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PERIODIC SOLUTIONS FOR NON-AUTONOMOUS SECOND-ORDER DIFFERENTIAL SYSTEMS WITH (q,p)-LAPLACIAN

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Abstract. Some existence theorems are obtained for periodic solutions of nonautonomous second-order differential systems with (q,p)-Laplacian by using the least action principle and the saddle point theorem.

1. Introduction

Consider the second-order system

$$\begin{gathered}\frac{d}{dt}\big(|\dot u_1(t)|^{q-2}\dot u_1(t)\big)=\nabla_{u_1}F(t,u_1(t),u_2(t)),\\ \frac{d}{dt}(|\dot u_2(t)|^{p-2}\dot u_2(t))=\nabla_{u_2}F(t,u_1(t),u_2(t)),\quad \text{a.e. } t\in[0,T],\\ u_1(0)-u_1(T)=\dot u_1(0)-\dot u_1(T)=0,\\ u_2(0)-u_2(T)=\dot u_2(0)-\dot u_2(T)=0,\end{gathered}\tag{1.1}$$

where $1<p,\ q<\infty$, $T>0$ and $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^N$. $F:[0,T]\times\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}$ satisfies the following assumption

(A1) – F is measurable in t for each $(x_1,x_2)\in\mathbb{R}^N\times\mathbb{R}^N$;
– F is continuously differentiable in (x_1,x_2) for a.e. $t\in[0,T]$;
– there exist $a_1,a_2\in C(\mathbb{R}_+,\mathbb{R}_+)$ and $b\in L^1(0,T;\mathbb{R}_+)$ such that

$$|F(t,x_1,x_2)|,\ |\nabla_{x_1}F(t,x_1,x_2)|,\ |\nabla_{x_2}F(t,x_1,x_2)|\le\big[a_1(|x_1|)+a_2(|x_2|)\big]b(t)$$

for all $(x_1,x_2)\in\mathbb{R}^N\times\mathbb{R}^N$ and a.e. $t\in[0,T]$.

We denote by $W_T^{1,p}$ the Sobolev space of functions $u\in L^p(0,T;\mathbb{R}^N)$ having a weak derivative $\dot u\in L^p(0,T;\mathbb{R}^N)$. The norm in $W_T^{1,p}$ is defined by

$$\|u\|_{W_T^{1,p}}=\Big(\int_0^T\big(|u(t)|^p+|\dot u(t)|^p\big)dt\Big)^{1/p}.$$

The corresponding functional $\varphi:W\to\mathbb{R}$ given is

$$\varphi(u_1,u_2)=\frac1q\int_0^T|\dot u_1(t)|^q dt+\frac1p\int_0^T|\dot u_2(t)|^p dt+\int_0^T F(t,u_1(t),u_2(t))dt,$$

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where $W = W_T^{1,q} \times W_T^{1,p}$ is a reflexive Banach space and endowed with the norm

$$\|(u_1,u_2)\|_W = \|u_1\|_{W_T^{1,q}} + \|u_2\|_{W_T^{1,p}}.$$

It follows from assumption (A1) that the functional φ is continuously differentiable and weakly lower semicontinuous on W. Moreover,

$$\begin{aligned}
&\langle \varphi'(u_1,u_2),(v_1,v_2)\rangle \\
&= \int_0^T [(|\dot u_1(t)|^{q-2}\dot u_1(t),\dot v_1(t)) + (\nabla_{u_1}F(t,u_1(t),u_2(t)),v_1(t))]dt \\
&\quad + \int_0^T [(|\dot u_2(t)|^{p-2}\dot u_2(t),\dot v_2(t)) + (\nabla_{u_2}F(t,u_1(t),u_2(t)),v_2(t))]dt
\end{aligned}$$

for all $(u_1,u_2),(v_1,v_2) \in W$.

For each $u \in W_T^{1,p}$ can be written as $u(t) = \bar u + \tilde u(t)$ with

$$\bar u = \frac{1}{T}\int_0^T u(t)dt, \quad \int_0^T \tilde u(t)dt = 0.$$

We have the Sobolev's inequality (for a proof and details see [4])

$$\|\tilde u\|_\infty \le C_1\|\dot u\|_p,\ \|\tilde v\|_\infty \le C_1\|\dot v\|_q \quad \text{for each } u \in W_T^{1,p},\ v \in W_T^{1,q},$$

and Wirtinger's inequality (see [4])

$$\|\tilde u\|_p \le C_2\|\dot u\|_p, \quad \|\tilde v\|_q \le C_2\|\dot v\|_q \quad \text{for each } u \in W_T^{1,p},\ v \in W_T^{1,q},$$

where

$$\|u\|_p = \Big(\int_0^T |u(t)|^p dt\Big)^{1/p}, \quad \|u\|_\infty = \max_{t\in[0,T]} |u(t)|.$$

A function $G : \mathbb{R}^N \to \mathbb{R}$ is called to be (λ,μ)-subconvex if

$$G(\lambda(x+y)) \le \mu(G(x)+G(y))$$

for some $\lambda,\mu > 0$ and all $x,y \in \mathbb{R}^N$ (see [19]).

The existence of periodic solutions for the second-order Hamiltonian system

$$\begin{gathered}
\ddot u(t) = \nabla F(t,u), \quad \text{a.e. } t \in [0,T], \\
u(0) - u(T) = \dot u(0) - \dot u(T) = 0,
\end{gathered} \tag{1.2}$$

has been extensively investigated in papers, such as [1, 2, 3, 4, 12, 13, 14, 15, 16, 18, 19] and the reference therein. Many solvability conditions are given, such as the coercive condition (see [1]), the periodicity condition (see [18]), the convexity condition (see [3]), the boundedness condition (see [4]), the subadditive condition (see [12]), and the sublinear condition (see [14]). When the gradient $\nabla F(t,x)$ is bounded; that is, there exists $g \in L^1(0,T;\mathbb{R}_+)$ such that

$$|\nabla F(t,x)| \le g(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$. Mawhin and Willem [4] obtained the existence of solutions for problem (1.2) under the condition

$$\int_0^T F(t,x)dt \to +\infty (\text{or } -\infty), \quad \text{as } |x| \to \infty.$$

Tang [14] proved the existence of solutions for problem (1.2) when

$$|\nabla F(t,x)| \le f(t)|x|^\alpha + g(t) \tag{1.3}$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$, where $f, g \in L^1(0,T;\mathbb{R}_+)$ and $\alpha \in [0,1)$. And, F satisfies the condition

$$|x|^{-2\alpha} \int_0^T F(t,x)dt \to +\infty \ (or \ -\infty), \text{ as } |x| \to \infty.$$

Tang and Meng [16] studied the existence of solutions for problem (1.2) under the conditions (1.3) or

$$|\nabla F(t,x)| \le f(t)|x| + g(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$, where $f, g \in L^1(0,T;\mathbb{R}_+)$. The results in [16] complement those in [14, Theorem 1 and 2].

