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REGULARITY OF PLANAR FLOWS FOR SHEAR-THICKENING FLUIDS UNDER PERFECT SLIP BOUNDARY CONDITIONS

JAKUB TICHÝ

Abstract. For evolutionary planar flows of shear-thickening fluids in bounded domains we prove the existence of a solution with the Hölder continuous velocity gradients and pressure. The problem is equipped with perfect slip boundary conditions. We also show L^q theory result for Stokes system under perfect slip boundary conditions.

1. Introduction

We study flows of incompressible shear-thickening fluids, which in evolutionary case are governed by the following initial value problem

$$\begin{gathered}\partial_t u - \operatorname{div}\mathcal{S}(Du) + (u\cdot\nabla)u + \nabla\pi = f, \quad \operatorname{div} u = 0 \quad \text{in } Q,\\ u(0,\cdot) = u_0 \quad \text{in } \Omega,\end{gathered} \tag{1.1}$$

where u is the velocity, π represents the pressure, f stands for the density of volume forces and $\mathcal{S}(Du)$ denotes the extra stress tensor. Du is the symmetric part of the velocity gradient; i.e., $Du = \frac{1}{2}[\nabla u + (\nabla u)^\top]$, $\Omega \subset \mathbb{R}^2$ is a bounded domain, $I = (0,T)$ denotes a finite time interval and $Q = I \times \Omega$. We are interested in the case, when (1.1) is equipped with the perfect slip boundary conditions

$$u\cdot\nu = 0, \quad [\mathcal{S}(Du)\nu]\cdot\tau = 0 \quad \text{on } I\times\partial\Omega, \tag{1.2}$$

where τ is the tangent vector and ν is the outward normal to $\partial\Omega$. The constitutive relation for $\mathcal{S}$ is given via the generalized viscosity μ and is of the form

$$\mathcal{S}(Du) := \mu(|Du|)Du.$$

The extra stress tensor $\mathcal{S}$ is assumed to possess p-potential structure with $p \geq 2$. More precisely, we can construct scalar potential $\Phi : [0,\infty) \mapsto [0,\infty)$ to the stress tensor $\mathcal{S}$; i.e.,

$$\mathcal{S}(A) = \partial_A \Phi(|A|) = \Phi'(|A|)\frac{A}{|A|} \quad \forall A \in \mathbb{R}^{2\times 2}_{\rm sym},$$

such that $\Phi \in \mathcal{C}^{1,1}((0,\infty)) \cap \mathcal{C}^1([0,\infty))$, $\Phi(0) = 0$ and there exist $p \in [2,\infty)$ and $0 < C_1 \leq C_2$ such that for all $A, B \in \mathbb{R}^{2\times 2}_{\rm sym}$

$$C_1(1+|A|^2)^{\frac{p-2}{2}}|B|^2 \leq \partial_A^2\Phi(|A|) : B\otimes B \leq C_2(1+|A|^2)^{\frac{p-2}{2}}|B|^2. \tag{1.3}$$

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In the analysis of equations of fluid motions the question of Hölder continuity of velocity gradients is an important issue. For instance, in optimal control problems, global regularity results that guarantee boundedness of velocity gradients are needed in order to establish the existence of the weak solution for adjoint equation to the original problem and for linearised models. These results are closely related to the regularity of the coefficients in the main part of the associated differential operators and enable to derive corresponding optimality conditions, as is done for example in [25]. For optimal control of flows with shear dependent viscosities in the stationary case where the author is dealing with the lack of the regularity result we refer to [4] and [5].

Hölder continuity of velocity gradients is also important when one studies exponential attractors. With such a regularity it is possible to show the differentiability of the solution operator with respect to the initial condition, which is the key technical step in the method of Lyapunov exponents. Differentiability of the solution is equivalent to the linearisation of the equation around particular solution which is used to study infinitesimal volume elements and leads to sharp dimension estimates of the global attractor. This is done for example in [17].

This article closely follows [13], where P. Kaplický shows Hölder continuity of velocity gradients and pressure for (1.1) with $p \in [2, 4)$ under no slip boundary conditions. Based on the same structure of the proof and using the results from [18] we extend the result to perfect slip boundary conditions and $p \in [2, \infty)$. Although some steps of the proof in [13] can be easily modified, we have to overcome a new difficulties connected to the another type of boundary conditions. First of all, the L^p theory result for the Stokes problem equipped with perfect slip boundary conditions has to be established. Keep at our disposal the paper [18], we are able to cover the case $p \geq 4$. From the point of application it would be very interesting to obtain also the result for the case $p \in (1, 2)$ for perfect slip or homogeneous Dirichlet boundary condition.

The idea of the proof goes back to [21], where the authors show that every weak solution u of $\partial_t u - \operatorname{div}(\mathcal{S}(\nabla u)) = 0$ in Q has locally Hölder continuous gradient in case that $\Omega \subset \mathbb{R}^2$ and $p = 2$. This result was extended in [12] to the case $p \in (1, 2)$. Regularity of $\partial_t u$ is shown first and after moving $\partial_t u$ to the right hand side the stationary L^q theory is applied.

