SOLUTION OF THE KDV EQUATION WITH FRACTIONAL TIME DERIVATIVE VIA VARIATIONAL METHOD

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ABSTRACT. This article presents a formulation of the time-fractional generalized Korteweg-de Vries (KdV) equation using the Euler-Lagrange variational technique in the Riemann-Liouville derivative sense. It finds an approximate solitary wave solution, and shows that He's variational iteration method is an efficient technique in finding the solution.

1. INTRODUCTION

During the past three decades or so, fractional calculus has obtained considerable popularity and importance as generalizations of integer-order evolution equation, and used to model some meaningful things, such as fractional calculus can model price volatility in finance [20, 44], in hydrology to model fast spreading of pollutants [48], the most common hydrologic application of fractional calculus is the generation of fractional Brownian motion as a representation of aquifer material with long-range correlation structure [8, 42]. Fractional differential equation is used to to model the particle motions in a heterogeneous environment and long particle jumps of the anomalous diffusion in physics [26, 33]. Other exact description of the applications of engineering, mechanics and mathematics et al., we can refer to [31, 34, 43, 47, 52]. If the Lagrangian of conservative system is constructed using fractional derivatives, the resulting equation of motion can be nonconservative. Therefore, in many cases, the real physical processes could be modeled in a reliable manner using fractional-order differential equation rather than integer-order equation [50]. Based on the stochastic embedding theory, Cresson [13] defined the fractional embedding of differential operators and provided a fractional Euler-Lagrange equation for Lagrangian systems, then investigated a fractional Noether-type theorem and a fractional Hamiltonian formulation of fractional Lagrangian systems. The fractional Noether-type theorem was proved by Frederico and Torres [18]. For the discussion of fractional constants of motion see also [14], and a more general version of Noether-type's theorem, valid for fractional problems of optimal control, in the Riemann-Liouville sense, can be found in [19].

The first necessary conditions of Euler-Lagrange were proved by Riewe in references [40, 41], and the first to obtain sufficient optimality conditions for the Euler-Lagrange fractional equation were Almeida and Torres in [4]. Herzallah and Baleanu [25] presented the necessary and sufficient optimality conditions for the Euler-Lagrange fractional equation of fractional variational problems, hereof the first discussion about the space of functions where fractional variational problems should be defined, in order to guarantee existence of solutions, is given in [9]. Euler-Lagrange equation for fractional variational problems with multiple integrals were studied before in [5, 15]. Malinowska [32] proves a fractional Noether-type theorem for multidimensional Lagrangians and proved the fractional Noether-type theorem for conservative and nonconservative generalized physical systems. Wu and Baleanu [51] developed some new variational iteration formulae to find approximate solutions of fractional differential equation and determined the Lagrange multiplier in a more accurate way. For generalized fractional Euler-Lagrange equation and fractional order Van der Pol-like oscillator, we can refer to the works by Odzijewicz [37, 38], Attari et al [6] respectively. Other the known results we can see Baleanu et al [7] and Inokuti et al [27]. In view of most of physical phenomena may be considered as nonconservative, then they can be described using fractional-order differential equation. Recently, several methods have been used to solve nonlinear fractional evolution equation using techniques of nonlinear analysis, such as Adomian decomposition method [45], homotopy analysis method [12, 30] and homotopy perturbation method [49]. It was mentioned that the variational iteration method has been used successfully to solve different types of integer and fractional nonlinear evolution equation.

The KdV equation has been used to describe a wide range of physics phenomena of the evolution and interaction to nonlinear waves. It was derived from the propagation of dispersive shallow water waves and is widely used in fluid dynamics, aerodynamics, continuum mechanics, as a model for shock wave formation, solitons, turbulence, boundary layer behavior, mass transport, and the solution representing the water's free surface over a flat bed [11, 28, 17]. Camassa and Holm [10] put forward the derivation of solution as a model for dispersive shallow water waves and discovered that it is formally integrable dimensional Hamiltonian system, and that its solitary waves are solitons. Most of classical mechanics techniques have studied conservative systems, but almost of the processes observed in the physical real world are nonconservative. In present paper, He's variational iteration method [21, 22, 35, 36] is applied to solve time-fractional generalized KdV equation

