Electronic Journal of Differential Equations, Vol. 2014 (2014), No. 61, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

SOLUTIONS TO THIRD-ORDER MULTI-POINT BOUNDARY-VALUE PROBLEMS AT RESONANCE WITH THREE DIMENSIONAL KERNELS

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Abstract. In this article, we consider the boundary-value problem

$$x'''(t) = f(t, x(t), x'(t), x''(t)), \quad t \in (0,1),$$

$$x''(0) = \sum_{i=1}^{m} \alpha_i x''(\xi_i), \quad x'(0) = \sum_{k=1}^{l} \gamma_k x'(\sigma_k), \quad x(1) = \sum_{j=1}^{n} \beta_j x(\eta_j),$$

where $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is a Carathéodory function, and the kernel to the linear operator has dimension three. Under some resonance conditions, by using the coincidence degree theorem, we show the existence of solutions. An example is given to illustrate our results.

1. Introduction

This concerns the third-order nonlinear differential equation

$$x'''(t) = f(t, x(t), x'(t), x''(t)), \quad t \in (0,1), \tag{1.1}$$

with the boundary conditions

$$x''(0) = \sum_{i=1}^{m} \alpha_i x''(\xi_i), \quad x'(0) = \sum_{k=1}^{l} \gamma_k x'(\sigma_k), \quad x(1) = \sum_{j=1}^{n} \beta_j x(\eta_j), \tag{1.2}$$

where $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is a Carathéodory function, $0 < \xi_1 < \cdots < \xi_m < 1$, $0 < \sigma_1 < \cdots < \sigma_l < 1$, $0 < \eta_1 < \cdots < \eta_n < 1$, $\alpha_i, \gamma_k, \beta_j \in \mathbb{R}$ ($i = 1, \ldots, m$; $k = 1, \ldots, l$; $j = 1, \ldots, n$) and $\sigma_1 > \{\xi_1, \ldots, \xi_m\}$.

The existence of solutions for multi-point boundary-value problems at resonance case has been extensively studied by many authors [1, 2, 3, 4, 5, 6, 7, 8, 10]. When the linear equation $Lx = x''' = 0$ with the boundary conditions (1.2) has a non-trivial solution, i.e. $\dim\ker L \ge 1$. we say that boundary value problem (BVP for short) (1.1) and (1.2) is a resonance problem.

The case of $\dim\ker L = 1$ has been discussed by many authors [1, 2, 4, 6]. For the case of $\dim\ker L = 2$, there are some results in [3, 7, 8, 10]., Lin, Du and Meng [7]

2000 *Mathematics Subject Classification.* 34B15.
Key words and phrases. Multi-point boundary-value problem; coincidence degree theory; resonance.

Submitted October 13, 2013. Published March 5, 2014.

discussed the third-order multi-point boundary value-problem with $\dim\ker L = 2$,

$$x'''(t) = f(t, x(t), x'(t), x''(t)), \quad t \in (0,1), \tag{1.3}$$

$$x''(0) = \sum_{i=1}^{m} \alpha_i x''(\xi_i), \quad x'(0) = 0, \quad x(1) = \sum_{j=1}^{n} \beta_j x(\eta_j). \tag{1.4}$$

Zhao, Liang and Ren [11] studied the nonlinear third-order boundary-value problems with $\dim\ker L = 3$,

$$x'''(t) = f(t, x(t), x'(t), x''(t)) + e(t), \quad t \in (0,1),$$

$$x'(0) = \sum_{j=1}^{n} \alpha_j x'(\xi_j), \quad x''(0) = x''(\eta), \quad x''(1) = x''(\zeta).$$

In [11], the results are obtained under the assumption that

$$M = \begin{vmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{vmatrix} \neq 0. \tag{1.5}$$

This condition is difficult to verify and looks superfluous.

Inspired by the above results, in this paper, we study (1.1)-(1.2) at resonance and show that the assumption

$$\begin{aligned} &\Lambda(p,q,r) \\ &= \begin{vmatrix} (p+1)(p+2)\sum_{i=1}^{m}\alpha_i\xi_i^{p} & (p+2)\sum_{k=1}^{l}\gamma_k\sigma_k^{p+1} & 2(1-\sum_{j=1}^{n}\beta_j\eta_j^{p+2}) \\ (q+1)(q+2)\sum_{i=1}^{m}\alpha_i\xi_i^{q} & (q+2)\sum_{k=1}^{l}\gamma_k\sigma_k^{q+1} & 2(1-\sum_{j=1}^{n}\beta_j\eta_j^{q+2}) \\ (r+1)(r+2)\sum_{i=1}^{m}\alpha_i\xi_i^{r} & (r+2)\sum_{k=1}^{l}\gamma_k\sigma_k^{r+1} & 2(1-\sum_{j=1}^{n}\beta_j\eta_j^{r+2}) \end{vmatrix} \neq 0 \end{aligned}$$

can replace (1.5) from [5, 11], by using Lemma 3.1 below. If there exists $\sigma_k (k = 1, 2, \dots, l)$ satisfying $x'(0) = \sum_{k=1}^{l} \gamma_k x'(\sigma_k) = 0$, then BVP (1.3)-(1.4) is a special case of BVP (1.1)-(1.2).

The remaining part of this article is organized as follows: In section 2, we will state some definitions and lemmas which would be useful in the proving of main results of this paper. In section 3, by applying Mawhin coincidence degree theory, we obtain some sufficient conditions which guarantee the existence of solution for BVP (1.1)-(1.2) at resonance case. In the last section, an example is given to illustrate our results.

2. Preliminaries

Now, some notation and an abstract existence result [9] are introduced. Let Y, Z be real Banach spaces and let $L : \operatorname{dom} L \subset Y \to Z$ be a linear operator which is a Fredholm map of index zero and $P : Y \to Y$, $Q : Z \to Z$ be continuous projectors such that $\operatorname{Im} P = \ker L$, $\ker Q = \operatorname{Im} L$ and $Y = \ker L \oplus \ker P$, $Z = \operatorname{Im} L \oplus \operatorname{Im} Q$. It follows that $L|_{\operatorname{dom} L \cap \ker P} : \operatorname{dom} L \cap \ker P \to \operatorname{Im} L$ is invertible, we denote the inverse of that map by K_P. Let Ω be an open bounded subset of Y such that $\operatorname{dom} L \cap \Omega \neq \emptyset$, the map $N : Y \to Z$ is said to be L-compact on $\bar\Omega$ if the map $QN(\bar\Omega)$ is bounded and $K_P(I-Q)N : \bar\Omega \to Y$ is compact.