In the case of generalized Newtonian fluids this method was modified in [16], where the authors consider the shear-thinning fluid model with periodic boundary conditions. In contrary to [21] the regularity of $\partial_t u$ and ∇u had to be obtained at once. The authors showed that velocity gradients are Hölder continuous for $p \in (4/3, 2]$. These results were extended to electro-rheological fluids and non-zero initial condition in [10].

Among many works concerning regularity theory for generalized Newtonian fluids we would like to mention two papers dealing with the stationary case. In [15] the stationary version of (1.1) under homogeneous Dirichlet boundary conditions is considered. The same authors later in [14] studied the problem equipped with non-homogeneous Dirichlet boundary conditions with two types of restriction on boundary data and perfect slip boundary conditions.

Let E be a Banach space and $\alpha \in (0, 1)$, $p, q \in [1, \infty)$, $s \in \mathbb{R}$. In this paper we use standard notation for Lebesgue spaces $L^q(\Omega)$, Sobolev-Slobodeckiĭspaces $W^{s,q}(\Omega)$, Bochner spaces $L^q(I,E)$ and $W^{\alpha,q}(I,E)$. (We do not use different notation for scalar, vector-valued or tensor-valued functions).

By $H^s_q(\Omega)$ we mean Bessel potential spaces and $B^s_{p,q}(\Omega)$ are Besov spaces. BUC stands for bounded and uniformly continuous functions.

Since the domain Ω is in our case at least $\mathcal{C}^{2,1}$, we can define $L^q_\sigma(\Omega)$ and $W^{1,q}_\sigma(\Omega)$ as follows:

$$L^q_\sigma(\Omega) = \{\varphi \in L^q(\Omega), \operatorname{div}\varphi = 0 \text{ in } \Omega,\ \varphi\cdot\nu = 0 \text{ on } \partial\Omega\},$$
$$W^{1,q}_\sigma(\Omega) = \{\varphi \in W^{1,q}(\Omega), \operatorname{div}\varphi = 0, \text{ in } \Omega,\ \varphi\cdot\nu = 0 \text{ on } \partial\Omega\}.$$

The duality between Banach space E and its dual E' is denoted by $\langle\cdot,\cdot\rangle$. Set $W^{-1,p'}_\sigma(\Omega) := (W^{1,p}_\sigma(\Omega))'$.

We begin with the definition of the weak solution to the problem (1.1) with (1.2).

Definition 1.1. Let $f \in L^{p'}(I, W^{-1,p'}_\sigma(\Omega))$, $p \in [2,\infty)$ and $u_0 \in L^2(\Omega)$. We say that the function $u : Q \mapsto \mathbb{R}^2$ is a weak solution to the problem (1.1) with (1.2), if $u \in L^\infty(I, L^2(\Omega)) \cap L^p(I, W^{1,p}_\sigma(\Omega))$, $\partial_t u \in L^{p'}(I, W^{-1,p'}_\sigma(\Omega))$, $u(0,\cdot) = u_0$ in $L^2(\Omega)$ and weak formulation

$$\int_I \langle\partial_t u,\varphi\rangle\,\mathrm{d}t + \int_Q \mathcal{S}(Du) : D\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_Q (u\cdot\nabla)u\varphi\,\mathrm{d}x\,\mathrm{d}t = \int_I \langle f,\varphi\rangle\,\mathrm{d}t$$

holds for all $\varphi \in L^p(I, W^{1,p}_\sigma(\Omega))$.

If we studied also the case $p \in (1,2)$, we would have to consider only test functions from the space of smooth functions. It is well known that the weak solution exists and is unique. It could be easily proven using the monotone operator theory. See for example [19, Chapter 5] for periodic boundary conditions. Now we formulate the main results of this paper.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded non-circular $\mathcal{C}^3$ domain and* (1.3) *holds for some $p \in [2,\infty)$. Let $u_0 \in W^{2+\beta,2}(\Omega)$ for $\beta \in (0,1/4)$, $\operatorname{div} u_0 = 0$, $f \in L^\infty(I, L^{q_0}(\Omega))$ and $\partial_t f \in L^{q_0}(I, W^{-1,q_0'}_\sigma(\Omega))$ for some $q_0 > 2$. Then there exists a unique solution (u,π) of* (1.1) *with* (1.2)*, such that for some $\alpha > 0$*

$$\nabla u, \pi \in \mathcal{C}^{0,\alpha}(\overline{Q}).$$

Remark 1.3. Perfect slip boundary conditions (1.2) are, as well as homogeneous Dirichlet boundary conditions, the limit case of partial slip boundary conditions which are are also often called Navier's slip boundary conditions:

$$u\cdot\nu = 0, \quad \alpha[\mathcal{S}(Du)\nu]\cdot\tau + (1-\alpha)u_\tau = 0 \quad \alpha \in [0,1] \text{ on } \partial\Omega.$$

It would be very interesting to obtain the same result as in Theorem 1.2 also for the Navier's boundary condition. In several parts of the proof of Theorem 1.2 we apply results from [18] that are formulated only for perfect slip boundary conditions. We don't know how to generalize these results also for partial slip boundary conditions.