$$
{}_0^R D_t^\alpha u(x,t) + a u^p(x,t) u_x(x,t) + b u_{xxx}(x,t) = 0,
$$

where a, b are constants, $u(x,t)$ is a field variable, the subscripts denote the partial differentiation of the function $u(x,t)$ with respect to the parameter x and t. $x \in \Omega(\Omega \subseteq \mathbb{R})$ is a space coordinate in the propagation direction of the field and $t \in T(= [0, t_0](t_0 > 0))$ is the time, which occur in different contexts in mathematical physics. a, b are constant coefficients and not equal to zero. The dissipative u_{xxx} term provides damping at small scales, and the non-linear term $u^p u_x$ $(p > 0)$ (which has the same form as that in the KdV or one-dimensional Navier-Stokes equation) stabilizes by transferring energy between large and small scales. For $p = 1$, we can refer to the known results of time-fractional KdV equation: formulation and solution using variational methods [16]. For $p > 0$, $p \neq 1$, there is a few of the formulation and solution to time-fractional KdV equation. Thus the present paper

considers that the formulation and solution to time-fractional KdV equation as the index of the nonlinear term satisfies $p > 0$, $p \neq 1$. $_0^R D_t^\alpha$ denotes the Riesz fractional derivative. Making use of the variational iteration method, this work motivation is devoted to formulate a time-fractional generalized KdV equation and derives an approximate solitary wave solution.

This paper is organized as follows: Section 2 states some background material from fractional calculus. Section 3 presents the principle of He's variational iteration method. Section 4 is devoted to describe the formulation of the time-fractional generalized KdV equation using the Euler-Lagrange variational technique and to derive an approximate solitary wave solution. Section 5 makes some analysis for the obtained graphs and discusses the present work.

2. Preliminaries

We recall the necessary definitions for the fractional calculus (see [29, 39, 46]) which is used throughout the remaining sections of this paper.

Definition 2.1. A real multivariable function $f(x,t)$, $t > 0$ is said to be in the space C_γ, $\gamma \in \mathbb{R}$ with respect to t if there exists a real number $r(> \gamma)$, such that $f(x,t) = t^r f_1(x,t)$, where $f_1(x,t) \in C(\Omega \times T)$. Obviously, $C_\gamma \subset C_\delta$ if $\delta \leq \gamma$.

Definition 2.2. The left-hand side Riemann-Liouville fractional integral of a function $f \in C_\gamma$, $(\gamma \geq -1)$ is defined by

$$_0 I_t^\alpha f(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(x,\tau)}{(t-\tau)^{1-\alpha}} d\tau, \quad \alpha > 0, \quad t \in T,$$
$$_0 I_t^0 f(x,t) = f(x,t).$$

Definition 2.3. The Riemann-Liouville fractional derivatives of the order α, with $n - 1 \leq \alpha < n$, of a function $f \in C_\gamma$, $(\gamma \geq -1)$ are defined as

$$_0 D_t^\alpha f(x,t) = \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial t^n} \int_0^t \frac{f(x,\tau)}{(t-\tau)^{\alpha+1-n}} d\tau,$$

$$_t D_{t_0}^\alpha f(x,t) = \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial t^n} \int_t^{t_0} \frac{f(x,\tau)}{(\tau-t)^{\alpha+1-n}} d\tau, \quad t \in T.$$

Lemma 2.4. *The integration formula of Riemann-Liouville fractional derivative, for order $0 < \alpha < 1$,*

$$\int_T f(x,t)\, _0 D_t^\alpha g(x,t) dt = \int_T g(x,t)\, _t D_{t_0}^\alpha f(x,t) dt$$

is valid under the assumption that $f,g \in C(\Omega \times T)$ and that for arbitrary $x \in \Omega$, $_t D_{t_0}^\alpha f$, $_0 D_t^\alpha g$ exist at every point $t \in T$ and are continuous in t.