Lemma 2.1 ([9, Theorem IV])**.** *Let L be a Fredholm map of index zero and let N be L-compact on $\bar\Omega$. Assume that the following conditions are satisfied:*

(i) *$Lx \neq \lambda Nx$ for every $(x, \lambda) \in ((\operatorname{dom} L \setminus \ker L) \cap \partial\Omega) \times [0, 1]$;*

(ii) $Nx \notin \operatorname{Im} L$ *for every* $x \in \ker L \cap \partial\Omega$;
(iii) $\deg(QN|_{\ker L}, \Omega \cap \ker L, 0) \neq 0$, *where* $Q : Z \to Z$ *is a continuous projector as above with* $\operatorname{Im} L = \ker Q$.

Then the abstract equation $Lx = Nx$ *has at least one solution in* $\operatorname{dom} L \cap \bar{\Omega}$.

We use the classical spaces $C[0,1]$, $C^1[0,1]$, $C^2[0,1]$ and $L^1[0,1]$. For x in $C^2[0,1]$, we use the norms $\|x\|_\infty = \max_{t\in(0,1)}\{|x(t)|\}$ and

$$\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty, \|x''\|_\infty\}.$$

For $L^1[0,1]$ we denote the norm by $\|\cdot\|_1$. We define the Sobolev space

$$\begin{aligned} W^{3,1}(0,1) = \big\{x : [0,1] \to R : x, x', x'' &\text{ are absolutely continuous on } [0,1] \\ &\text{with } x''' \in L^1[0,1]\big\}. \end{aligned}$$

Let $Y = C^2[0,1]$, $Z = L^1[0,1]$, and define the linear operator $L : \operatorname{dom} L \subset Y \to Z$ as $Lx = x'''$, $x \in \operatorname{dom} L$, where

$$\operatorname{dom} L = \{x \in W^{3,1}(0,1) : x \text{ satisfies boundary conditions (1.2)}\}.$$

We define $N : Y \to Z$ as

$$Nx = f(t, x(t), x'(t), x''(t)), \quad t \in (0,1),$$

then (1.1)-(1.2) can be written as $Lx = Nx$.

The resonance conditions of (1.1)-(1.2) are as follows

$$\begin{gathered} \sum_{i=1}^m \alpha_i = 1, \quad \sum_{k=1}^l \gamma_k = 1, \quad \sum_{j=1}^n \beta_j = 1, \\ \sum_{j=1}^n \beta_j\eta_j = 1, \quad \sum_{j=1}^n \beta_j\eta_j^2 = 1, \quad \sum_{k=1}^l \gamma_k\sigma_k = 0. \end{gathered} \tag{2.1}$$

Define the following symbols:

$$\begin{aligned} &\Lambda(p,q,r) \\ &= \begin{vmatrix} (p+1)(p+2)\sum_{i=1}^m \alpha_i\xi_i^p & (p+2)\sum_{k=1}^l \gamma_k\sigma_k^{p+1} & 2(1-\sum_{j=1}^n \beta_j\eta_j^{p+2}) \\ (q+1)(q+2)\sum_{i=1}^m \alpha_i\xi_i^q & (q+2)\sum_{k=1}^l \gamma_k\sigma_k^{q+1} & 2(1-\sum_{j=1}^n \beta_j\eta_j^{q+2}) \\ (r+1)(r+2)\sum_{i=1}^m \alpha_i\xi_i^r & (r+2)\sum_{k=1}^l \gamma_k\sigma_k^{r+1} & 2(1-\sum_{j=1}^n \beta_j\eta_j^{r+2}) \end{vmatrix}, \end{aligned}$$

$$M_{11} = \begin{vmatrix} (q+2)\sum_{k=1}^l \gamma_k\sigma_k^{q+1} & 2(1-\sum_{j=1}^n \beta_j\eta_j^{q+2}) \\ (r+2)\sum_{k=1}^l \gamma_k\sigma_k^{r+1} & 2(1-\sum_{j=1}^n \beta_j\eta_j^{r+2}) \end{vmatrix},$$

$$M_{12} = -\begin{vmatrix} (q+1)(q+2)\sum_{i=1}^m \alpha_i\xi_i^q & 2(1-\sum_{j=1}^n \beta_j\eta_j^{q+2}) \\ (r+1)(r+2)\sum_{i=1}^m \alpha_i\xi_i^r & 2(1-\sum_{j=1}^n \beta_j\eta_j^{r+2}) \end{vmatrix},$$

$$M_{13} = \begin{vmatrix} (q+1)(q+2)\sum_{i=1}^m \alpha_i\xi_i^q & (q+2)\sum_{k=1}^l \gamma_k\sigma_k^{q+1} \\ (r+1)(r+2)\sum_{i=1}^m \alpha_i\xi_i^r & (r+2)\sum_{k=1}^l \gamma_k\sigma_k^{r+1} \end{vmatrix},$$

$$M_{21} = -\begin{vmatrix} (p+2)\sum_{k=1}^l \gamma_k\sigma_k^{p+1} & 2(1-\sum_{j=1}^n \beta_j\eta_j^{p+2}) \\ (r+2)\sum_{k=1}^l \gamma_k\sigma_k^{r+1} & 2(1-\sum_{j=1}^n \beta_j\eta_j^{r+2}) \end{vmatrix},$$

$$M_{22} = \begin{vmatrix} (p+1)(p+2)\sum_{i=1}^m \alpha_i\xi_i^p & 2(1-\sum_{j=1}^n \beta_j\eta_j^{p+2}) \\ (r+1)(r+2)\sum_{i=1}^m \alpha_i\xi_i^r & 2(1-\sum_{j=1}^n \beta_j\eta_j^{r+2}) \end{vmatrix},$$

$$M_{23} = -\begin{vmatrix} (p+1)(p+2)\sum_{i=1}^m \alpha_i\xi_i^p & (p+2)\sum_{k=1}^l \gamma_k\sigma_k^{p+1} \\ (r+1)(r+2)\sum_{i=1}^m \alpha_i\xi_i^r & (r+2)\sum_{k=1}^l \gamma_k\sigma_k^{r+1} \end{vmatrix},$$

$$M_{31} = \begin{vmatrix} (p+2)\sum_{k=1}^l \gamma_k\sigma_k^{p+1} & 2(1-\sum_{j=1}^n \beta_j\eta_j^{p+2}) \\ (q+2)\sum_{k=1}^l \gamma_k\sigma_k^{q+1} & 2(1-\sum_{j=1}^n \beta_j\eta_j^{q+2}) \end{vmatrix},$$

$$M_{32} = -\begin{vmatrix} (p+1)(p+2)\sum_{i=1}^m \alpha_i\xi_i^p & 2(1-\sum_{j=1}^n \beta_j\eta_j^{p+2}) \\ (q+1)(q+2)\sum_{i=1}^m \alpha_i\xi_i^q & 2(1-\sum_{j=1}^n \beta_j\eta_j^{q+2}) \end{vmatrix},$$

$$M_{33} = \begin{vmatrix} (p+1)(p+2)\sum_{i=1}^m \alpha_i\xi_i^p & (p+2)\sum_{k=1}^l \gamma_k\sigma_k^{p+1} \\ (q+1)(q+2)\sum_{i=1}^m \alpha_i\xi_i^q & (q+2)\sum_{k=1}^l \gamma_k\sigma_k^{q+1} \end{vmatrix}$$