The paper is organized as follows: Section 2 contains preliminaries needed later, in Section 3 we gather L^q theory results for the classical Stokes system. Further we extend L^q theory results to generalized Stokes system where the Laplace operator is replaced by a general elliptic operator in divergence form with bounded measurable coefficients.

Section 4 is devoted to the proof of the main theorem in the case of quadratic growth, i.e. $p = 2$. In Section 5 we introduce the quadratic approximation of the stress tensor $\mathcal{S}(Du)$ which is done by the truncation of the generalized viscosity from above; i.e., $\mu^\varepsilon(|Du^\varepsilon|) := \min\{\mu(|Du|), 1/\varepsilon\}$ for $\varepsilon \in (0,1)$. We prove the main result for the approximated problem and pass from the approximated problem to the original one at the end.

2. Preliminary general material

2.1. **Function spaces.** Let E and F be reflexive Banach spaces. Although it is not necessary to have reflexive spaces in all definitions, for convenience we assume it. By $\mathcal{L}(E,F)$ we mean the Banach space of all bounded linear operators from E to F and $\mathcal{L}(E) := \mathcal{L}(E,E)$. If E is a linear subspace of F and the natural injection $i : x \mapsto x$ belongs to $\mathcal{L}(E,F)$, we write $E \hookrightarrow F$. In the case E is also dense in F, it will be denoted by $E \overset{d}{\hookrightarrow} F$. Furthermore, $\mathcal{L}is(E,F)$ consists of all topological linear isomorphisms from E onto F. We also write $E \doteq F$ if $E \hookrightarrow F$ and $F \hookrightarrow E$, i.e. E equals F with equivalent norms.

A Banach space E is said to be of class $\mathcal{HT}$, if the Hilbert transform is bounded on $L^p(\mathbb{R},E)$ for some (and then for all) $p \in (1,\infty)$. Here the Hilbert transform H of a function $f \in \mathcal{S}(\mathbb{R},E)$, the Schwartz space of rapidly decreasing E-valued functions, is defined by $Hf := \frac{1}{\pi} PV(\frac{1}{t}) * f$. It is well known theorem that the set of Banach spaces of class $\mathcal{HT}$ coincides with the class of UMD spaces, where the UMD stands for the property of unconditional martingale differences. Note that all closed subspaces of $L^q(\Omega)$ are UMD spaces provided $q \in (1,\infty)$.

2.2. **Semigroups and interpolation-extrapolation scales.** For a linear operator A in E_0 we denote the domain of A by $\mathcal{D}(A)$. $A \in \mathcal{H}(E_1,E_0)$ means that A is the negative infinitesimal generator of a bounded analytic semigroup in E_0 and $E_1 \doteq \mathcal{D}(A)$. It holds

$$\mathcal{H}(E_1,E_0) = \cup_{\kappa\geq 1,\,\omega>0}\mathcal{H}(E_1,E_0,\kappa,\omega),$$

where $A \in \mathcal{H}(E_1,E_0,\kappa,\omega)$ if $\omega + A \in \mathcal{L}is(E_1,E_0)$ and

$$\kappa^{-1} \leq \frac{\|(\lambda + A)u\|_{E_0}}{|\lambda|\|u\|_{E_0} + \|u\|_{E_1}} \leq \kappa, \quad Re(\lambda) \geq \omega, \quad u \in E_1.$$

By $\sigma(A)$ we mean the spectrum of A and $\varrho(A)$ denotes the resolvent set. A linear operator A in E is said to be of positive type if it belongs to $\mathcal{P}(E) := \cup_{K>1} P_K(E)$. $A \in P_K(E)$ if it is closed, densely defined, $\mathbb{R}^+ \subset \varrho(-A)$ and $(1+s)\|(s+A)^{-1}\|_{\mathcal{L}(E)} \leq K$ for $s \in \mathbb{R}^+$, where $K \geq 1$.

We say that a linear operator A in E is of type (E,K,ϑ), denoted by $A \in \mathcal{P}(E,K,\vartheta)$, if it is densely defined and if

$$\Sigma_\vartheta := \{|\arg z| \leq \vartheta\} \cup \{0\} \subset \varrho(-A) \quad \text{and} \quad (1+|\lambda|)\|(\lambda+A)^{-1}\|_{\mathcal{L}(E)} \leq K, \quad \lambda \in \Sigma_\vartheta.$$

Put $\mathcal{P}(E,\vartheta) := \cup_{K>1}\mathcal{P}(E,K,\vartheta)$.

A linear operator A in E is said to have bounded imaginary powers, in symbols,

$$A \in \mathcal{BIP}(E) := \cup_{K\geq 1,\,\theta\geq 0}\mathcal{BIP}(E,K,\theta),$$

provided $A \in \mathcal{P}(E)$ and there exist $\theta \geq 0$ and $K \geq 1$ such that $A^{is} \in \mathcal{L}(E)$ and $\|A^{is}\|_{\mathcal{L}(E)} \leq Ke^{\theta|s|}$ for $s \in \mathbb{R}$.