Recently, Paşca and Tang [9] established the existence results for problem (1.1) which extend [14, Theorems 1 and 2]. By applying the least action principle, Paşca [7] proved some existence theorems for problem (1.1) which generalize the corresponding Theorems of [19]. Using the Saddle Point Theorem, Paşca and Tang [10] obtained some existence results for problem (1.1). Paşca [8] studied the existence of periodic solutions for nonautonomous second-order differential inclusions systems with (q,p)-Laplacian which extend the results of [5, 6, 9, 14].

In this paper, motivated by references [7, 9, 14, 16], we consider the existence of periodic solutions for problem (1.1) by using the least action principle and the Saddle Point Theorem. Our main results are the following theorems.

Theorem 1.1. *Suppose that $F = F_1 + F_2$, where F_1 and F_2 satisfy assumption* (A1) *and the following conditions:*

- (H0) *$F_1(t,\cdot,\cdot)$ is (λ,μ)-subconvex with $\lambda > 1/2$ and $1/2 < \mu < 2^{r-1}\lambda^r$ for a.e.$t \in [0,T]$, where $r = \min\{p,q\}$;*
- (H1) *there exist $f_i, g_i, h_i \in L^1(0,T;\mathbb{R}_+)$, $i = 1,2$, $\alpha_1 \in [0,q-1)$, $\alpha_2 \in [0,p-1)$, $\beta_1 \in [0,p/q')$, $\beta_2 \in [0,q/p')$, $q' = q/(q-1)$ and $p' = p/(p-1)$*
$$|\nabla_{x_1} F_2(t,x_1,x_2)| \le f_1(t)|x_1|^{\alpha_1} + g_1(t)|x_2|^{\beta_1} + h_1(t)$$
$$|\nabla_{x_2} F_2(t,x_1,x_2)| \le f_2(t)|x_2|^{\alpha_2} + g_2(t)|x_1|^{\beta_2} + h_2(t)$$
for all $(x_1,x_2) \in \mathbb{R}^N \times \mathbb{R}^N$ and a.e. $t \in [0,T]$;
- (H2)
$$\lim_{|x|\to\infty} \frac{1}{|x_1|^{\gamma_1} + |x_2|^{\gamma_2}} \Big(\frac{1}{\mu} \int_0^T F_1(t,\lambda x_1,\lambda x_2)dt + \int_0^T F_2(t,x_1,x_2)dt \Big) > 2K,$$
where $|x| = \sqrt{|x_1|^2 + |x_2|^2}$, $\gamma_1 = \max\{q'\alpha_1, \beta_2 p'\}$, $\gamma_2 = \max\{p'\alpha_2, \beta_1 q'\}$ and
$$K = \max\Big\{ \frac{4^{q'/q}(2^{q-1}\|f_1\|_{L^1}C_1)^{q'}}{q'}, \frac{4^{p'/p}(2^{p-1}\|f_2\|_{L^1}C_1)^{p'}}{p'}, \frac{4^{q'/q}(2^{\beta_1}\|g_1\|_{L^1}C_1)^{q'}}{q'}, \frac{4^{p'/p}(2^{\beta_2}\|g_2\|_{L^1}C_1)^{p'}}{p'} \Big\}.$$

Then problem (1.1) *has at least one solution in W.*

Corollary 1.2. *Suppose that $F = F_1 + F_2$, satisfies* (H0), (H1) *and*

(H2')
$$\frac{1}{|x_1|^{\gamma_1} + |x_2|^{\gamma_2}} \Big(\frac{1}{\mu} \int_0^T F_1(t,\lambda x_1,\lambda x_2)dt + \int_0^T F_2(t,x_1,x_2)dt \Big) \to \infty$$

as $|x| \to +\infty$. Then problem (1.1) has at least one solution in W.

Remark 1.3. Corollary 1.2 generalizes Theorem 1 of [7]. In fact, it follows from Corollary 1.2 by letting $\beta_1 = \beta_2 = 0$. There are functions satisfying the assumptions of our Corollary 1.2 and not satisfying the assumptions in [7, 9]. For example, Let $\alpha_1 = \alpha_2 = 15/4$, $\beta_1 = \beta_2 = 11/4$, $p = q = 5$, $p' = q' = 5/4$, and

$$F_1(t, x_1) = 5 + \sin(|x_1|^6 + |x_2|^6),$$
$$F_2(t, x_1, x_2) = \big(\frac{2T}{3} - t\big)(|x_1|^{19/4} + |x_2|^{19/4} + |x_1|^{5/4}|x_2|^{5/4}).$$

Theorem 1.4. *Suppose that $F(t, x_1, x_2)$ satisfies* (H1) *and*

(H3)
$$\lim_{|x|\to\infty} \frac{1}{|x_1|^{\gamma_1} + |x_2|^{\gamma_2}} \int_0^T F(t, x_1, x_2)dt < -(2q' + 2p' + 1)2K.$$

Then problem (1.1) has at least one solution in W.

Corollary 1.5. *Suppose that $F(t, x_1, x_2)$ satisfies* (H1) *and*

(H3')
$$\frac{1}{|x_1|^{\gamma_1} + |x_2|^{\gamma_2}} \int_0^T F(t, x_1, x_2)dt \to -\infty$$

as $|x| \to \infty$. Then problem (1.1) has at least one solution in W.

Remark 1.6. Corollary 1.5 extends [9, Theorem 2]. In fact, it follows from Corollary 1.5 by letting $\beta_1 = \beta_2 = 0$. There are functions satisfying the assumptions of our Corollary 1.5 and not satisfying the assumptions in [9]. For example, Let $\alpha_1 = \alpha_2 = 15/4$, $\beta_1 = \beta_2 = 11/4$, $p = q = 5$, $p' = q' = 5/4$, and

$$F(t, x_1, x_2) = \big(\frac{T}{3} - t\big)(|x_1|^{19/4} + |x_2|^{19/4} + |x_1|^{5/4}|x_2|^{5/4}).$$

2. Proofs of main results

Tian and Ge [17] proved the following result which generalizes a very well known result proved by Jean Mawhin and Michel Willem [4, Theorem 1.4].