3. Main results

Lemma 3.1. *Assume condition* (2.1) *holds, then there exist* $p \in \{1, 2, \dots, n\}$, $q \in Z^+$, $q \geq p+1$ *and* $r \in Z^+$ *large enough number, such that* $\Lambda(p, q, r) \neq 0$.

Proof. Clearly there exists $p \in \{1, 2, \dots, n\}$ such that $\sum_{i=1}^m \alpha_i\xi_i^{p+2} \neq 0$. Otherwise, we have $\sum_{i=1}^m \alpha_i\xi_i^{p+2} = 0$ and $p \in \{1, 2, \dots, n\}$, then

$$\sum_{i=1}^m \alpha_i\xi_i^j(1-\xi_i) = 0, \quad j = 1, 2, \dots, n-1;$$

i.e.,

$$\begin{pmatrix} \xi_1(1-\xi_1) & \dots & \xi_m(1-\xi_m) \\ \vdots & \ddots & \vdots \\ \xi_1^{n-1}(1-\xi_1) & \dots & \xi_m^{n-1}(1-\xi_m) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Because

$$\begin{vmatrix} \xi_1(1-\xi_1) & \dots & \xi_m(1-\xi_m) \\ \vdots & \ddots & \vdots \\ \xi_1^{n-1}(1-\xi_1) & \dots & \xi_m^{n-1}(1-\xi_m) \end{vmatrix} = \prod_{i=1}^m \xi_i(1-\xi_i) \prod_{1\leq i\leq j\leq m-1} (\xi_j - \xi_i) \neq 0.$$

So we have $\alpha_i = 0$, $i \in \{1, 2, \dots, m\}$. Which is a contradiction to $\sum_{i=1}^m \alpha_i = 1$ of condition (2.1).

Similarly, there exists $q \in \{1, 2, \dots, n+1\}$, such that $\sum_{i=1}^m \alpha_i\xi_i^q \neq 0$. And for each $s \in Z, s \geq 0$, there exists $k_s \in \{sn+1, \dots, (s+1)n\}$ such that

$$\sum_{k=1}^l \gamma_k\sigma_k^{k_s+1} \neq 0, \quad \sum_{i=1}^m \alpha_i\xi_i^{k_s} \neq 0.$$

Set

$$S = \Big\{k_s \in Z : \sum_{k=1}^l \gamma_k\sigma_k^{q+1} = \frac{(q+1)\sum_{i=1}^m \alpha_i\xi_i^q \sum_{k=1}^l \gamma_k\sigma_k^{k_s+1}}{(k_s+1)\sum_{i=1}^m \alpha_i\xi_i^{k_s}}\Big\},$$

then S is a finite set. If else, there exists a monotone sequence $\{k_{s_t}\}$, $t = 1, 2, \dots$, $k_{s_t} \leq k_{s_{t+1}}$, such that

$$\sum_{k=1}^l \gamma_k\sigma_k^{q+1} = \frac{(q+1)\sum_{i=1}^m \alpha_i\xi_i^q \sum_{k=1}^l \gamma_k\sigma_k^{k_{s_t}+1}}{(k_{s_t}+1)\sum_{i=1}^m \alpha_i\xi_i^{k_{s_t}}}.$$

For $\sigma_1 > \{\xi_1, \dots, \xi_m\}$,

$$\sum_{k=1}^{l} \gamma_k \sigma_k^{q+1} = \lim_{k_{s_t}\to\infty} \frac{(q+1)\sum_{i=1}^m \alpha_i \xi_i^q \sum_{k=1}^l \gamma_k \sigma_k^{k_{s_t}+1}}{(k_{s_t}+1)\sum_{i=1}^m \alpha_i \xi_i^{k_{s_t}}} = \infty,$$

which is a contradiction. Then

$$\begin{vmatrix} (p+1)(p+2)\sum_{i=1}^m \alpha_i \xi_i^p & (p+2)\sum_{k=1}^l \gamma_k \sigma_k^{p+1} \\ (q+1)(q+2)\sum_{i=1}^m \alpha_i \xi_i^q & (q+2)\sum_{k=1}^l \gamma_k \sigma_k^{q+1} \end{vmatrix} \neq 0$$

Thus, we have

$$\begin{aligned} &\lim_{r\to\infty} \Lambda(p,q,r) \\ &= \begin{vmatrix} (p+1)(p+2)\sum_{i=1}^m \alpha_i \xi_i^p & (p+2)\sum_{k=1}^l \gamma_k \sigma_k^{p+1} & 2(1-\sum_{j=1}^n \beta_j \eta_j^{p+2}) \\ (q+1)(q+2)\sum_{i=1}^m \alpha_i \xi_i^q & (q+2)\sum_{k=1}^l \gamma_k \sigma_k^{q+1} & 2(1-\sum_{j=1}^n \beta_j \eta_j^{q+2}) \\ 0 & 0 & 2 \end{vmatrix} \\ &= 2\begin{vmatrix} (p+1)(p+2)\sum_{i=1}^m \alpha_i \xi_i^p & (p+2)\sum_{k=1}^l \gamma_k \sigma_k^{p+1} \\ (q+1)(q+2)\sum_{i=1}^m \alpha_i \xi_i^q & (q+2)\sum_{k=1}^l \gamma_k \sigma_k^{q+1} \end{vmatrix} \neq 0. \end{aligned}$$

So there exist $p \in \{1, 2, \dots, n\}$, $q \in Z^+$, $q \geq p+1$ and $r \in Z^+$ large enough number, such that $\Lambda(p,q,r) \neq 0$. $\square$