We introduce an interpolation-extrapolation scale which is essential in the proof of Theorem 3.9. Let $p,q \in (1,\infty)$, $\theta \in (0,1)$ and $[\cdot,\cdot]_\theta$ denotes the complex and $(\cdot,\cdot)_{\theta,q}$ the real interpolation functor. Let $A \in \mathcal{H}(E_1,E_0)$. Then we denote by

$[(E_\alpha, A_\alpha); \alpha \in \mathbb{R}]$ the interpolation-extrapolation scale generated by (E, A) and $[\cdot,\cdot]_\theta$ or $(\cdot,\cdot)_{\theta,q}$, where we set $E_k := \mathcal{D}(A^k)$ for $k \in \mathbb{N}$ with $k \geq 2$. Also set $E^\sharp := E'$ and $A^\sharp := A'$, where A' is the dual of A in E in the sense of unbounded linear operators. Finally let $E_k^\sharp := \mathcal{D}((A^\sharp)^k)$ for $k \in \mathbb{N}$. Then we define E_{-k} for $k \in \mathbb{N}$ by $E_{-k} := (E_k^\sharp)'$. We put $E_{k+\theta} := [E_k, E_{k+1}]_\theta$ (and similarly for the real interpolation functor). If $\alpha \geq 0$ we denote by A_α the maximal restriction of A to E_α whose domain equals $\{u \in E_\alpha \cap E_1;\ Au \in E_\alpha\}$. If $\alpha < 0$ then A_α is the closure of A in E_α.

For the dual interpolation functor $(\cdot,\cdot)_\theta^\sharp$ (which is equal to $[\cdot,\cdot]_\theta$ for the complex interpolation and $(\cdot,\cdot)_{\theta,q'}$ for real interpolation) we abbreviate the interpolation-extrapolation scale generated by $(E^\sharp, A^\sharp)$ and $(\cdot,\cdot)_\theta^\sharp$, by $[(E_\alpha^\sharp, A_\alpha^\sharp); \alpha \in \mathbb{R}]$ and call it interpolation-extrapolation scale dual to $[(E_\alpha, A_\alpha); \alpha \in \mathbb{R}]$. It holds $(E_{-\alpha})' \doteq E_\alpha^\sharp$ and $(A_{-\alpha})' = A_\alpha^\sharp$. For more details see [2, Section V.2].

3. L^q theory for Stokes system

In this section we collect facts about L^q theory for the Stokes system

$$\begin{gathered} \partial_t u - \Delta u + \nabla \pi = f, \quad \operatorname{div} u = 0 \quad \text{in } Q, \\ u(0,\cdot) = u_0 \quad \text{on } \Omega, \end{gathered} \tag{3.1}$$

equipped with the perfect slip boundary conditions

$$u \cdot \nu = 0, \quad [(Du)\nu] \cdot \tau = 0 \quad \text{on } I \times \partial\Omega. \tag{3.2}$$

Unlike the main theorem of this paper which is formulated for $\Omega \subset \mathbb{R}^n$, $n = 2$, results of this sections are valid for $n \geq 2$. Let P denote the projection operator from $L^q(\Omega)$ to $L^q_\sigma(\Omega)$ associated with the Helmholtz decomposition. By $Bu = 0$ we mean that (3.2) holds in the sense of traces. Using the projection P we shall define the Stokes operator $\mathbb{A}$ by $\mathbb{A}u = -P\Delta u$ for $u \in \mathcal{D}(\mathbb{A})$, where

$$\mathcal{D}(\mathbb{A}) = L^q_\sigma(\Omega) \cap H^2_{q,B}(\Omega), \quad H^2_{q,B}(\Omega) := \{u \in H^2_q(\Omega),\ Bu = 0,\ \text{on } \partial\Omega\}.$$

Applying the Helmholtz projection P to (3.1) with (3.2), we eliminate the pressure from equations and with the help of the newly established notation the Stokes system reduces to

$$\begin{gathered} \partial_t u + \mathbb{A}u = Pf, \quad \operatorname{div} u = 0 \quad \text{in } Q, \\ u(0,\cdot) = u_0 \quad \text{on } \Omega, \quad Bu = 0 \quad \text{on } I \times \partial\Omega. \end{gathered} \tag{3.3}$$

At first we mention some basic properties of the Stokes operator $\mathbb{A}$. From [23] we know that $\mathbb{A} \in \mathcal{H}(L^q_\sigma(\Omega) \cap H^2_{q,B}(\Omega), L^q_\sigma(\Omega))$. This also tells us that $\mathbb{A} \in \mathcal{P}(L^q_\sigma(\Omega), \omega)$ for $\omega \in [0, \pi/2)$ (see [11, Theorem II.4.6]). Shimada later showed in [22] the L^q-maximal regularity for $\mathbb{A}$. In [1, Theorem 1] Abels and Terasawa proved the following result.