Lemma 2.1 ([17]). *Let $L : [0, T] \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, $(t, x_1, x_2, y_1, y_2) \to L(t, x_1, x_2, y_1, y_2)$ be measurable in t for each (x_1, x_2, y_1, y_2), and continuously differentiable in (x_1, x_2, y_1, y_2) for a.e. $t \in [0, T]$. If there exist $a_i \in C(\mathbb{R}_+, \mathbb{R}_+)$, $i = 1, 2$, $b \in L^1(0, T; \mathbb{R}_+)$, and $c_1 \in L^p(0, T; \mathbb{R}_+)$, $c_2 \in L^q(0, T; \mathbb{R}_+)$, $1 < p, q < \infty$, such that for a.e. $t \in [0, T]$ and every $(x_1, x_2, y_1, y_2) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$, one has*

$$|L(t, x_1, x_2, y_1, y_2)| \le (a_1(|x_1|) + a_2(|x_2|))(b(t) + |y_1|^q + |y_2|^p),$$
$$|D_{x_1} L(t, x_1, x_2, y_1, y_2)| \le (a_1(|x_1|) + a_2(|x_2|))(b(t) + |y_2|^p),$$
$$|D_{x_2} L(t, x_1, x_2, y_1, y_2)| \le (a_1(|x_1|) + a_2(|x_2|))(b(t) + |y_1|^q),$$
$$|D_{y_1} L(t, x_1, x_2, y_1, y_2)| \le (a_1(|x_1|) + a_2(|x_2|))(c_1(t) + |y_1|^{q-1}),$$
$$|D_{y_2} L(t, x_1, x_2, y_1, y_2)| \le (a_1(|x_1|) + a_2(|x_2|))(c_2(t) + |y_2|^{p-1}),$$

then the function $\varphi : W_T^{1,q} \times W_T^{1,p} \to \mathbb{R}$ defined by

$$\varphi(u_1, u_2) = \int_0^T L(t, u_1(t), u_2(t), \dot u_1(t), \dot u_2(t))dt$$

is continuously differentiable on $W_T^{1,q} \times W_T^{1,p}$ *and*

$$\begin{aligned}\langle \varphi'(u_1,u_2),(v_1,v_2)\rangle = \int_0^T &((D_{x_1}L(t,u_1(t),u_2(t),\dot u_1(t),\dot u_2(t)),v_1(t))\\ &+ (D_{y_1}L(t,u_1(t),u_2(t),\dot u_1(t),\dot u_2(t)),\dot v_1(t))\\ &+ (D_{x_2}L(t,u_1(t),u_2(t),\dot u_1(t),\dot u_2(t)),v_2(t))\\ &+ (D_{y_2}L(t,u_1(t),u_2(t),\dot u_1(t),\dot u_2(t)),\dot v_2(t)))dt.\end{aligned}$$

Corollary 2.2. *Let* $L : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ *be defined by*

$$L(t,x_1,x_2,y_1,y_2) = \frac{1}{q}|y_1|^q + \frac{1}{p}|y_2|^p + F(t,x_1,x_2)$$

where $F : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ *satisfies condition* (A1). *If* $(u_1,u_2) \in W_T^{1,q} \times W_T^{1,p}$ *is a solution of the corresponding Euler equation* $\varphi'(u_1,u_2) = 0$*, then* (u_1,u_2) *is a solution of problem* (1.1).

Remark 2.3. The function φ is weakly lower semi-continuous (w.l.s.c.) on W as the sum of two convex continuous functions and of a weakly continuous one.

We will prove Theorem 1.1 by using the least action principle [4, Theorem 1.1], and Theorem 1.4 by using the saddle point theorem [11, Theorem 4.6].

Proof of Theorem 1.1. Let $\beta = \log_{2\lambda}(2\mu)$. Then $0 < \beta < r$. For $|x| > 1$, there exists a positive integer n such that

$$n-1 < \log_{2\lambda}|x| \le n.$$

So, we have $|x|^\beta > (2\lambda)^{(n-1)\beta} = (2\mu)^{n-1}$ and $|x| \le (2\lambda)^n$. Then, by (A1) and (H0), one has

$$\begin{aligned}F_1(t,x_1,x_2) &\le 2\mu F_1(t,x_1/(2\lambda),x_2/(2\lambda)) \le \dots\\ &\le (2\mu)^n F_1(t,x_1/(2\lambda),x_2/(2\lambda))\\ &\le 2\mu|x|^\beta(a_{10}+a_{20})b(t)\end{aligned}$$

for a.e. $t \in [0,T]$ and all $|x| > 1$, where $a_{i0} = \max_{0\le s\le 1} a_i(s)$, $i = 1,2$. Therefore,

$$F_1(t,x_1,x_2) \le (2^{\beta/2+1}\mu(|x_1|^\beta + |x_2|^\beta) + 1)(a_{10}+a_{20})b(t) \tag{2.1}$$

for a.e. $t \in [0,T]$ and all $(x_1,x_2) \in \mathbb{R}^N \times \mathbb{R}^N$.

It follows from (H1), Sobolev's inequality and Young's inequality that

$$\begin{aligned}&\Big|\int_0^T (F_2(t,u_1(t),\bar u_2) - F_2(t,\bar u_1,\bar u_2))dt\Big|\\ &= \Big|\int_0^T\int_0^1 (\nabla_{x_1}F_2(t,\bar u_1 + s\tilde u_1(t),\bar u_2),\tilde u_1(t))dsdt\Big|\\ &\le \int_0^T\int_0^1 f_1(t)|\bar u_1 + s\tilde u_1(t)|^{\alpha_1}|\tilde u_1(t)|dsdt + \int_0^T\int_0^1 g_1(t)|\bar u_2|^{\beta_1}|\tilde u_1(t)|dsdt\\ &\quad + \int_0^T\int_0^1 h_1(t)|\tilde u_1(t)|dsdt\\ &\le 2^{q-1}(|\bar u_1|^{\alpha_1} + \|\tilde u_1\|_\infty^{\alpha_1})\|\tilde u_1\|_\infty\|f_1\|_{L^1} + |\bar u_2|^{\beta_1}\|\tilde u_1\|_\infty\|g_1\|_{L^1} + \|\tilde u_1\|_\infty\|h_1\|_{L^1}\\ &\le 2^{q-1}C_1^{\alpha_1+1}\|f_1\|_{L^1}\|\dot u_1\|_q^{\alpha_1+1} + 2^{q-1}\|f_1\|_{L^1}C_1|\bar u_1|^{\alpha_1}\|\dot u_1\|_q\end{aligned}$$