Lemma 3.2. *If condition* (2.1) *holds, then* $L : \operatorname{dom} L \subset Y \to Z$ *is a Fredholm operator of index zero. Furthermore, the continuous projector operator* $Q : Z \to Z$ *can be defined by*

$$Q(y) = (T_1 y(t))t^{p-1} + (T_2 y(t))t^{q-1} + (T_3 y(t))t^{r-1},$$

where p, q, r *are given by Lemma 3.1 and*

$$\begin{gathered} T_1 y = \frac{p(p+1)(p+2)}{\Lambda(p,q,r)}[M_{11}Q_1 y + M_{12}Q_2 y + M_{13}Q_3 y], \\ T_2 y = \frac{q(q+1)(q+2)}{\Lambda(p,q,r)}[M_{21}Q_1 y + M_{22}Q_2 y + M_{23}Q_3 y], \\ T_3 y = \frac{r(r+1)(r+2)}{\Lambda(p,q,r)}[M_{31}Q_1 y + M_{32}Q_2 y + M_{33}Q_3 y], \\ Q_1 y = \sum_{i=1}^m \int_0^{\xi_i} y(s)ds, Q_2 y = \sum_{k=1}^l \gamma_k \int_0^{\sigma_k} (\sigma_k - s)y(s)ds, \\ Q_3 y = \int_0^1 (1-s)^2 y(s)ds - \sum_{j=1}^n \beta_j \int_0^{\eta_j} (\eta_j - s)^2 y(s)ds. \end{gathered}$$

The linear operator $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$ *can be written as*

$$K_P y(t) = \frac{1}{2}\int_0^t (t-s)^2 y(s)ds, \; y \in \operatorname{Im} L.$$

Furthermore $\|K_P y\| \leq \|y\|_1$, $y \in \operatorname{Im} L$.

Proof. It is clear that

$$\ker L = \{x \in \operatorname{dom} L : x = a + bt + ct^2, \; a, b, c \in \mathbb{R}\}.$$

Now we show that

$$\operatorname{Im} L = \{y \in Z : Q_1 y = Q_2 y = Q_3 y = 0\}. \tag{3.1}$$

Since the problem

$$x''' = y \tag{3.2}$$

has a solution $x(t)$, such that boundary conditions (1.2) hold; i.e.,

$$x''(0) = \sum_{i=1}^{m} \alpha_i x''(\xi_i), \quad x'(0) = \sum_{k=1}^{l} \gamma_k x'(\sigma_k), \quad x(1) = \sum_{j=1}^{n} \beta_j x(\eta_j),$$

if and only if

$$Q_1 y = Q_2 y = Q_3 y = 0. \tag{3.3}$$

In fact, if (3.2) has a solution $x(t)$ that satisfies the boundary conditions (1.2), then we have

$$x(t) = x(0) + x'(0)t + \frac{1}{2}x''(0)t^2 + \frac{1}{2}\int_0^t (t-s)^2 y(s)ds.$$

According to condition (2.1), we have $Q_1 y = Q_2 y = Q_3 y = 0$.

On the other hand, we let

$$x(t) = a + bt + ct^2 + \frac{1}{2}\int_0^t (t-s)^2 y(s)ds,$$

where a, b, c are arbitrary constants. If (3.3) holds, then $x(t)$ is a solution of (3.2) and (1.2). Hence (3.1) holds. By Lemma 3.1, there exists $p \in \{1, 2, \dots, n\}$, $q \in Z^+$, $q \geq p+1$ and $r \in Z^+$ is a large enough number, such that $\Lambda(p,q,r) \neq 0$. Set

$$\begin{aligned}
T_1 y &= \frac{p(p+1)(p+2)}{\Lambda(p,q,r)}[M_{11}Q_1 y + M_{12}Q_2 y + M_{13}Q_3 y], \\
T_2 y &= \frac{q(q+1)(q+2)}{\Lambda(p,q,r)}[M_{21}Q_1 y + M_{22}Q_2 y + M_{23}Q_3 y], \\
T_3 y &= \frac{r(r+1)(r+2)}{\Lambda(p,q,r)}[M_{31}Q_1 y + M_{32}Q_2 y + M_{33}Q_3 y].
\end{aligned}$$

And we define

$$Q(y) = (T_1 y(t))t^{p-1} + (T_2 y(t))t^{q-1} + (T_3 y(t))t^{r-1},$$

then $\dim \operatorname{Im} Q = 3$. So we have

$$\begin{aligned}
&T_1((T_1 y)t^{p-1}) \\
&= \frac{p(p+1)(p+2)}{\Lambda(p,q,r)}[M_{11}Q_1((T_1 y)t^{p-1}) + M_{12}Q_2((T_1 y)t^{p-1}) + M_{13}Q_3((T_1 y)t^{p-1})] \\
&= \frac{p(p+1)(p+2)}{\Lambda(p,q,r)}[M_{11}Q_1(t^{p-1}) + M_{12}Q_2(t^{p-1}) + M_{13}Q_3(t^{p-1})](T_1 y) \\
&= \frac{1}{\Lambda(p,q,r)}[M_{11}(p+1)(p+2)\sum_{i=1}^{m}\alpha_i \xi_i^p + M_{12}(p+2)\sum_{k=1}^{l}\gamma_k \sigma_k^{p+1} \\
&\quad + 2M_{13}(1 - \sum_{j=1}^{n}\beta_j \eta_j^{p+2})](T_1 y) \\
&= T_1 y,
\end{aligned}$$

$$
\begin{aligned}
&T_1((T_2y)t^{q-1}) \\
&= \frac{q(q+1)(q+2)}{\Lambda(p,q,r)}[M_{11}Q_1((T_2y)t^{q-1}) + M_{12}Q_2((T_2y)t^{q-1}) + M_{13}Q_3((T_2y)t^{q-1})] \\
&= \frac{q(q+1)(q+2)}{\Lambda(p,q,r)}[M_{11}Q_1(t^{q-1}) + M_{12}Q_2(t^{q-1}) + M_{13}Q_3(t^{q-1})](T_2y) \\
&= \frac{1}{\Lambda(p,q,r)}[M_{11}(q+1)(q+2)\sum_{i=1}^{m}\alpha_i\xi_i^q + M_{12}(q+2)\sum_{k=1}^{l}\gamma_k\sigma_k^{q+1} \\
&\quad + 2M_{13}(1-\sum_{j=1}^{n}\beta_j\eta_j^{q+2})](T_2y) = 0,
\end{aligned}
$$