Definition 2.5. The Riesz fractional integral of order α, $n - 1 \leq \alpha < n$, of a function $f \in C_\gamma$, $(\gamma \geq -1)$ is defined as

$$_0^R I_t^\alpha f(x,t) = \frac{1}{2}\big(_0 I_t^\alpha f(x,t) + \,_t I_{t_0}^\alpha f(x,t)\big) = \frac{1}{2\Gamma(\alpha)} \int_0^{t_0} |t-\tau|^{\alpha-1} f(x,\tau) d\tau,$$

where $_0 I_t^\alpha$ and $_t I_{t_0}^\alpha$ are respectively the left- and right-hand side Riemann-Liouville fractional integral operators.

Definition 2.6. The Riesz fractional derivative of the order α, $n - 1 \leq \alpha < n$ of a function $f \in C_\gamma$, $(\gamma \geq -1)$ is defined by

$$
\begin{aligned}
{}^R_0 D^\alpha_t f(x,t) &= \frac{1}{2}\big({}_0 D^\alpha_t f(x,t) + (-1)^n {}_t D^\alpha_{t_0} f(x,t)\big) \\
&= \frac{1}{2\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^{t_0} |t-\tau|^{n-\alpha-1} f(x,\tau) d\tau,
\end{aligned}
$$

where ${}_0 D^\alpha_t$ and ${}_t D^\alpha_{t_0}$ are respectively the left- and right-hand side Riemann-Liouville fractional differential operators.

Lemma 2.7. *Let $\alpha > 0$ and $\beta > 0$ be such that $n - 1 < \alpha < n$, $m - 1 < \beta < m$ and $\alpha + \beta < n$, and let $f \in L_1(\Omega \times T)$ and $_0 I^{m-\alpha}_t f \in AC^m(\Omega \times T)$. Then we have the following index rule:*

$$
{}^R_0 D^\alpha_t \big({}^R_0 D^\beta_t f(x,t)\big) = {}^R_0 D^{\alpha+\beta}_t f(x,t) - \sum_{i=1}^m {}^R_0 D^{\beta-i}_t f(x,t)|_{t=0} \frac{t^{-\alpha-i}}{\Gamma(1-\alpha-i)}.
$$

Remark 2.8. One can express the Riesz fractional differential operator ${}^R_0 D^{\alpha-1}_t$ of the order $0 < \alpha < 1$ as the Riesz fractional integral operator ${}^R_0 I^{1-\alpha}_\tau$, i.e.

$$
{}^R_0 D^{\alpha-1}_t f(x,t) = {}^R_0 I^{1-\alpha}_t f(x,t), \quad t \in T.
$$

3. Variational iteration method

The variational iteration method provides an effective procedure for explicit and solitary wave solutions of a wide and general class of differential systems representing real physical problems. Moreover, the variational iteration method can overcome the foregoing restrictions and limitations of approximate techniques so that it provides us with a possibility to analyze strongly nonlinear evolution equation. Therefore, we extend this method to solve the time-fractional KdV equation. The basic features of the variational iteration method outlined as follows.

Considering a nonlinear evolution equation that consists of a linear part $\mathcal{L}u(x,t)$, nonlinear part $\mathcal{N}u(x,t)$, and a free term $g(x,t)$ represented as

$$
\mathcal{L}u(x,t) + \mathcal{N}u(x,t) = g(x,t). \tag{3.1}
$$

According to the variational iteration method, the $n + 1$-th approximate solution of (3.1) can be read using iteration correction functional as

$$
u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\tau)\big(\mathcal{L}\tilde{u}(x,\tau) + \mathcal{N}\tilde{u}(x,\tau) - g(x,\tau)\big) d\tau, \tag{3.2}
$$

where $\lambda(\tau)$ is a Lagrangian multiplier and $\tilde{u}(x,t)$ is considered as a restricted variation function, i.e., $\delta\tilde{u}(x,t) = 0$. Extreming the variation of the correction functional (3.2) leads to the Lagrangian multiplier $\lambda(\tau)$. The initial iteration $u_0(x,t)$ can be used as the initial value $u(x,0)$. As n tends to infinity, the iteration leads to the solitary wave solution of (3.1), i.e.