$$
\begin{aligned}
&+ C_1\|g_1\|_{L^1}|\bar u_2|^{\beta_1}\|\dot u_1\|_q + C_1\|h_1\|_{L^1}\|\dot u_1\|_q \\
&\le 2^{q-1}C_1^{\alpha_1+1}\|f_1\|_{L^1}\|\dot u_1\|_q^{\alpha_1+1} + \frac{1}{4q}\|\dot u_1\|_q^q + \frac{4^{q'/q}(2^{q-1}\|f_1\|_{L^1}C_1)^{q'}}{q'}|\bar u_1|^{q'\alpha_1} \\
&\quad + \frac{1}{4q}\|\dot u_1\|_q^q + \frac{4^{q'/q}(\|g_1\|_{L^1}C_1)^{q'}}{q'}|\bar u_2|^{q'\beta_1} + C_1\|h_1\|_{L^1}\|\dot u_1\|_q \\
&= \frac{1}{2q}\|\dot u_1\|_q^q + 2^{q-1}C_1^{\alpha_1+1}\|f_1\|_{L^1}\|\dot u_1\|_q^{\alpha_1+1} + \frac{4^{q'/q}(2^{q-1}\|f_1\|_{L^1}C_1)^{q'}}{q'}|\bar u_1|^{q'\alpha_1} \\
&\quad + \frac{4^{q'/q}(\|g_1\|_{L^1}C_1)^{q'}}{q'}|\bar u_2|^{q'\beta_1} + C_1\|h_1\|_{L^1}\|\dot u_1\|_q
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big|\int_0^T (F_2(t,u_1(t),u_2(t)) - F_2(t,u_1(t),\bar u_2))dt\Big| \\
&= \Big|\int_0^T\int_0^1 (\nabla_{x_2}F_2(t,u_1(t),\bar u_2 + s\tilde u_2(t)), \tilde u_2(t))ds dt\Big| \\
&\le \int_0^T\int_0^1 f_2(t)|\bar u_2 + s\tilde u_2(t)|^{\alpha_2}|\tilde u_2(t)|ds dt + \int_0^T\int_0^1 g_2(t)|u_1|^{\beta_2}|\tilde u_2(t)|ds dt \\
&\quad + \int_0^T\int_0^1 h_2(t)|\tilde u_2(t)|ds dt \\
&\le 2^{p-1}(|\bar u_2|^{\alpha_2} + \|\tilde u_2\|_\infty^{\alpha_2})\|\tilde u_2\|_\infty\|f_2\|_{L^1} + 2^{\beta_2}(|\bar u_1|^{\beta_2} + \|\tilde u_1\|_\infty^{\beta_2})\|\tilde u_2\|_\infty\|g_2\|_{L^1} \\
&\quad + \|\tilde u_2\|_\infty\|h_2\|_{L^1} \\
&\le 2^{p-1}C_1^{\alpha_2+1}\|f_2\|_{L^1}\|\dot u_2\|_p^{\alpha_2+1} + 2^{p-1}C_1\|f_2\|_{L^1}|\bar u_2|^{\alpha_2}\|\dot u_2\|_p + C_1\|h_2\|_{L^1}\|\dot u_2\|_p \\
&\quad + 2^{\beta_2}\|g_2\|_{L^1}C_1^{\beta_2+1}\|\dot u_2\|_p\|\dot u_1\|_q^{\beta_2} + 2^{\beta_2}\|g_2\|_{L^1}C_1\|\dot u_2\|_p|\bar u_1|^{\beta_2} \\
&\le 2^{p-1}C_1^{\alpha_2+1}\|f_2\|_{L^1}\|\dot u_2\|_p^{\alpha_2+1} + \frac{1}{4p}\|\dot u_2\|_p^p + \frac{4^{p'/p}(2^{p-1}\|f_2\|_{L^1}C_1)^{p'}}{p'}|\bar u_2|^{p'\alpha_2} \\
&\quad + \frac{1}{4p}\|\dot u_2\|_p^p + \frac{4^{p'/p}(2^{\beta_2}\|g_2\|_{L^1}C_1^{\beta_2+1})^{p'}}{p'}\|\dot u_1\|_q^{\beta_2 p'} + \frac{1}{4p}\|\dot u_2\|_p^p \\
&\quad + \frac{4^{p'/p}(2^{\beta_2}\|g_2\|_{L^1}C_1)^{p'}}{p'}|\bar u_1|^{\beta_2 p'} + C_1\|h_2\|_{L^1}\|\dot u_2\|_p \\
&= \frac{3}{4p}\|\dot u_2\|_p^p + 2^{p-1}C_1^{\alpha_2+1}\|f_2\|_{L^1}\|\dot u_2\|_p^{\alpha_2+1} + \frac{4^{p'/p}(2^{p-1}\|f_2\|_{L^1}C_1)^{p'}}{p'}|\bar u_2|^{p'\alpha_2} \\
&\quad + \frac{4^{p'/p}(2^{\beta_2}\|g_2\|_{L^1}C_1^{\beta_2+1})^{p'}}{p'}\|\dot u_1\|_q^{\beta_2 p'} + \frac{4^{p'/p}(2^{\beta_2}\|g_2\|_{L^1}C_1)^{p'}}{p'}|\bar u_1|^{\beta_2 p'} \\
&\quad + C_1\|h_2\|_{L^1}\|\dot u_2\|_p
\end{aligned}
$$

for all $(u_1,u_2)\in W$. So, one has

$$
\begin{aligned}
&\Big|\int_0^T (F_2(t,u_1(t),u_2(t)) - F_2(t,\bar u_1,\bar u_2))dt\Big| \\
&\le \Big|\int_0^T (F_2(t,u_1(t),\bar u_2) - F_2(t,\bar u_1,\bar u_2))dt\Big|
\end{aligned}
$$

$$
\begin{aligned}
&+\Big|\int_0^T(F_2(t,u_1(t),u_2(t))-F_2(t,u_1(t),\bar u_2))dt\Big|\\
&\le \frac{1}{2q}\|\dot u_1\|_q^q+\frac{3}{4p}\|\dot u_2\|_p^p+C_1\|h_1\|_{L^1}\|\dot u_1\|_q+C_1\|h_2\|_{L^1}\|\dot u_2\|_p\\
&+2^{q-1}C_1^{\alpha_1+1}\|f_1\|_{L^1}\|\dot u_1\|_q^{\alpha_1+1}+2^{p-1}C_1^{\alpha_2+1}\|f_2\|_{L^1}\|\dot u_2\|_p^{\alpha_2+1}\\
&+\frac{4^{q'/q}(2^{q-1}\|f_1\|_{L^1}C_1)^{q'}}{q'}|\bar u_1|^{q'\alpha_1}+\frac{4^{q'/q}(\|g_1\|_{L^1}C_1)^{q'}}{q'}|\bar u_2|^{q'\beta_1}\\
&+\frac{4^{p'/p}(2^{p-1}\|f_2\|_{L^1}C_1)^{p'}}{p'}|\bar u_2|^{p'\alpha_2}+\frac{4^{p'/p}(2^{\beta_2}\|g_2\|_{L^1}C_1)^{p'}}{p'}|\bar u_1|^{p'\beta_2}\\
&+\frac{4^{p'/p}(2^{\beta_2}\|g_2\|_{L^1}C_1^{\beta_2+1})^{p'}}{p'}\|\dot u_1\|_q^{\beta_2p'}
\end{aligned}
$$

for all $(u_1,u_2)\in W$. Hence, we obtain from (H0), (2.1) and the above expression that