$$
\begin{aligned}
&T_1((T_3y)t^{r-1}) \\
&= \frac{r(r+1)(r+2)}{\Lambda(p,q,r)}[M_{11}Q_1((T_3y)t^{r-1}) + M_{12}Q_2((T_3y)t^{r-1}) + M_{13}Q_3((T_3y)t^{r-1})] \\
&= \frac{r(r+1)(r+2)}{\Lambda(p,q,r)}[M_{11}Q_1(t^{r-1}) + M_{12}Q_2(t^{r-1}) + M_{13}Q_3(t^{r-1})](T_3y) \\
&= \frac{1}{\Lambda(p,q,r)}[M_{11}(r+1)(r+2)\sum_{i=1}^{m}\alpha_i\xi_i^r + M_{12}(r+2)\sum_{k=1}^{l}\gamma_k\sigma_k^{r+1} \\
&\quad + 2M_{13}(1-\sum_{j=1}^{n}\beta_j\eta_j^{r+2})](T_3y) = 0.
\end{aligned}
$$

Similarly, we obtain

$$
\begin{gathered}
T_2((T_1y)t^{p-1}) = 0, \quad T_2((T_2y)t^{q-1}) = T_2y, \quad T_2((T_3y)t^{r-1}) = 0, \\
T_3((T_1y)t^{p-1}) = 0, \quad T_3((T_2y)t^{q-1}) = 0, \quad T_3((T_3y)t^{r-1}) = T_3y.
\end{gathered}
$$

So we have

$$
\begin{aligned}
Q^2y &= Q(T_1y(t))t^{p-1} + (T_2y(t))t^{q-1} + (T_3y(t))t^{r-1}) \\
&= T_1((T_1y(t))t^{p-1} + (T_2y(t))t^{q-1} + (T_3y(t))t^{r-1})t^{p-1} \\
&\quad + T_2((T_1y(t))t^{p-1} + (T_2y(t))t^{q-1} + (T_3y(t))t^{r-1})t^{q-1} \\
&\quad + T_3((T_1y(t))t^{p-1} + (T_2y(t))t^{q-1} + (T_3y(t))t^{r-1})t^{r-1} \\
&= (T_1y(t))t^{p-1} + (T_2y(t))t^{q-1} + (T_3y(t))t^{r-1} \\
&= Qy.
\end{aligned}
$$

Thus, Q is a well defined operator.

Now we need to show that $\ker Q = \operatorname{Im} L$. If $y \in \ker Q$, then $Qy = 0$. By the definition of Qy, we have

$$
\begin{gathered}
M_{11}Q_1y + M_{12}Q_2y + M_{13}Q_3y = 0, \\
M_{21}Q_1y + M_{22}Q_2y + M_{23}Q_3y = 0, \\
M_{31}Q_1y + M_{32}Q_2y + M_{33}Q_3y = 0.
\end{gathered}
$$

Because of

$$
\begin{vmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{vmatrix} = \Lambda^2(p,q,r) \neq 0,
$$

we have $Q_1y = Q_2y = Q_3y = 0$, i.e. $y \in ImL$.

If $y \in \operatorname{Im} L$, we have $Q_1y = Q_2y = Q_3y = 0$. From the definition of Qy, it is obvious that $Qy = 0$, thus $y \in KerQ$. Hence, $\ker Q = \operatorname{Im} L$.

For $y \in Z$, let $y = (y - Qy) + Qy$, since $Q(y - Qy) = Qy - Q^2y = 0$, we know $y - Qy \in \ker Q = \operatorname{Im} L$, and we have $Qy \in \operatorname{Im} Q$. Thus $Z = \operatorname{Im} L + \operatorname{Im} Q$. And for any $y \in \operatorname{Im} L \cap \operatorname{Im} Q$, from $y \in \operatorname{Im} Q$, there exists constants $a, b, c \in \mathbb{R}$, such that $y(t) = at^{p-1} + bt^{q-1} + ct^{r-1}$. From $y \in \operatorname{Im} L$, we obtain

$$
\begin{aligned}
&\frac{a}{p}\sum_{i=1}^{m}\alpha_i\xi_i^p + \frac{b}{q}\sum_{i=1}^{m}\alpha_i\xi_i^q + \frac{c}{r}\sum_{i=1}^{m}\alpha_i\xi_i^r = 0,\\
&\frac{a}{p(p+1)}\sum_{k=1}^{l}\gamma_k\sigma_k^{p+1} + \frac{b}{q(q+1)}\sum_{k=1}^{l}\gamma_k\sigma_k^{q+1} + \frac{c}{r(r+1)}\sum_{k=1}^{l}\gamma_k\sigma_k^{r+1} = 0,\\
&a(\frac{1}{p} - \frac{2}{p+1} + \frac{1}{p+2})(1 - \sum_{j=1}^{n}\beta_j\eta_j^{p+2}) + b(\frac{1}{q} - \frac{2}{q+1} + \frac{1}{q+2})(1 - \sum_{j=1}^{n}\beta_j\eta_j^{q+2})\\
&+ c(\frac{1}{r} - \frac{2}{r+1} + \frac{1}{r+2})(1 - \sum_{j=1}^{n}\beta_j\eta_j^{r+2}) = 0.
\end{aligned}
\tag{3.4}
$$

In view of

$$
\begin{aligned}
&\begin{vmatrix}
\frac{1}{p}\sum_{i=1}^{m}\alpha_i\xi_i^p & \frac{1}{q}\sum_{i=1}^{m}\alpha_i\xi_i^q & \frac{1}{r}\sum_{i=1}^{m}\alpha_i\xi_i^r, \\
\frac{1}{p(p+1)}\sum_{k=1}^{l}\gamma_k\sigma_k^{p+1} & \frac{1}{q(q+1)}\sum_{k=1}^{l}\gamma_k\sigma_k^{q+1} & \frac{1}{r(r+1)}\sum_{k=1}^{l}\gamma_k\sigma_k^{r+1} \\
A_{31} & A_{32} & A_{33}
\end{vmatrix}\\
&= \frac{1}{p(p+1)(p+2)q(q+1)(q+2)r(r+1)(r+2)}\Lambda(p,q,r) \neq 0,
\end{aligned}
$$

where the entries of the third row are

$$
\begin{aligned}
A_{31} &= (\frac{1}{p} - \frac{2}{p+1} + \frac{1}{p+2})(1 - \sum_{j=1}^{n}\beta_j\eta_j^{p+2})\\
A_{32} &= (\frac{1}{q} - \frac{2}{q+1} + \frac{1}{q+2})(1 - \sum_{j=1}^{n}\beta_j\eta_j^{q+2})\\
A_{33} &= (\frac{1}{r} - \frac{2}{r+1} + \frac{1}{r+2})(1 - \sum_{j=1}^{n}\beta_j\eta_j^{r+2}).
\end{aligned}
$$

I split the above matrix, please check it

Therefore (3.4) has an unique solution $a = b = c = 0$, which implies $\operatorname{Im} Q \cap \operatorname{Im} L = \{0\}$ and $Z = \operatorname{Im} L \oplus \operatorname{Im} Q$. Since $\dim\ker L = \mathrm{dimIm}Q = \mathrm{codimIm}L = 3$, thus L is a Fredholm map of index zero.