$$
u(x,t) = \lim_{n \to \infty} u_n(x,t).
$$

4. Time fractional generalized KdV equation

The generalized KdV equation in (1+1) dimensions is given as

$$u_t(x,t) + au^p(x,t)u_x(x,t) + bu_{xxx}(x,t) = 0, \tag{4.1}$$

where $p > 0$ a, b are constants, $u(x,t)$ is a field variable, $x \in \Omega$ is a space coordinate in the propagation direction of the field and $t \in T$ is the time. Employing a potential function $v(x,t)$ on the field variable, set $u(x,t) = v_x(x,t)$ yields the potential equation of the generalized KdV equation (4.1) in the form,

$$v_{xt}(x,t) + av_x^p(x,t)v_{xx}(x,t) + bv_{xxxx}(x,t) = 0. \tag{4.2}$$

The Lagrangian of this generalized KdV equation (4.1) can be defined using the semi-inverse method [23, 24] as follows. The functional of the potential equation (4.2) can be represented as

$$J(v) = \int_\Omega dx \int_T \Big(v(x,t)\big(c_1 v_{xt}(x,t) + c_2 av_x^p(x,t)v_{xx}(x,t) + c_3 bv_{xxxx}(x,t)\big)\Big)dt, \tag{4.3}$$

with $c_i(i = 1, 2, 3)$ is unknown constant to be determined later. Integrating (4.3) by parts and taking $v_t|_\Omega = v_x|_\Omega = v_x|_T = 0$ yield

$$J(v) = \int_\Omega dx \int_T \Big(-c_1 v_x(x,t)v_t(x,t) - \frac{c_2 a}{p+1}v_x^{p+2}(x,t) - c_3 bv_x(x,t)v_{xxx}(x,t)\Big)dt. \tag{4.4}$$

The constants $c_i(i = 1, 2, \ldots, 6)$ can be determined taking the variation of the functional (4.4) to make it optimal. By applying the variation of the functional, integrating each term by parts, and making use of the variation optimum condition of the functional $J(v)$, it yields the following expression

$$-2c_1 v_{xt}(x,t) - c_2 a(p+2)v_x^p(x,t)v_{xx}(x,t) - 2c_3 bv_{xxxx}(x,t) = 0. \tag{4.5}$$

We notice that the obtained result (4.5) is equivalent to (4.2), so one has that the constants $c_i(i = 1, 2, \ldots, 6)$ are respectively

$$c_1 = -\frac{1}{2}, \quad c_2 = -\frac{1}{p+2}, \quad c_3 = -\frac{1}{2}.$$

In addition, the functional expression given by (4.4) obtains directly the Lagrangian form of the generalized KdV equation,

$$L(v_t, v_x, v_{xxx}) = \frac{1}{2}v_x(x,t)v_t(x,t) + \frac{a}{(p+1)(p+2)}v_x^{p+2}(x,t) + \frac{b}{2}v_x(x,t)v_{xxx}(x,t).$$

Similarly, the Lagrangian of the time-fractional version of the generalized KdV equation could be read as

$$\begin{aligned} &F({}_0D_t^\alpha v, v_x, v_{xxx}) \\ &= \frac{1}{2}v_x(x,t){}_0D_t^\alpha v(x,t) + \frac{a}{(p+1)(p+2)}v_x^{p+2}(x,t) + \frac{b}{2}v_x(x,t)v_{xxx}(x,t), \end{aligned} \tag{4.6}$$

where $\alpha \in]0, 1]$. Then the functional of the time-fractional generalized KdV equation will take the expression

$$J(v) = \int_\Omega dx \int_T F({}_0D_t^\alpha v, v_x, v_{xxx})dt, \tag{4.7}$$

where the time-fractional Lagrangian $F(_0D_t^\alpha v, v_x, v_{xxx})$ is given by (4.6). Following Agrawal's method [1, 2, 3], the variation of functional (4.7) with respect to $v(x,t)$ leads to

$$\delta J(v) = \int_\Omega dx \int_T \Big(\frac{\partial F}{\partial {_0D_t^\alpha v}} \delta(_0D_t^\alpha v(x,t)) + \frac{\partial F}{\partial v_x} \delta v_x(x,t) + \frac{\partial F}{\partial v_{xxx}} \delta v_{xxx}(x,t) \Big) dt. \tag{4.8}$$

By Lemma 2.4, upon integrating the right-hand side of (4.8), one has

$$\delta J(v) = \int_\Omega dx \int_T \Big({_tD_T^\alpha}\big(\frac{\partial F}{\partial {_0D_t^\alpha v}}\big) - \frac{\partial}{\partial x}\big(\frac{\partial F}{\partial v_x}\big) - \frac{\partial^3}{\partial x^3}\big(\frac{\partial F}{\partial v_{xxx}}\big) \Big) \delta v\, dt,$$

noting that $\delta v|_T = \delta v|_\Omega = \delta v_x|_\Omega = 0$.