$$
\begin{aligned}
\varphi(u_1,u_2)&=\frac1q\int_0^T|\dot u_1(t)|^qdt+\frac1p\int_0^T|\dot u_2(t)|^pdt+\int_0^TF_1(t,u_1(t),u_2(t))dt\\
&\quad+\int_0^T(F_2(t,u_1(t),u_2(t))-F_2(t,\bar u_1,\bar u_2))dt+\int_0^TF_2(t,\bar u_1,\bar u_2)dt\\
&\ge\frac{1}{2q}\|\dot u_1\|_q^q+\frac{1}{4p}\|\dot u_2\|_p^p-C_1\|h_1\|_{L^1}\|\dot u_1\|_q-C_1\|h_2\|_{L^1}\|\dot u_2\|_p\\
&\quad-2^{q-1}C_1^{\alpha_1+1}\|f_1\|_{L^1}\|\dot u_1\|_q^{\alpha_1+1}-2^{p-1}C_1^{\alpha_2+1}\|f_2\|_{L^1}\|\dot u_2\|_p^{\alpha_2+1}\\
&\quad-\frac{4^{p'/p}(2^{\beta_2}\|g_2\|_{L^1}C_1^{\beta_2+1})^{p'}}{p'}\|\dot u_1\|_q^{p'\beta_2}-\frac{4^{q'/q}(2^{q-1}\|f_1\|_{L^1}C_1)^{q'}}{q'}|\bar u_1|^{q'\alpha_1}\\
&\quad-\frac{4^{q'/q}(\|g_1\|_{L^1}C_1)^{q'}}{q'}|\bar u_2|^{q'\beta_1}-\frac{4^{p'/p}(2^{p-1}\|f_2\|_{L^1}C_1)^{p'}}{p'}|\bar u_2|^{p'\alpha_2}\\
&\quad-\frac{4^{p'/p}(2^{\beta_2}\|g_2\|_{L^1}C_1)^{p'}}{p'}|\bar u_1|^{\beta_2p'}+\int_0^TF_2(t,\bar u_1,\bar u_2)dt\\
&\quad+\frac1\mu\int_0^TF_1(t,\lambda\bar u_1,\lambda\bar u_2)dt-\int_0^TF_1(t,-\tilde u_1,-\tilde u_2)dt\\
&\ge\frac{1}{2q}\|\dot u_1\|_q^q+\frac{1}{4p}\|\dot u_2\|_p^p-\frac{4^{p'/p}(2^{\beta_2}\|g_2\|_{L^1}C_1^{\beta_2+1})^{p'}}{p'}\|\dot u_1\|_q^{p'\beta_2}\\
&\quad-2^{q-1}C_1^{\alpha_1+1}\|f_1\|_{L^1}\|\dot u_1\|_q^{\alpha_1+1}-2^{p-1}C_1^{\alpha_2+1}\|f_2\|_{L^1}\|\dot u_2\|_p^{\alpha_2+1}\\
&\quad-C_1\|h_1\|_{L^1}\|\dot u_1\|_q-C_1\|h_2\|_{L^1}\|\dot u_2\|_p\\
&-(2^{\beta/2+1}C_1^\beta\mu(\|\dot u_1\|_q^\beta+\|\dot u_2\|_p^\beta)+1)(a_{10}+a_{20})\int_0^Tb(t)dt\\
&\quad+(|\bar u_1|^{\gamma_1}+|\bar u_2|^{\gamma_2})(\frac{1}{|\bar u_1|^{\gamma_1}+|\bar u_2|^{\gamma_2}}(\frac1\mu\int_0^TF_1(t,\lambda\bar u_1,\lambda\bar u_2)dt\\
&\quad+\int_0^TF_2(t,\bar u_1,\bar u_2)dt)-2K)-K_0
\end{aligned}
$$

for all $(u_1, u_2) \in W$ and some positive constants K and K_0. It follows that $\varphi(u_1, u_2) \to +\infty$ as $\|(u_1, u_2)\|_W \to \infty$ due to (H2). By [4, Theorem 1.1] and Corollary 2.2, The proof is complete. $\square$

Proof of Theorem 1.4. Firstly, we prove that φ satisfies the (PS) condition. Suppose that $\{(u_{1n}, u_{2n})\}$ is a (PS) sequence for φ, that is, $\varphi'(u_{1n}, u_{2n}) \to 0$ as $n \to \infty$ and $\{\varphi(u_{1n}, u_{2n})\}$ is bounded. In a way similar to the proof of Theorem 1.1, we have

$$\begin{aligned}
&\Big|\int_0^T (\nabla_{x_1} F(t, u_{1n}(t), u_{2n}(t)), \tilde{u}_{1n}(t))dt\Big| \\
&\leq 2^{q-1} C_1^{\alpha_1+1} \|f_1\|_{L^1} \|\dot{u}_{1n}\|_q^{\alpha_1+1} + \frac{3}{4q}\|\dot{u}_{1n}\|_q^q + \frac{4^{q'/q}(2^{q-1}\|f_1\|_{L^1} C_1)^{q'}}{q'} |\bar{u}_{1n}|^{q'\alpha_1} \\
&\quad + \frac{4^{q'/q}(2^{\beta_1}\|g_1\|_{L^1} C_1^{\beta_1+1})^{q'}}{q'} \|\dot{u}_{2n}\|_p^{\beta_1 q'} + \frac{4^{q'/q}(2^{\beta_1}\|g_1\|_{L^1} C_1)^{q'}}{q'} |\bar{u}_{2n}|^{q'\beta_1} \\
&\quad + C_1 \|h_1\|_{L^1} \|\dot{u}_{1n}\|_q
\end{aligned}$$

and

$$\begin{aligned}
&\Big|\int_0^T (\nabla_{x_2} F(t, u_{1n}(t), u_{2n}(t)), \tilde{u}_{2n}(t))dt\Big| \\
&\leq 2^{p-1} C_1^{\alpha_2+1} \|f_2\|_{L^1} \|\dot{u}_{2n}\|_p^{\alpha_2+1} + \frac{3}{4p}\|\dot{u}_{2n}\|_p^p + \frac{4^{p'/p}(2^{p-1}\|f_2\|_{L^1} C_1)^{p'}}{p'} |\bar{u}_{2n}|^{p'\alpha_2} \\
&\quad + \frac{4^{p'/p}(2^{\beta_2}\|g_2\|_{L^1} C_1^{\beta_2+1})^{p'}}{p'} \|\dot{u}_{1n}\|_q^{\beta_2 p'} + \frac{4^{p'/p}(2^{\beta_2}\|g_2\|_{L^1} C_1)^{p'}}{p'} |\bar{u}_{1n}|^{p'\beta_2} \\
&\quad + C_1 \|h_2\|_{L^1} \|\dot{u}_{2n}\|_p
\end{aligned}$$