Let $P : Y \to Y$ be defined by

$$
Px(t) = x(0) + x'(0)t + \frac{1}{2}x''(0)t^2, \quad t \in (0,1).
$$

then, the generalized inverse operator $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$ be defined by

$$
K_P(y(t)) = \frac{1}{2}\int_0^t (t-s)^2 y(s)ds, \quad y \in \operatorname{Im} L.
$$

If $y \in \operatorname{Im} L$, we have $(LK_P)(y(t)) = (K_P y)''' = y(t)$. If $x(t) \in \operatorname{dom} L \cap \ker P$, we have

$$\begin{aligned}(K_P L)x(t) &= (K_P)(x'''(t)) \\ &= \frac{1}{2}\int_0^t (t-s)^2 x'''(s)ds \\ &= x(t) - [x(0) + x'(0)t + \frac{1}{2}x''(0)t^2] \\ &= x(t) - Px(t).\end{aligned}$$

Because $x \in \operatorname{dom} L \cap \ker P$, we know that $Px(t) = 0$. Thus $(K_P L)(x(t)) = x(t)$. It is obviously that $\|K_P y\| \le \|y\|_1$. □

For the next theorem we use the assumptions:

- (H1) There exists $\alpha(t), \beta(t), \gamma(t), \theta(t) \in L^1[0,1]$, such that for all (x_1, x_2, x_3) in $\mathbb{R}^3$, $t \in (0,1)$,
$$|f(t, x_1, x_2, x_3)| \le \alpha(t)|x_1| + \beta(t)|x_2| + \gamma(t)|x_3| + \theta(t).$$
- (H2) There exists a constant $A > 0$ such that for $x(t) \in \operatorname{dom} L$, if $|x(t)| > A$ or $|x'(t)| > A$ or $|x''(t)| > A$ for all $t \in (0,1)$, then $Q_1 N(x(t)) \ne 0$ or $Q_2 N(x(t)) \ne 0$ or $Q_3 N(x(t)) \ne 0$.
- (H3) There exists a constant $B > 0$ such that for $a, b, c \in \mathbb{R}$, if $|a| > B$, $|b| > B$, $|c| > B$, then either
$$Q_1 N(a + bt + ct^2) + Q_2 N(a + bt + ct^2) + Q_3 N(a + bt + ct^2) > 0, \tag{3.5}$$
or
$$Q_1 N(a + bt + ct^2) + Q_2 N(a + bt + ct^2) + Q_3 N(a + bt + ct^2) < 0. \tag{3.6}$$

Theorem 3.3. *Let the conditions* (2.1), (H1), (H2), (H3) *hold. Then BVP* (1.1)-(1.2) *have at least one solution in* $C^2[0,1]$, *provided that* $\|\alpha\|_1 + \|\beta\|_1 + \|\gamma\|_1 < 1$.

Proof. We divide the proof into four steps.

Step 1: Let
$$\Omega_1 = \{x \in \operatorname{dom} L \setminus \ker L : Lx = \lambda Nx, \lambda \in [0,1]\}.$$
Then Ω_1 is bounded. Suppose that $x \in \Omega_1$, we have $Lx = \lambda Nx$. Thus $\lambda \ne 0$, $Nx \in \operatorname{Im} L = \ker Q$, hence
$$Q_1(Nx(t)) = Q_2(Nx(t)) = Q_3(Nx(t)) = 0.$$
From (H2), there exists $t_1, t_2, t_3 \in (0,1)$, such that $|x(t_1)| \le A$, $|x'(t_2)| \le A$, $|x''(t_3)| \le A$. Since x, x', x'' are absolutely continuous for all $t \in (0,1)$, and
$$x'(t) = x'(t_2) + \int_{t_2}^t x''(s)ds, \quad x''(t) = x''(t_3) + \int_{t_3}^t x'''(s)ds.$$
Thus
$$\|x''\|_\infty \le A + \|x'''\|_1, \ \|x'\|_\infty \le 2A + \|x'''\|_1.$$
And
$$\begin{aligned}\|x\|_\infty &\le \|(I-P)x\|_\infty + \|Px\|_\infty = \|Px\|_\infty + \|K_P L(I-P)x\|_\infty \\ &= \|Px\|_\infty + \|K_P Lx\|_\infty \le \|Px\|_\infty + \|K_P\|\|Lx\|_1 \\ &\le \|Px\|_\infty + \|Lx\|_1 = \|Px\|_\infty + \|x'''\|_1.\end{aligned}$$

From (H_1), we obtain

$$\begin{aligned}\|x'''\|_1 &= \|Lx\|_1 \le \|Nx\|_1 \\ &\le \|\alpha\|_1\|x\|_\infty + \|\beta\|_1\|x'\|_\infty + \|\gamma\|_1\|x''\|_\infty + \|\theta\|_1 \\ &\le (\|\alpha\|_1 + \|\beta\|_1 + \|\gamma\|_1)\|x'''\|_1 + (2\|\beta\|_1 + \|\gamma\|_1)A + \|\alpha\|_1\|Px\|_\infty + \|\theta\|_1.\end{aligned}$$

Then

$$\|x'''\|_1 \le \frac{1}{1 - (\|\alpha\|_1 + \|\beta\|_1 + \|\gamma\|_1)}[(2\|\beta\|_1 + \|\gamma\|_1)A + \|\alpha\|_1\|Px\|_\infty + \|\theta\|_1].$$

So there exists a constant $M_1 > 0$ such that $\|x\| \le M_1$. Hence we show that Ω_1 is bounded.