Obviously, optimizing the variation of the functional $J(v)$, i.e., $\delta J(v) = 0$, yields the Euler-Lagrange equation for time-fractional generalized KdV equation in the following expression

$$_tD_T^\alpha\big(\frac{\partial F}{\partial {_0D_t^\alpha v}}\big) - \frac{\partial}{\partial x}\big(\frac{\partial F}{\partial v_x}\big) - \frac{\partial^3}{\partial x^3}\big(\frac{\partial F}{\partial v_{xxx}}\big) = 0. \tag{4.9}$$

Substituting the Lagrangian of the time-fractional generalized KdV equation (4.6) into Euler-Lagrange formula (4.9) obtains

$$_tD_T^\alpha\big(\frac{1}{2}v_x(x,t)\big) - {_0D_t^\alpha}\big(\frac{1}{2}v_x(x,t)\big) - av_x^p(x,t)v_{xx}(x,t) - bv_{xxxx}(x,t) = 0.$$

Once again, substituting the potential function $v_x(x,t)$ for $u(x,t)$, yields the time-fractional generalized KdV equation for the state function $u(x,t)$ as

$$\frac{1}{2}\big(_0D_t^\alpha u(x,t) - {_tD_T^\alpha}u(x,t)\big) + au^p(x,t)u_x(x,t) + bu_{xxx}(x,t) = 0. \tag{4.10}$$

According to the Riesz fractional derivative $_0^R D_t^\alpha u(x,t)$, the time-fractional generalized KdV equation represented in (4.10) can write as

$$_0^R D_t^\alpha u(x,t) + au^p(x,t)u_x(x,t) + bu_{xxx}(x,t) = 0. \tag{4.11}$$

Acting from left-hand side by the Riesz fractional operator $_0^R D_t^{1-\alpha}$ on (4.11) leads to

$$\frac{\partial}{\partial t}u(x,t) - {_0^R D_t^{\alpha-1}}u(x,t)|_{t=0}\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}$$
$$+ {_0^R D_t^{1-\alpha}}\Big(au^p(x,t)\frac{\partial}{\partial x}u(x,t) + b\frac{\partial^3}{\partial x^3}u(x,t)\Big) = 0, \tag{4.12}$$

from Lemma 2.7. In view of the variational iteration method, combining with (4.12), the $n+1$-th approximate solution of (4.11) can be read using iteration correction functional as

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\tau)\Big[\frac{\partial}{\partial \tau}u_n(x,\tau) - {_0^R D_\tau^{\alpha-1}}u_n(x,\tau)|_{\tau=0}\frac{\tau^{\alpha-2}}{\Gamma(\alpha-1)}$$
$$+ {_0^R D_\tau^{1-\alpha}}\Big(a\tilde u_n^p(x,\tau)\frac{\partial}{\partial x}\tilde u_n(x,\tau) + b\frac{\partial^3}{\partial x^3}\tilde u_n(x,\tau)\Big)\Big]d\tau, \quad n\geq 0, \tag{4.13}$$

where the function $\tilde u_n(x,t)$ is considered as a restricted variation function, i.e., $\delta \tilde u_n(x,t) = 0$. The extreme of the variation of (4.13) subject to the restricted variation function straightforwardly yields

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \int_0^t \lambda(\tau)\delta\frac{\partial}{\partial \tau}u_n(x,\tau)d\tau$$

$$= \delta u_n(x,t) + \lambda(\tau)\delta u_n(x,\tau)|_{\tau=t} - \int_0^t \frac{\partial}{\partial \tau}\lambda(\tau)\delta u_n(x,\tau)d\tau = 0.$$