for all n. Hence, one has

$$\begin{aligned}
&\|(\tilde{u}_{1n}, \tilde{u}_{2n})\|_W \\
&\geq \langle \varphi'(u_{1n}, u_{2n}), (\tilde{u}_{1n}, \tilde{u}_{2n})\rangle \\
&= \int_0^T ((\nabla_{x_1} F(t, u_{1n}(t), u_{2n}(t)), \tilde{u}_{1n}(t)) + (|\dot{u}_{1n}(t)|^{q-2}\dot{u}_{1n}(t), \dot{u}_{1n}(t)) \\
&\quad + (\nabla_{x_2} F(t, u_{1n}(t), u_{2n}(t)), \tilde{u}_{2n}(t)) + (|\dot{u}_{2n}(t)|^{p-2}\dot{u}_{2n}(t), \dot{u}_{2n}(t)))dt \\
&\geq \frac{4q-3}{4q}\|\dot{u}_{1n}\|_q^q + \frac{4p-3}{4p}\|\dot{u}_{2n}\|_p^p - 2^{q-1} C_1^{\alpha_1+1}\|f_1\|_{L^1}\|\dot{u}_{1n}\|_q^{\alpha_1+1} \\
&\quad - \frac{4^{q'/q}(2^{\beta_1}\|g_1\|_{L^1} C_1^{\beta_1+1})^{q'}}{q'} \|\dot{u}_{2n}\|_p^{\beta_1 q'} - \frac{4^{q'/q}(2^{q-1}\|f_1\|_{L^1} C_1)^{q'}}{q'} |\bar{u}_{1n}|^{q'\alpha_1} \\
&\quad - \frac{4^{q'/q}(2^{\beta_1}\|g_1\|_{L^1} C_1)^{q'}}{q'} |\bar{u}_{2n}|^{q'\beta_1} - 2^{p-1} C_1^{\alpha_2+1}\|f_2\|_{L^1}\|\dot{u}_{2n}\|_p^{\alpha_2+1} \\
&\quad - \frac{4^{p'/p}(2^{\beta_2}\|g_2\|_{L^1} C_1^{\beta_2+1})^{p'}}{p'} \|\dot{u}_{1n}\|_q^{\beta_2 p'} - \frac{4^{p'/p}(2^{p-1}\|f_2\|_{L^1} C_1)^{p'}}{p'} |\bar{u}_{2n}|^{p'\alpha_2} \\
&\quad - \frac{4^{p'/p}(2^{\beta_2}\|g_2\|_{L^1} C_1)^{p'}}{p'} |\bar{u}_{1n}|^{p'\beta_2} - C_1\|h_2\|_{L^1}\|\dot{u}_{2n}\|_p - C_1\|h_1\|_{L^1}\|\dot{u}_{1n}\|_q
\end{aligned} \tag{2.2}$$

for large n. It follows from Wirtinger's inequality that

$$\begin{aligned}\|(\tilde u_{1n},\tilde u_{2n})\|_W &= \|\tilde u_{1n}\|_{W_T^{1,q}}+\|\tilde u_{2n}\|_{W_T^{1,p}}\\ &\le (1+C_2^q)^{1/q}\|\dot u_{1n}\|_q+(1+C_2^p)^{1/p}\|\dot u_{2n}\|_p\\ &\le \max\big\{(1+C_2^q)^{1/q},(1+C_2^p)^{1/p}\big\}\big(\|\dot u_{1n}\|_q+\|\dot u_{2n}\|_p\big)\end{aligned}\tag{2.3}$$

for all n. So, it follows from (2.2) and (2.3) that

$$\begin{aligned}&K(|\bar u_{1n}|^{p'\beta_2}+|\bar u_{2n}|^{p'\alpha_2}+|\bar u_{1n}|^{q'\alpha_1}+|\bar u_{2n}|^{q'\beta_1})\\
&\ge \frac{4^{q'/q}(2^{q-1}\|f_1\|_{L^1}C_1)^{q'}}{q'}|\bar u_{1n}|^{q'\alpha_1}+\frac{4^{q'/q}(2^{\beta_1}\|g_1\|_{L^1}C_1)^{q'}}{q'}|\bar u_{2n}|^{q'\beta_1}\\
&\quad+\frac{4^{p'/p}(2^{p-1}\|f_2\|_{L^1}C_1)^{p'}}{p'}|\bar u_{2n}|^{p'\alpha_2}+\frac{4^{p'/p}(2^{\beta_2}\|g_2\|_{L^1}C_1)^{p'}}{p'}|\bar u_{1n}|^{p'\beta_2}\\
&\ge \frac{4q-3}{4q}\|\dot u_{1n}\|_q^q+\frac{4p-3}{4p}\|\dot u_{2n}\|_p^p\\
&\quad-2^{q-1}C_1^{\alpha_1+1}\|f_1\|_{L^1}\|\dot u_{1n}\|_q^{\alpha_1+1}-\frac{4^{q'/q}(2^{\beta_1}\|g_1\|_{L^1}C_1^{\beta_1+1})^{q'}}{q'}\|\dot u_{2n}\|_p^{\beta_1q'}\\
&\quad-2^{p-1}C_1^{\alpha_2+1}\|f_2\|_{L^1}\|\dot u_{2n}\|_p^{\alpha_2+1}-\frac{4^{p'/p}(2^{\beta_2}\|g_2\|_{L^1}C_1^{\beta_2+1})^{p'}}{p'}\|\dot u_{1n}\|_q^{\beta_2p'}\\
&\quad-C_1\|h_2\|_{L^1}\|\dot u_{2n}\|_p-C_1\|h_1\|_{L^1}\|\dot u_{1n}\|_q-\|(\tilde u_{1n},\tilde u_{2n})\|_W\\
&\ge \frac{4q-3}{4q}\|\dot u_{1n}\|_q^q+\frac{4p-3}{4p}\|\dot u_{2n}\|_p^p-C_1\|h_2\|_{L^1}\|\dot u_{2n}\|_p-C_1\|h_1\|_{L^1}\|\dot u_{1n}\|_q\\
&\quad-2^{q-1}C_1^{\alpha_1+1}\|f_1\|_{L^1}\|\dot u_{1n}\|_q^{\alpha_1+1}-\frac{4^{q'/q}(2^{\beta_1}\|g_1\|_{L^1}C_1^{\beta_1+1})^{q'}}{q'}\|\dot u_{2n}\|_p^{\beta_1q'}\\
&\quad-2^{p-1}C_1^{\alpha_2+1}\|f_2\|_{L^1}\|\dot u_{2n}\|_p^{\alpha_2+1}-\frac{4^{p'/p}(2^{\beta_2}\|g_2\|_{L^1}C_1^{\beta_2+1})^{p'}}{p'}\|\dot u_{1n}\|_q^{\beta_2p'}\\
&\quad-(1+C_2^p)^{1/p}\|\dot u_{2n}\|_p-(1+C_2^q)^{1/q}\|\dot u_{1n}\|_q\\
&\ge \frac{q-1}{q}\|\dot u_{1n}\|_q^q+\frac{p-1}{p}\|\dot u_{2n}\|_p^p-K_1\\
&=\frac{1}{q'}\|\dot u_{1n}\|_q^q+\frac{1}{p'}\|\dot u_{2n}\|_p^p-K_1\end{aligned}$$