Step 2: Let $\Omega_2 = \{x \in \ker L : Nx \in \operatorname{Im} L\}$. Then Ω_2 is bounded. Since $x \in \Omega_2$, $x \in \ker L = \{x \in \operatorname{dom} L : x = a + bt + ct^2,\ a, b, c \in \mathbb{R}\}$, and $QNx = 0$, thus,

$$Q_1(N(a + bt + ct^2)) = Q_2(N(a + bt + ct^2)) = Q_3(N(a + bt + ct^2)) = 0.$$

From (H3),

$$\|x\| \le |a| + |b| + |c| \le 3B.$$

So Ω_2 is bounded.

Step 3: Let

$$\Omega_3 = \{x \in \ker L : \lambda Jx + (1 - \lambda)QNx = 0,\ \lambda \in [0, 1]\}.$$

Here $J : \ker L \to \operatorname{Im} Q$ is the linear isomorphism given by

$$J(a + bt + ct^2) = \frac{1}{\Lambda(p, q, r)}(a_1 t^{p-1} + b_1 t^{q-1} + c_1 t^{r-1}), \quad a, b, c \in \mathbb{R}$$

where

$$\begin{aligned}a_1 &= p(p + 1)(p + 2)(M_{11}|a| + M_{12}|b| + M_{13}|c|), \\ b_1 &= q(q + 1)(q + 2)(M_{21}|a| + M_{22}|b| + M_{23}|c|), \\ c_1 &= r(r + 1)(r + 2)(M_{31}|a| + M_{32}|b| + M_{33}|c|).\end{aligned}$$

Then Ω_3 is bounded.

Set

$$B_1 = \frac{p(p + 1)(p + 2)}{\Lambda(p, q, r)}, \quad B_2 = \frac{q(q + 1)(q + 2)}{\Lambda(p, q, r)}, \quad B_3 = \frac{r(r + 1)(r + 2)}{\Lambda(p, q, r)},$$

$$X_1 = \lambda|a| + (1 - \lambda)Q_1N(a + bt + ct^2), \quad X_2 = \lambda|b| + (1 - \lambda)Q_2N(a + bt + ct^2),$$

$$X_3 = \lambda|c| + (1 - \lambda)Q_3N(a + bt + ct^2).$$

Since $x(t) = a + bt + ct^2 \in \Omega_3$, then we have $\lambda Jx + (1 - \lambda)QNx = 0$; i.e.,

$$\begin{aligned}B_1M_{11}X_1 + B_1M_{12}X_2 + B_1M_{13}X_3 = 0, \\ B_2M_{21}X_1 + B_2M_{22}X_2 + B_2M_{23}X_3 = 0, \\ B_3M_{31}X_1 + B_3M_{32}X_2 + B_3M_{33}X_3 = 0.\end{aligned}$$

Because

$$\begin{aligned}\begin{vmatrix} B_1M_{11} & B_1M_{12} & B_1M_{13} \\ B_2M_{21} & B_2M_{22} & B_2M_{23} \\ B_3M_{31} & B_3M_{32} & B_3M_{33} \end{vmatrix} &= (B_1B_2B_3)\begin{vmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{vmatrix} \\ &= \frac{p(p + 1)(p + 2)q(q + 1)(q + 2)r(r + 1)(r + 2)}{\Lambda(p, q, r)} \ne 0.\end{aligned}$$

Then $X_1 = X_2 = X_3 = 0$. If $\lambda = 1$, then $|a| = |b| = |c| = 0$. If $\lambda \neq 1$ and $|a| > B$ or $|b| > B$ or $|c| > B$ and (3.5) hold, then

$$\begin{aligned}\lambda(|a|+|b|+|c|) = -(1-\lambda)\big[&Q_1(N(a+bt+ct^2)) + Q_2(N(a+bt+ct^2)) \\ &+ Q_3(N(a+bt+ct^2))\big] < 0,\end{aligned}$$

which contradicts $\lambda(|a|+|b|+|c|) > 0$. Thus

$$\|x\| \le |a|+|b|+|c| \le 3B.$$

So Ω_3 is bounded.

If (3.6) holds, we let

$$\Omega_3 = \{x \in \ker L : -\lambda Jx + (1-\lambda)QNx = 0,\ \lambda \in [0,1]\}.$$

Similarly, we can proof that Ω_3 is bounded.

Step 4: In the following, we shall prove that all the conditions of Lemma 2.1 are satisfied. Let $\Omega \subset Y$ be a bounded open set such that $\cup_{i=1}^{3}\overline{\Omega_i} \subset \Omega$. By the Ascoli-Arzela theorem, we can show that $K_P(I-Q)N : \overline{\Omega} \to Y$ is compact, thus N is $L-$compact on $\overline{\Omega}$. Then by the above argument, we obtain

(i) $Lx \neq \lambda Nx$ for each $(x,\lambda) \in ((\operatorname{dom} L \setminus \ker L) \cap \partial\Omega) \times [0,1]$;
(ii) $Nx \notin \operatorname{Im} L$ for each $x \in \ker L \cap \partial\Omega$.

At last we prove that (iii) of the Lemma 2.1 is satisfied. Let $H(x,\lambda) = \pm\lambda Jx + (1-\lambda)QNx$. According to the above argument, we have $H(x,\lambda) \neq 0$, for $x \in \ker L \cap \partial\Omega$. Thus, by the homotopy property of degree, we get

$$\begin{aligned}\deg(QN|_{\ker L}, \Omega\cap\ker L, 0) &= \deg(H(\cdot,0), \Omega\cap\ker L, 0)\\ &= \deg(H(\cdot,1), \Omega\cap\ker L, 0)\\ &= \deg(\pm J, \Omega\cap\ker L, 0) \neq 0.\end{aligned}$$

Then by Lemma 2.1, $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$, so that BVP (1.1)-(1.2) has at least one solution in $C^2[0,1]$. □

4. Applications

We consider the boundary value problem

$$x'''(t) = x''(t) + \sin x(t)(1-\cos x'(t)), t \in (0,1), \tag{4.1}$$

$$x''(0) = 2x''(\frac{1}{5}) - x''(\frac{1}{4}), \quad x'(0) = 3x'(\frac{1}{3}) - 2x'(\frac{1}{2}), \tag{4.2}$$

$$x(1) = \frac{9}{5}x(\frac{1}{3}) - 4x(\frac{1}{2}) + \frac{16}{5}x(\frac{3}{4}). \tag{4.3}$$

So $f(t,x(t),x'(t),x''(t)) = x''(t) + \sin x(t)(1-\cos x'(t))$, $\alpha_1 = 2$, $\alpha_2 = -1$, $\xi_1 = \frac{1}{5}$, $\xi_2 = \frac{1}{4}$, $\gamma_1 = 3$, $\gamma_2 = -2$, $\sigma_1 = \frac{1}{3}$, $\sigma_2 = \frac{1}{2}$, $\beta_1 = \frac{9}{5}$, $\beta_2 = -4$, $\beta_3 = \frac{16}{5}$, $\eta_1 = \frac{1}{3}$, $\eta_2 = \frac{1}{2}$, $\eta_3 = \frac{3}{4}$. Then we have $\alpha_1 - \alpha_2 = 1$, $\gamma_1 + \gamma_2 = 1$, $\beta_1 + \beta_2 + \beta_3 = 1$, $\beta_1\eta_1 + \beta_2\eta_2 + \beta_3\eta_3 = 1$, $\beta_1\eta_1^2 + \beta_2\eta_2^2 + \beta_3\eta_3^2 = 1$, $\gamma_1\sigma_1 + \gamma_2\sigma_2 = 0$. So the condition (2.1) holds.