This expression reduces to the stationary conditions

$$\frac{\partial}{\partial \tau}\lambda(\tau) = 0, \quad 1 + \lambda(\tau) = 0,$$

which converted to the Lagrangian multiplier at $\lambda(\tau) = -1$. Therefore, the correction functional (4.13) takes the following form

$$\begin{aligned}
u_{n+1}(x,t) = u_n(x,t) - \int_0^t \Big[& \frac{\partial}{\partial \tau}u_n(x,\tau) - {}_0^R I_\tau^{1-\alpha}u_n(x,\tau)|_{\tau=0}\frac{\tau^{\alpha-2}}{\Gamma(\alpha-1)} \\
& + {}_0^R D_\tau^{1-\alpha}\Big(au_n^p(x,\tau)\frac{\partial}{\partial x}u_n(x,\tau) + b\frac{\partial^3}{\partial x^3}u_n(x,\tau)\Big)\Big]d\tau, \quad n \geq 0,
\end{aligned} \tag{4.14}$$

since $\alpha - 1 < 0$, the fractional derivative operator ${}_0^R D_t^{\alpha-1}$ reduces to fractional integral operator ${}_0^R I_t^{1-\alpha}$ by Remark 2.8.

In view of the right-hand side Riemann-Liouville fractional derivative is interpreted as a future state of the process in physics. For this reason, the right-derivative is usually neglected in applications, when the present state of the process does not depend on the results of the future development, and so the right-derivative is used equal to zero in the following calculations. The zero order solitary wave solution can be taken as the initial value of the state variable, which is taken in this case as

$$u_0(x,t) = u(x,0) = k\operatorname{sech}^{2/p}\big(\frac{p}{2\sqrt{b}}(x+\eta_0)\big),$$

where $k = \big(\frac{(p+1)(p+2)}{2a}\big)^{\frac{1}{p}}$, η_0 is a constant.

Substituting this zero order approximate solitary wave solution into (4.14) and using the Definition 2.6 leads to the first order approximate solitary wave solution

$$u_1(x,t) = k\operatorname{sech}^{2/p}\big(\frac{p}{2\sqrt{b}}(x+\eta_0)\big)\Big(1 + \frac{t^\alpha}{\sqrt{b}\Gamma(\alpha+1)}\tanh\big(\frac{p}{2\sqrt{b}}(x+\eta_0)\big)\Big).$$

Substituting the first order approximate solitary wave solution into (4.14), using the Definition 2.6 then leads to the second order approximate solitary wave solution in the following form

$$u_2(x,t)$$

$$\begin{aligned}
= {}& k\operatorname{sech}^{2/p}\big(\frac{p}{2\sqrt{b}}(x+\eta_0)\big) + \frac{t^\alpha}{\Gamma(\alpha+1)}\Big[\frac{ak^{p+1}}{\sqrt{b}}\operatorname{sech}^{\frac{2}{p}+2}\big(\frac{p}{2\sqrt{b}}(x+\eta_0)\big) \\
& \times \tanh\big(\frac{p}{2\sqrt{b}}(x+\eta_0)\big) + \frac{k}{\sqrt{b}}\operatorname{sech}^{2/p}\big(\frac{p}{2\sqrt{b}}(x+\eta_0)\big)\tanh^3\big(\frac{p}{2\sqrt{b}}(x+\eta_0)\big) \\
& - \frac{3kp}{2\sqrt{b}}\operatorname{sech}^{\frac{2}{p}+2}\big(\frac{p}{2\sqrt{b}}(x+\eta_0)\big) - \frac{kp^2}{2\sqrt{b}}\operatorname{sech}^{\frac{2}{p}+2}\big(\frac{p}{2\sqrt{b}}(x+\eta_0)\big) \\
& \times \tanh\big(\frac{p}{2\sqrt{b}}(x+\eta_0)\big)\Big] \\
& - \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\Big[\frac{ak^{p+1}}{b}\operatorname{sech}^{\frac{2}{p}+2}\big(\frac{p}{2\sqrt{b}}(x+\eta_0)\big)\big(\frac{p}{2} - \frac{3p+2}{2}\tanh^2\big(\frac{p}{2\sqrt{b}}(x+\eta_0)\big)\big) \\
& - \frac{3kp^2(p+2)}{8b}\operatorname{sech}^{2/p}\big(\frac{p}{2\sqrt{b}}(x+\eta_0)\big)\tanh^2\big(\frac{p}{2\sqrt{b}}(x+\eta_0)\big)
\end{aligned}$$