for large n and some positive constant K_1. Hence, by the above expression, we obtain

$$2K(|\bar u_{1n}|^{\gamma_1}+|\bar u_{2n}|^{\gamma_2})\ge \min\big\{\frac{1}{q'},\frac{1}{p'}\big\}\big(\|\dot u_{1n}\|_q^q+\|\dot u_{2n}\|_p^p\big)-K_2\tag{2.4}$$

for large n and some positive constant K_2. By the proof of Theorem 1.1, we have

$$\begin{aligned}&\Big|\int_0^T(F(t,u_{1n}(t),u_{2n}(t))-F(t,\bar u_{1n},\bar u_{2n}))dt\Big|\\
&\le \frac{1}{2q}\|\dot u_{1n}\|_q^q+\frac{3}{4p}\|\dot u_{2n}\|_p^p+C_1\|h_1\|_{L^1}\|\dot u_{1n}\|_q+C_1\|h_2\|_{L^1}\|\dot u_{2n}\|_p\\
&\quad+2^{q-1}C_1^{\alpha_1+1}\|f_1\|_{L^1}\|\dot u_{1n}\|_q^{\alpha_1+1}+2^{p-1}C_1^{\alpha_2+1}\|f_2\|_{L^1}\|\dot u_{2n}\|_p^{\alpha_2+1}\\
&\quad+\frac{4^{q'/q}(2^{q-1}\|f_1\|_{L^1}C_1)^{q'}}{q'}|\bar u_{1n}|^{q'\alpha_1}+\frac{4^{q'/q}(\|g_1\|_{L^1}C_1)^{q'}}{q'}|\bar u_{2n}|^{q'\beta_1}\end{aligned}$$

$$
\begin{aligned}
&+\frac{4^{p'/p}(2^{p-1}\|f_2\|_{L^1}C_1)^{p'}}{p'}|\bar u_{2n}|^{p'\alpha_2}+\frac{4^{p'/p}(2^{\beta_2}\|g_2\|_{L^1}C_1)^{p'}}{p'}|\bar u_{1n}|^{p'\beta_2}\\
&+\frac{4^{p'/p}(2^{\beta_2}\|g_2\|_{L^1}C_1^{\beta_2+1})^{p'}}{p'}\|\dot u_{1n}\|_q^{\beta_2 p'}
\end{aligned}
$$

for all n. It follows from the boundedness of $\{\varphi(u_{1n},u_{2n})\}$, (2.4) and the above inequality that

$$
\begin{aligned}
K_3 &\le \varphi(u_{1n},u_{2n})\\
&=\frac{1}{q}\int_0^T|\dot u_{1n}(t)|^q dt+\frac{1}{p}\int_0^T|\dot u_{2n}(t)|^p dt\\
&\quad+\int_0^T\big[F(t,u_{1n}(t),u_{2n}(t))-F(t,\bar u_{1n},\bar u_{2n})\big]dt+\int_0^T F(t,\bar u_{1n},\bar u_{2n})dt\\
&\le \frac{3}{2q}\|\dot u_{1n}\|_q^q+\frac{7}{4p}\|\dot u_{2n}\|_p^p+C_1\|h_1\|_{L^1}\|\dot u_{1n}\|_q+C_1\|h_2\|_{L^1}\|\dot u_{2n}\|_p\\
&\quad+2^{q-1}C_1^{\alpha_1+1}\|f_1\|_{L^1}\|\dot u_{1n}\|_q^{\alpha_1+1}+2^{p-1}C_1^{\alpha_2+1}\|f_2\|_{L^1}\|\dot u_{2n}\|_p^{\alpha_2+1}\\
&\quad+\frac{4^{q'/q}(2^{q-1}\|f_1\|_{L^1}C_1)^{q'}}{q'}|\bar u_{1n}|^{q'\alpha_1}+\frac{4^{q'/q}(\|g_1\|_{L^1}C_1)^{q'}}{q'}|\bar u_{2n}|^{q'\beta_1}\\
&\quad+\frac{4^{p'/p}(2^{p-1}\|f_2\|_{L^1}C_1)^{p'}}{p'}|\bar u_{2n}|^{p'\alpha_2}+\frac{4^{p'/p}(2^{\beta_2}\|g_2\|_{L^1}C_1)^{p'}}{p'}|\bar u_{1n}|^{p'\beta_2}\\
&\quad+\frac{4^{p'/p}(2^{\beta_2}\|g_2\|_{L^1}C_1^{\beta_2+1})^{p'}}{p'}\|\dot u_{1n}\|_q^{\beta_2 p'}+\int_0^T F(t,\bar u_{1n},\bar u_{2n})dt\\
&\le 2(\|\dot u_{1n}\|_q^q+\|\dot u_{2n}\|_p^p)+2K(|\bar u_{1n}|^{\gamma_1}+|\bar u_{2n}|^{\gamma_2})+\int_0^T F(t,\bar u_{1n},\bar u_{2n})dt+K_4\\
&\le (2\max\{q',p'\}+1)2K(|\bar u_{1n}|^{\gamma_1}+|\bar u_{2n}|^{\gamma_2})+\int_0^T F(t,\bar u_{1n},\bar u_{2n})dt+K_5\\
&\le (2q'+2p'+1)2K(|\bar u_{1n}|^{\gamma_1}+|\bar u_{2n}|^{\gamma_2})+\int_0^T F(t,\bar u_{1n},\bar u_{2n})dt+K_5\\
&\le (|\bar u_{1n}|^{\gamma_1}+|\bar u_{2n}|^{\gamma_2})\Big(\frac{1}{|\bar u_{1n}|^{\gamma_1}+|\bar u_{2n}|^{\gamma_2}}\int_0^T F(t,\bar u_{1n},\bar u_{2n})dt\\
&\quad+(2q'+2p'+1)2K\Big)+K_5
\end{aligned}
$$

for large n and some real constants K_3, K_4 and K_5. The above inequality and (H3) imply that $(|\bar u_{1n}|^{\gamma_1}+|\bar u_{2n}|^{\gamma_2})$ is bounded. Hence, (u_{1n},u_{2n}) is bounded by (2.3) and (2.4). By the compactness of the embedding $W_T^{1,p}$(or $W_T^{1,q}$) $\subset C(0,T;\mathbb{R}^N)$, the sequence $\{u_{1n}\}$ (or $\{u_{2n}\}$) has a subsequence, still denoted by $\{u_{1n}\}$ (or $\{u_{2n}\}$), such that

$$u_{1n}\ (\text{or } u_{2n})\ \rightharpoonup u_1\ (\text{or } u_2)\quad \text{weakly in } W_T^{1,p}\ (\text{or in } W_T^{1,q}), \tag{2.5}$$

$$u_{1n}\ (\text{or } u_{2n})\ \to u_1\ (\text{or } u_2)\quad \text{strongly in } C(0,T;\mathbb{R}^N). \tag{2.6}$$