Proposition 3.1. *Let $q \in (1,\infty)$, $n \geq 2$, $r \in (n,\infty]$ such that $q, q' \leq r$. Let $\Omega \subset \mathbb{R}^n$ be a domain with $W^{2-\frac{1}{r},r}$-boundary and $\vartheta \in (0,\pi)$. Then there is some $R > 0$ such that $(\lambda + \mathbb{A})^{-1}$ exists and*

$$(1 + |\lambda|)\|(\lambda + \mathbb{A})^{-1}\|_{\mathcal{L}(L^q(\Omega))} \leq C$$

for all $\lambda \in \Sigma_\vartheta$ with $|\lambda| \geq R$. Moreover,

$$\Big\| \int_{\Gamma_R} h(-\lambda)(\lambda + \mathbb{A})^{-1}\,\mathrm{d}\lambda \Big\|_{\mathcal{L}(L^q(\Omega))} \leq C\|h\|_{L^\infty(\Sigma_{\pi-\vartheta})}$$

for every $h \in H^\infty(\vartheta)$, where $\Gamma = \partial\Sigma_\vartheta$, $\Gamma_R = \Gamma \setminus \overline{B_R(0)}$ and $H^\infty(\vartheta)$ denotes the Banach algebra of all bounded holomorphic functions $h : \Sigma_{\pi-\vartheta} \to \mathbb{C}$. In particular, for every $\omega \in \mathbb{R}$ and $\vartheta' \in (0, \vartheta]$ such that $\omega + \Sigma_{\vartheta'} \subset \varrho(-\mathbb{A})$ the shifted Stokes operator $\omega + \mathbb{A}$ admits a bounded H^∞-calculus with respect to ϑ'; i.e.,

$$h(\omega + \mathbb{A}) := \frac{1}{2\pi i} \int_\Gamma h(-\lambda)(\lambda + \omega + \mathbb{A})^{-1}\, \mathrm{d}\lambda$$

is a bounded operator satisfying

$$\|h(\omega + \mathbb{A})\|_{\mathcal{L}(L^q(\Omega))} \le C\|h\|_{L^\infty(\Sigma_{\pi-\vartheta})}$$

for all $h \in H^\infty(\vartheta')$.

Note that the class of operators with a bounded H^∞-calculus is a subclass of the operators which have $\mathcal{BIP}$, therefore these operators admit all important properties which has operators with bounded imaginary powers. For another properties of a bounded H^∞-calculus we refer for example to [8, Section 2, Subsection 2.4].

From the result of Shibata and Shimada in [23] follows that $\omega + \Sigma_{\vartheta'} \subset \varrho(-\mathbb{A})$ even for $\omega = 0$ provided the domain Ω is bounded and non-axisymmetric (see Definition 3.8). Thus, Proposition 3.1 and [23, Theorem 1.3] gives $\mathbb{A} \in \mathcal{BIP}$. The Stokes operator $\mathbb{A}$ has realizations $\mathbb{A}_\alpha$ on $\mathbb{E}_\alpha$ for some α. Concretely, from [24, Section 2.2] we know that $\mathbb{A}_\alpha \in \mathcal{H}(\mathbb{E}_{\alpha+1}, \mathbb{E}_\alpha)$ for $\alpha \ge -1$. Steiger in [24] provides the characterization of spaces $\mathbb{E}_\alpha$:

Proposition 3.2 ([24, Corollary 2.6])**.** *Set $s_\alpha := \{-2+1/q, -1+1/q, 1/q, 1+1/q\}$ and $F^s_q(\Omega) := H^s_p(\Omega)$ for the complex interpolation functor and $F^s_q(\Omega) := B^s_{q,q}(\Omega)$ for the real interpolation functor. Define*

$$F^s_{q,B}(\Omega) := \begin{cases} \{u \in F^s_q(\Omega),\ Bu = 0 \ \text{on } \partial\Omega\}, & s \in (1 + 1/q, 2], \\ \{u \in F^s_q(\Omega),\ u \cdot \nu = 0 \ \text{on } \partial\Omega\}, & s \in [1/q, 1 + 1/q), \\ F^s_q(\Omega), & s \in [0, 1/q), \\ \big(F^{-s}_{q',B,\sigma}(\Omega)\big)', & s \in [-2, 0) \setminus s_\alpha \end{cases} \tag{3.4}$$

and

$$F^s_{q,B,\sigma}(\Omega) := \begin{cases} F^s_{q,B}(\Omega) \cap L^q_\sigma(\Omega), & s \in [0, 2] \setminus s_\alpha, \\ \big(F^{-s}_{q',B,\sigma}(\Omega)\big)', & s \in [-2, 0) \setminus s_\alpha. \end{cases} \tag{3.5}$$

Then $\mathbb{E}_\alpha \doteq F^{2\alpha}_{q,B,\sigma}(\Omega)$ for $2\alpha \in [-2, 2] \setminus s_\alpha$.

This gives

$$A_\alpha \in \mathcal{H}(F^{2\alpha+2}_{q,B,\sigma}(\Omega), F^{2\alpha}_{q,B,\sigma}(\Omega)), \quad 2\alpha \in [-2, 2] \setminus s_\alpha. \tag{3.6}$$

Remark 3.3 ([24, Remark 2.3c])**.** The Helmholtz projection P enjoys following continuity properties:

$$P \in \mathcal{L}(F^s_{q,B}(\Omega)) \cap \mathcal{L}(F^s_{q,B}(\Omega), F^s_{q,B,\sigma}(\Omega)), \quad s \in (-1 + 1/q, 1 + 1/q) \setminus s_\alpha. \tag{3.7}$$

We will use the fact, that the property of bounded imaginary powers can be carried over the interpolation-extrapolation scales.