By calculations, we obtain

$$Q_1 y = 2\int_0^{1/5} y(s)ds, \quad Q_2 y = 3\int_0^{\frac{1}{3}} (\frac{1}{3} - s)y(s)ds - 2\int_0^{1/2} (\frac{1}{2} - s)y(s)ds,$$

$$
\begin{aligned}
Q_3y &= \int_0^1 (1-s)^2 y(s)ds - \frac{9}{5}\int_0^{\frac{1}{3}} (\frac{1}{3}-s)^2 y(s)ds \\
&\quad + 4\int_0^{1/2} (\frac{1}{2}-s)^2 y(s)ds - \frac{16}{5}\int_0^{3/4} (\frac{3}{4}-s)^2 y(s)ds, \\
&\Lambda(1,2,3) = \frac{1225231}{1834400}, \\
M_{11} &= -\frac{5731015}{1508544}, \quad M_{12} = \frac{45787}{311040}, \quad M_{13} = -\frac{3429}{38880}, \\
M_{21} &= \frac{70758}{279935}, \quad M_{22} = \frac{3089}{9600}, \quad M_{23} = \frac{8649}{2260}, \\
M_{31} &= -\frac{477}{3888}, \quad M_{32} = -\frac{1269}{3600}, \quad M_{33} = -\frac{79}{200}. \\
T_1y &= \frac{6}{\Lambda(p,q,r)}[M_{11}Q_1y + M_{12}Q_2y + M_{13}Q_3y], \\
T_2y &= \frac{24}{\Lambda(p,q,r)}[M_{21}Q_1y + M_{22}Q_2y + M_{23}Q_3y], \\
T_3y &= \frac{60}{\Lambda(p,q,r)}[M_{31}Q_1y + M_{32}Q_2y + M_{33}Q_3y].
\end{aligned}
$$

Define Qy by $Qy = T_1y + (T_2y)t + (T_3y)t^2$, and we take $K_Py(t)$ as in Lemma 3.2, then Lemma 3.2 holds.

On the other hand, we have

$$
|f(t, x(t), x'(t), x''(t))| \le |x''(t)| + 2, \quad t \in (0,1).
$$

And let $\alpha(t) = 0$, $\beta(t) = 0$, $\gamma(t) = 1$, $\theta(t) = 2$, then the condition (H1) of Theorem 3.3 is satisfied. If $x''(t) > 8 = A$, $t \in (0,1)$, then

$$
\begin{aligned}
Q_1y &= 2\int_0^{1/5} (x''(t) + \sin x(t)(1-\cos x'(t)))dt \\
&\quad - \int_0^{1/4} (x''(t) + \sin x(t)(1-\cos x'(t)))dt \\
&> 2\int_0^{1/5} 7dt - \int_0^{1/4} 10dt > \frac{9}{10},
\end{aligned}
$$

If $x''(t) < -8 = -A$, $t \in (0,1)$, then

$$
\begin{aligned}
Q_1y &= 2\int_0^{1/5} (x''(t) + \sin x(t)(1-\cos x'(t)))dt \\
&\quad - \int_0^{1/4} (x''(t) + \sin x(t)(1-\cos x'(t)))dt \\
&< 2\int_0^{1/5} (-6)dt - \int_0^{1/4} (-9)dt < -\frac{3}{20},
\end{aligned}
$$

So condition (H2) is satisfied.

If $|a| > 16 = B$, $|b| > 16 = B$, $|c| > 16 = B$, then

$$
\begin{aligned}
&Q_1N(a+bt+ct^2) + Q_2N(a+bt+ct^2) + Q_3N(a+bt+ct^2) \\
&= 2\int_0^{1/5} (|2c| + \sin(a+bt+ct^2)[1-\cos(b+2ct)])dt
\end{aligned}
$$

$$
\begin{aligned}
&-\int_0^{1/4}(|2c|+\sin(a+bt+ct^2)[1-\cos(b+2ct)])dt\\
&+3\int_0^{\frac{1}{3}}(\frac{1}{3}-t)(|2c|+\sin(a+bt+ct^2)[1-\cos(b+2ct)])dt\\
&-2\int_0^{1/2}(\frac{1}{2}-t)(|2c|+\sin(a+bt+ct^2)[1-\cos(b+2ct)])dt\\
&+\int_0^{1}(1-t)^2(|2c|+\sin(a+bt+ct^2)[1-\cos(b+2ct)])dt\\
&-\frac{9}{5}\int_0^{\frac{1}{3}}(\frac{1}{3}-t)^2(|2c|+\sin(a+bt+ct^2)[1-\cos(b+2ct)])dt\\
&+4\int_0^{1/2}(\frac{1}{2}-t)^2(|2c|+\sin(a+bt+ct^2)[1-\cos(b+2ct)])dt\\
&-\frac{16}{5}\int_0^{3/4}(\frac{3}{4}-t)^2(|2c|+\sin(a+bt+ct^2)[1-\cos(b+2ct)])dt>0.
\end{aligned}
$$

So condition (H3) is satisfied. Thus all the conditions of Theorem 3.3 are satisfied. Hence BVP (4.1)-(4.3) has at least one solution in $C^2[0,1]$.

Acknowledgments. This research was sponsored by the Natural Science Foundation of China (11071205, 11101349) and by PAPD of Jiangsu Higher Education Institutions, the Humanistic and Social Scientific Research Planning Program in Ministry of Education of China (No. 12YJA630063), the Social Science Foundation of Jiangsu Province of China (No. 10GLB001).

The authors express their sincere thanks to the anonymous reviewer for his or her valuable comments and suggestions for improving the quality of this article.

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