Note that

$$\begin{aligned}
&\langle \varphi'(u_{1n},u_{2n}),(u_1-u_{1n},0)\rangle\\
&=\int_0^T |\dot u_{1n}(t)|^{p-2}(\dot u_{1n}(t),\dot u_1-\dot u_{1n}(t))dt\\
&\quad-\int_0^T (\nabla_{x_1}F(t,u_{1n}(t),u_{2n}(t)),u_1(t)-u_{1n}(t))dt\to 0
\end{aligned}\tag{2.7}$$

as $n\to\infty$. From (2.6), $\{u_{1n}\}$ is bounded in $C(0,T;\mathbb{R}^N)$. Then we have

$$\begin{aligned}
&\Big|\int_0^T (\nabla_{x_1}F(t,u_{1n}(t),u_{2n}(t)),u_1(t)-u_{1n}(t))dt\Big|\\
&\le \int_0^T |\nabla_{x_1}F(t,u_{1n}(t),u_{2n}(t))|\cdot|u_1(t)-u_{1n}(t)|dt\\
&\le K_6\int_0^T b(t)|u_1(t)-u_{1n}(t)|dt\\
&\le K_6\|b\|_{L^1}\|u_1-u_{1n}\|_\infty
\end{aligned}$$

for some positive constant K_6, which combines with (2.6) implies that

$$\int_0^T (\nabla_{x_1}F(t,u_{1n}(t),u_{2n}(t)),u_1(t)-u_{1n}(t))dt\to 0 \quad\text{as } n\to\infty.$$

Hence, by (2.7), one has

$$\int_0^T |\dot u_{1n}(t)|^{p-2}(\dot u_{1n}(t),\dot u_1(t)-\dot u_{1n}(t))dt\to 0\quad\text{as } n\to\infty\ .$$

Moreover, from (2.6) we obtain

$$\int_0^T |u_{1n}(t)|^{p-2}(u_{1n}(t),u_1(t)-u_{1n}(t))dt\to 0\quad\text{as } n\to\infty\,.$$

Setting

$$\psi(u_1,u_2)=\frac1p\int_0^T(|u_1(t)|^p+|\dot u_1(t)|^p)dt+\frac1q\int_0^T(|u_2(t)|^q+|\dot u_2(t)|^q)dt,$$

one obtains

$$\begin{aligned}
\langle \psi'(u_{1n},u_{2n}),(u_1-u_{1n},0)\rangle&=\int_0^T |u_{1n}(t)|^{p-2}(u_{1n}(t),u_1(t)-u_{1n}(t))dt\\
&\quad+\int_0^T |\dot u_{1n}(t)|^{p-2}(\dot u_{1n}(t),\dot u_1(t)-\dot u_{1n}(t))dt
\end{aligned}$$

and

$$\langle \psi'(u_{1n},u_{2n}),(u_1-u_{1n},0)\rangle\to 0\ \text{ as } n\to\infty\,. \tag{2.8}$$

By the Hölder's inequality, we have

$$0\le(\|u_{1n}\|^{p-1}-\|u_1\|^{p-1})(\|u_{1n}\|-\|u_1\|)\le\langle \psi'(u_{1n},u_{2n})-\psi'(u_1,u_2),(u_1-u_{1n},0)\rangle,$$

which together with (2.8) yields $\|u_{1n}\|\to\|u_1\|$. It follows that $u_{1n}\to u_1$ strongly in $W_T^{1,p}$ by the uniform convexity of $W_T^{1,p}$. Similarly, we have $u_{2n}\to u_2$ strongly in $W_T^{1,q}$. Hence, the (PS) condition is satisfied.

Let $\widetilde{W}=\widetilde{W}_T^{1,q}\times\widetilde{W}_T^{1,p}$ be the subspace of W given by

$$\widetilde{W}=\{(u_1,u_2)\in W \ \mid \ (\bar{u}_1,\bar{u}_2)=(0,0)\}.$$

Then

$$\varphi(u_1,u_2)\to+\infty \tag{2.9}$$

as $\|(u_1,u_2)\|_W\to\infty$ in $\widetilde{W}$. In fact, by the proof of Theorem 1.1, one has

$$\begin{aligned}
\varphi(u_1,u_2)&=\frac{1}{q}\int_0^T|\dot{u}_1(t)|^q dt+\frac{1}{p}\int_0^T|\dot{u}_2(t)|^p dt\\
&\quad+\int_0^T(F(t,u_1(t),u_2(t))-F(t,\bar{u}_1,\bar{u}_2))dt+\int_0^T F(t,\bar{u}_1,\bar{u}_2)dt\\
&\geq\frac{1}{2q}\|\dot{u}_1\|_q^q+\frac{1}{4p}\|\dot{u}_2\|_p^p-2^{q-1}C_1^{\alpha_1+1}\|f_1\|_{L^1}\|\dot{u}_1\|_q^{\alpha_1+1}\\
&\quad-2^{p-1}C_1^{\alpha_2+1}\|f_2\|_{L^1}\|\dot{u}_2\|_p^{\alpha_2+1}-\frac{4^{p'/p}(2^{\beta_2}\|g_2\|_{L^1}C_1^{\beta_2+1})^{p'}}{p'}\|\dot{u}_1\|_q^{\beta_2p'}\\
&\quad-C_1\|h_1\|_{L^1}\|\dot{u}_1\|_q-C_1\|h_2\|_{L^1}\|\dot{u}_2\|_p+\int_0^T F(t,\bar{u}_1,\bar{u}_2)dt
\end{aligned}$$

for all $(u_1,u_2)\in\widetilde{W}$. By Wirtinger's inequality, the norm

$$\||(u_1,u_2)\||=\|(\dot{u}_1,\dot{u}_2)\|_{L^q\times L^p}=\|\dot{u}_1\|_q+\|\dot{u}_2\|_p$$

is an equivalent norm on $\widetilde{W}$. Hence, (2.9) follows from the above inequality.

On the other hand, one has

$$\varphi(x_1,x_2)\to-\infty \tag{2.10}$$

as $|(x_1,x_2)|\to\infty$ in $\mathbb{R}^N\times\mathbb{R}^N$, which follows from (H3). Now, Theorem 1.4 is proved by (2.9), (2.10) and the Saddle Point Theorem (see [11, Theorem 4.6]). □

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