Proposition 3.4 ([2, Proposition V.1.5.5])**.** *Let $A \in \mathcal{P}(E)$ and let $[(E_\alpha, A_\alpha); \alpha \in (-n, \infty)]$ be the interpolation-extrapolation scale generated by (E, A) and an exact functor. If $A \in \mathcal{BIP}(E, M, \sigma)$ then $A_\alpha \in \mathcal{BIP}(E_\alpha, M, \sigma)$.*

The reiteration property will be needed.

Proposition 3.5 ([2, Theorem V.1.5.4]). *Suppose that $A \in \mathcal{BIP}(E)$. Then the interpolation-extrapolation scale $[(E_\alpha, A_\alpha); \alpha \in [-n, \infty)]$ generated by (E, A) and complex interpolation functor possesses the reiteration property*

$$[E_\alpha, E_\beta]_\eta \doteq E_{(1-\eta)\alpha+\eta\beta}, \quad -n \le \alpha \le \beta < \infty, \quad \eta \in (0,1).$$

Let us define the maximal L^q-regularity for the operator A (compare [2, Section III.1, Subsection 1.5 and Section III.4, Remark 4.10.9.c])

Definition 3.6. Let $A \in \mathcal{H}(E_1, E_0)$ and $q \in (1, \infty)$. We say that the pair $\big(L^q(I, E_0), L^q(I, E_1) \cap W^{1,q}(I, E_0)\big)$ is a pair of maximal regularity for A (or A has maximal regularity), if for $u_0 \in (E_0, E_1)_{1-1/q,q}$ and $f \in L^q(I, E_0)$ there exists a unique solution $u \in L^q(I, E_1) \cap W^{1,q}(I, E_0)$ of (3.3), and

$$\|\partial_t u\|_{L^q(I,E_0)} + \|u\|_{L^q(I,E_0)} + \|Au\|_{L^q(I,E_0)} \le C\Big(\|f\|_{L^q(I,E_0)} + \|u_0\|_{(E_0,E_1)_{1-1/q,q}}\Big). \tag{3.8}$$

Further we mention the relation between maximal regularity and negative infinitesimal generators of a bounded analytic semigroup.

Proposition 3.7 ([2, Theorem III.4.10.7]). *Suppose that E_0 is a UMD space, $A \in \mathcal{H}(E_1, E_0)$ and there are constants $M > 0$, $\vartheta \in (0, \pi/2)$ such that $\Sigma_\vartheta \subset \varrho(-A)$ and for $\lambda \in \Sigma_\vartheta$ and $j = 0, 1$ holds*

$$\|A\|_{\mathcal{L}(E_1,E_0)} + (1 + |\lambda|)^{1-j}\|(\lambda + A)^{-1}\|_{\mathcal{L}(E_0,E_j)} \le M$$

and suppose that there exist constants $N \ge 1$ and $\theta \in [0, \pi/2)$ such that $A \in \mathcal{BIP}(E_0, N, \theta)$. Then A has maximal regularity and the estimate (3.8) *holds uniformly with respect to T.*

To specify the shape of the domain Ω we add the definition of axisymmetric domain in the same way as in [9, Definition-Lemma 1].

Definition 3.8. Let Ω be a smooth bounded open subset of $\mathbb{R}^n$, $n \ge 2$. We say that Ω is axisymmetric if and only if there exists a nontrivial rigid motion R which is tangent to $\partial\Omega$; or equivalently, which satisfies for all $t \in \mathbb{R}$ $e^{tR}\Omega = \Omega$. Here e^{tR} is the isometry defined via $\frac{d}{dt}e^{tR}(x) = Re^{tR}(x)$.

By rigid motions R we understand affine maps $R : \Omega \to \mathbb{R}^n$ whose linear part is antisymmetric. If we consider the most common dimensions $n = 2$ and $n = 3$ we can use simpler definition. A domain in $\mathbb{R}^2$ is axisymmetric if it has a circular symmetry around some point. A domain in $\mathbb{R}^3$ is axisymmetric if it admits an axis of symmetry; i.e., the domain is preserved by a rotation of arbitrary angle around this axis. If the domain admits two nonparallel axes of symmetry, then it is spherically symmetric around some point.

The main result of this section is the following.

Theorem 3.9. *Let $\Omega \subset \mathbb{R}^n$ be a bounded non-axisymmetric $\mathcal{C}^{2,1}$ domain, $q \in [2, \infty)$, $f \in L^q(I, W_\sigma^{-1,q'}(\Omega))$, $u_0 \in B^{1-2/q}_{q,q,B,\sigma}(\Omega)$ then there exists a constant $C > 0$ and the unique weak solution of* (3.3) *satisfying*

$$\|\nabla u\|_{L^q(Q)} + \|u\|_{BUC(I,B^{1-2/q}_{q,q,B,\sigma}(\Omega))} \le C\Big(\|f\|_{L^q(I,W_\sigma^{-1,q'}(\Omega))} + \|u_0\|_{B^{1-2/q}_{q,q,B,\sigma}(\Omega)}\Big).$$

The constant C is independent of T, u, f and u_0.