$$- \frac{3kp^2(p+2)}{8b} \operatorname{sech}^{\frac{2}{p}+2}(\frac{p}{2\sqrt{b}}(x+\eta_0)) + \frac{3kp(p+2)^2}{8b} \operatorname{sech}^{2/p}(\frac{p}{2\sqrt{b}}(x+\eta_0))$$

$$\times \tanh^2(\frac{p}{2\sqrt{b}}(x+\eta_0)) + \frac{kp^3}{8b} \operatorname{sech}^{2/p}(\frac{p}{2\sqrt{b}}(x+\eta_0))$$

$$- \frac{k(p+2)^3}{8b} \operatorname{sech}^{2/p}(\frac{p}{2\sqrt{b}}(x+\eta_0)) \tanh^4(\frac{p}{2\sqrt{b}}(x+\eta_0))$$

$$+ \frac{3k(p+2)^2}{8b} \operatorname{sech}^{\frac{2}{p}+2}(\frac{p}{2\sqrt{b}}(x+\eta_0)) \tanh^2(\frac{p}{2\sqrt{b}}(x+\eta_0))$$

$$+ \frac{kp^2(p+2)}{4b} \operatorname{sech}^{\frac{2}{p}+2}(\frac{p}{2\sqrt{b}}(x+\eta_0)) \tanh^2(\frac{p}{2\sqrt{b}}(x+\eta_0))\Big]$$

$$- \frac{\Gamma(2\alpha+1)t^{3\alpha}}{\Gamma(3\alpha+1)(\Gamma(\alpha+1))^2}\Big[\frac{apk^{p+1}}{2b\sqrt{b}} \operatorname{sech}^{\frac{2}{p}+2}(\frac{p}{2\sqrt{b}}(x+\eta_0)) \tanh(\frac{p}{2\sqrt{b}}(x+\eta_0))$$

$$\times \big(p - \frac{p+2}{2} \tanh^2(\frac{p}{2\sqrt{b}}(x+\eta_0)))\big)\Big].$$

Using Definition 2.6 and the Maple package or Mathematics, we obtain $u_3(x,t)$, $u_4(x,t)$ and so on, substituting $n-1$ order approximate solitary wave solution into (4.14), there leads to the n order approximate solitary wave solution. As n tends to infinity, the iteration leads to the solitary wave solution of the time-fractional generalized KdV equation

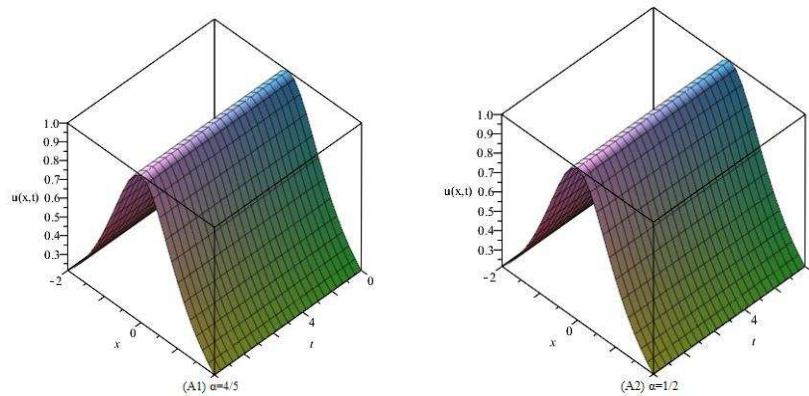$$u(x,t) = k \operatorname{sech}^{2/p}\Big(\frac{p}{2\sqrt{b}}(x - t + \eta_0)\Big).$$

FIGURE 1. The function u as a 3-dimensions graph for order α: (A1) $\alpha = 4/5$, (A2) $\alpha = 1/2$

