5}$$

On the other hand, $\lambda c^2 > 0$ which contradicts to (3.4). Therefore, Ω_3 is bounded.

Case 2, suppose $\Lambda > 0$, it is similar to case 1 to proof Ω_3 is bounded. So, we omit it. □

Proof of Theorem 3.1. Assume that Ω is a bounded open set of X with $\cup_{i=1}^3 \overline{\Omega_i} \subset \Omega$. By Lemma 3.3, we obtain that N is M-compact on $\overline{\Omega}$. Then by Lemmas 3.4 and 3.5, we have

(i) $Mx \neq N_\lambda x$ for each $(u,\lambda) \in (\operatorname{dom} M \setminus \ker M) \times (0,1)$,
(ii) $QNu \neq 0$, for all $u \in \partial\Omega \cap \ker M$.

Thus, we need to prove that (iii) of Lemma 2.1 is true, Let I be the identity operator in the Banach space X, and $H(u,\lambda) = (-1)^m \lambda J^{-1}(u) + (1-\lambda)QN(u)$. According to Lemma 3.6 we know that for each $u \in \partial\Omega \cap \ker M$, $H(u,\lambda) \neq 0$. Thus, by the homotopic property of degree, we have

$$\begin{aligned}\deg(JQN|_{\ker M}, \Omega \cap \ker M, 0) &= \deg(H(\cdot,0), \Omega \cap \ker M, 0) \\ &= \deg(H(\cdot,1), \Omega \cap \ker M, 0) \\ &= \deg(\pm I, \Omega \cap \ker M, 0) \neq 0.\end{aligned}$$

which means (iii) of Lemma 2.1 is satisfied. Consequently, by Lemma 2.1, the equation $Mu = Nu$ has at least one solution in $\operatorname{dom} M \cap \Omega$. Namely, BVP (1.2) have at least one solution in the space X. □

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