Proof. We consider the system (3.3) instead of (3.1) with (3.2). Since for UMD space E, E' is one as well and for an interpolation couple of UMD spaces the interpolation spaces are also UMD (see [2, Theorem III.4.5.2]), $\mathbb{E}_{-1/2}$ is a UMD space. Proposition 3.4 gives us $\mathbb{A}_{-1/2}$ has $\mathcal{BIP}$. Together with (3.6), [2, Corollary I.1.4.3] and [23, Theorem 1.3] we can see that assumptions of Proposition 3.7 are fulfilled for $\mathbb{A}_{-1/2}$. Therefore we obtain (3.8) for $\mathbb{A}_{-1/2}$ and $E_0 = \mathbb{E}_{-1/2}$:

$$\begin{aligned} &\|\partial_t u\|_{L^q(I,\mathbb{E}_{-1/2})} + \|u\|_{L^q(I,\mathbb{E}_{-1/2})} + \|\mathbb{A}_{-1/2}u\|_{L^q(I,\mathbb{E}_{-1/2})} \\ &\le C\Big(\|f\|_{L^q(I,\mathbb{E}_{-1/2})} + \|u_0\|_{(\mathbb{E}_{-1/2},\mathbb{E}_{1/2})_{1-1/q,q}}\Big). \end{aligned} \tag{3.9}$$

It remains to determine the correct spaces in (3.9). For the space of initial condition u_0 we get by Proposition 3.2 for the complex interpolation functor

$$u_0 \in (H^{-1}_{q,B,\sigma}(\Omega), H^1_{q,B,\sigma}(\Omega))_{1-1/q,q}.$$

This space equals (with equivalent norms) to $B^{1-2/q}_{q,q,B,\sigma}(\Omega)$ since for $q \ge 2$,

$$\begin{aligned} B^{1-2/q}_{q,q,B,\sigma}(\Omega) &\doteq (L^q_\sigma(\Omega), H^1_{q,B,\sigma}(\Omega))_{1-2/q,q} \\ &\doteq ([H^{-1}_{q,B,\sigma}(\Omega), H^1_{q,B,\sigma}(\Omega)]_{1/2}, H^1_{q,B,\sigma}(\Omega))_{1-2/q,q} \\ &\doteq (H^{-1}_{q,B,\sigma}(\Omega), H^1_{q,B,\sigma}(\Omega))_{1-1/q,q}, \end{aligned} \tag{3.10}$$

where we used Proposition 3.5. The similar interpolation of the solenoidal functions in case of Dirichlet boundary conditions is done in [3, Proof of Lemma 9.1]. From the embedding [2, Theorem V.4.10.2]

$$L^q(I,E_1) \cap W^{1,q}(I,E_0) \hookrightarrow BUC(I,(E_0,E_1)_{1-1/q,q}),$$

we obtain $u \in BUC(I, B^{1-2/q}_{q,q,B,\sigma}(\Omega))$. Due to $\|\mathbb{A}_{-1/2}u\|_{\mathbb{E}_{-1/2}} = \|u\|_{\mathbb{E}_{1/2}}$ and $\mathbb{E}_{1/2} \doteq W^{1,q}_\sigma(\Omega)$ we have boundedness of ∇u in $L^q(Q)$. It remains to find the space for f. By Proposition 3.2,

$$f \in L^q(I, W^{-1,q'}_\sigma(\Omega)),$$

since $H^s_q(\Omega) \doteq W^{s,q}(\Omega)$ for $s \in \mathbb{Z}$. □

Without loss of generality we may assume that there exists a symmetric tensor $G \in L^q(Q)$, such that the weak formulation of the right hand side of (3.1) can be written in the form

$$\int_Q G : D\varphi \,\mathrm{d}x\,\mathrm{d}t = \int_I \langle f, \varphi\rangle \,\mathrm{d}t \quad \forall \varphi \in L^{q'}(I, W^{1,q'}_\sigma(\Omega)). \tag{3.11}$$

To prove it, we proceed in the same way like in [16, Proof of Proposition 2.1, Step 1] where the authors are dealing with periodic boundary conditions. Consider the Stokes system which can be formulated in the weak form for a. a. $t \in I$ as follows

$$\int_\Omega Dw(t) : D\varphi \,\mathrm{d}x = \langle f(t), \varphi\rangle \quad \forall \varphi \in W^{1,q}_\sigma(\Omega). \tag{3.12}$$