5. DISCUSSION

The target of present work is to explore the effect of the fractional order derivative on the structure and propagation of the resulting solitary waves obtained from time-fractional generalized KdV equation. We derive the Lagrangian of the generalized KdV equation by the semiinverse method, then take a similar form of Lagrangian to the time-fractional generalized KdV equation. Using the Euler-Lagrange variational
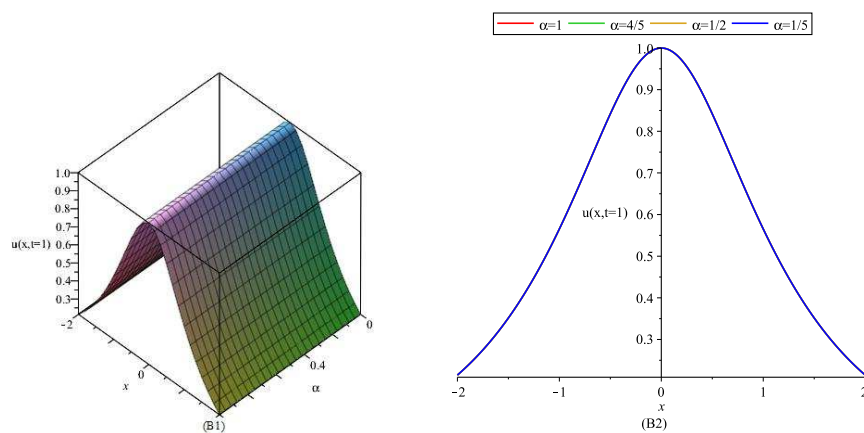
FIGURE 2. The function u as a function of space x at time $t = 1$ for order α: (B1) 3-dimensions graph, (B2) 2-dimensions graph

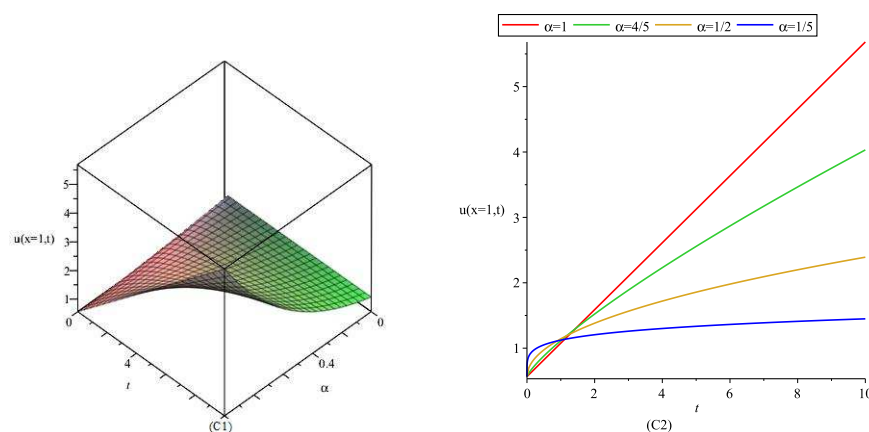
FIGURE 3. The amplitude of the function u as a function of time t at space $x = 1$ for order α: (C1) 3-dimensions graph, (C2) 2-dimensions graph

technique, we continue our calculations until the three-order iteration. During this period, our approximate calculations are carried out concerning the solution of the time-fractional generalized KdV equation taking into account the values of the coefficients and some meaningful values namely, $\frac{4}{5}$, $\frac{1}{2}$ and $p = 3, a = 10, b = 1, \eta_0 = 0$. The solitary wave solution of time-fractional generalized KdV equation are obtained. In addition, 3-dimensional representation of the solution $u(x, t)$ for the time-fractional generalized KdV equation with space x and time t for different values of the order α is presented respectively in Figure 1, the solution u is still a single soliton wave solution for all values of the order α. It shows that the balancing scenario between nonlinearity and dispersion is still valid. Figure 2 presents the change of amplitude and width of the soliton due to the variation of the order α, 2-and 3-dimensional graphs depicted the behavior of the solution $u(x, t)$ at time $t = 1$

corresponding to different values of the order α. This behavior indicates that the increasing of the value α is uniform both the height and the width of the solitary wave solution. That is, the order α can be used to modify the shape of the solitary wave without change of the nonlinearity and the dispersion effects in the medium. Figure 3 devoted to study the expression between the amplitude of the soliton and the fractional order at different time values. These figures show that at the same time, the increasing of the fractional α increases the amplitude of the solitary wave to some value of α.

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