As $f \in L^q(I, W^{-1,q}_\sigma(\Omega))$, there exists a solution $w(t) \in W^{1,q}_\sigma(\Omega)$ of (3.12) enjoying the estimate

$$\|w(t)\|_{W^{1,q}(\Omega)} \le C\|f\|_{W^{-1,q}_\sigma(\Omega)}$$

with the positive constant C independent of t. Consequently, $w \in L^q(I, W^{1,q}_\sigma(\Omega))$ and

$$\|w\|_{L^q(I,W^{1,q}(\Omega))} \le C\|f\|_{L^q(I,W^{-1,q}_\sigma(\Omega))}.$$

Defining $G = Dw$ we conclude (3.11) from (3.12) by density arguments. Therefore, for all $f \in L^q(I, W_\sigma^{-1,q}(\Omega))$ there exists $G \in L^q(Q)$ such that (3.11) and estimate

$$\|G\|_{L^q(Q)} \le C\|f\|_{L^q(I,W_\sigma^{-1,q}(\Omega))}$$

holds. We would like to point out that the perfect slip boundary conditions are hidden in the weak formulation. If G is smooth enough then it holds

$$\int_I \langle f, \varphi\rangle \,\mathrm{d}t = -\int_Q \operatorname{div} G \cdot \varphi \,\mathrm{d}x\,\mathrm{d}t + \int_I \int_{\partial\Omega} (G\nu)\tau(\varphi\cdot\tau)\,\mathrm{d}\sigma\,\mathrm{d}t \quad \forall \varphi \in L^q(I, W_\sigma^{1,q}(\Omega)).$$

The Stokes system (3.1) with (3.2) can be formulated in the weak form as follows

$$\int_I \langle \partial_t u, \varphi\rangle \,\mathrm{d}t + \int_Q Du : D\varphi \,\mathrm{d}x\,\mathrm{d}t = \int_Q G : D\varphi \,\mathrm{d}x\,\mathrm{d}t \quad \forall \varphi \in L^q(I, W_\sigma^{1,q}(\Omega)). \tag{3.13}$$

Introducing the solution operator $S : (G, u_0) \mapsto Du$, we conclude first from the existence theory, that S is continuous from $L^2(Q) \times L^2_\sigma(\Omega)$ to $L^2(Q)$ with the norm less or equal to 1. By Theorem 3.9 we know that S is continuous from $L^{q_1}(Q) \times B^{1-2/q_1}_{q_1,q_1,B,\sigma}(\Omega)$ to $L^{q_1}(Q)$ with norm estimated by $C_q > 1$. Since $S(G, u_0) = S(G, 0) + S(0, u_0)$, Riesz-Thorin theorem and the real interpolation method implies following assertion, see for example [7, Theorem 5.2.1 and Theorem 6.4.5].

Lemma 3.10. *Let Ω be a bounded non-axisymmetric $\mathcal{C}^{2,1}$ domain and $q_1 > 2$. There exist constant $C > 0$ and $K := C_{q_1}^{q_1/(q_1-2)}$ such that for every $q \in (2, q_1)$, arbitrary $G \in L^q(I, L^q_\sigma(\Omega))$, $u_0 \in B^{1-2/q}_{q,q,B,\sigma}(\Omega)$ there exists a unique solution u of* (3.13) *satisfying*

$$\|Du\|_{L^q(Q)} \le K^{1-\frac{2}{q}} \Big(\|G\|_{L^q(Q)} + C\|u_0\|_{B^{1-2/q}_{q,q,B,\sigma}(\Omega)}\Big).$$

For $q > 2$ small enough Lemma 3.10 allows us to prove the L^q theory for a generalized Stokes system, where the Stokes operator is replaced by a general elliptic operator with bounded measurable coefficients. More precisely, let $0 < \gamma_1 \le \gamma_2$ and suppose that the coefficient matrix $\mathbb{M} \in L^\infty(Q)$ is symmetric in the sense $M_{ij}^{kl} = M_{kl}^{ij} = M_{kl}^{ji}$ for $i, j, k, l = 1, 2$ and fulfils for all $B \in \mathbb{R}^{2\times 2}$, $x \in \Omega$ and $t \in I$,

$$\gamma_1 |B|^2 \le \mathbb{M}(t, x) : B \otimes B \le \gamma_2 |B|^2.$$

We consider the system

$$\begin{aligned} &\int_I \langle \partial_t u, \varphi\rangle \,\mathrm{d}t + \int_Q \mathbb{M} : Du \otimes D\varphi \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_Q G : D\varphi \,\mathrm{d}x\,\mathrm{d}t \quad \forall \varphi \in L^q(I, W_\sigma^{1,q}(\Omega)). \end{aligned} \tag{3.14}$$

The following lemma states the L^q theory result.

Lemma 3.11. *Let Ω be a bounded non-axisymmetric $\mathcal{C}^{2,1}$ domain and $q > 2$. There exist constants $K, L > 0$ such that if $q \in [2, 2 + L\frac{\gamma_1}{\gamma_2})$, $G \in L^q(Q)$ and $u_0 \in B^{1-2/q}_{q,q,B,\sigma}(\Omega)$ then the unique weak solution $u \in L^q(I, W_\sigma^{1,q}(\Omega))$ of* (3.14) *satisfies*

$$\|Du\|_{L^q(Q)} + \gamma_2^{-\frac{1}{q}} \|u\|_{BUC(I, B^{1-2/q}_{q,q,B,\sigma}(\Omega))} \le \frac{K}{\gamma_1} \Big(\|G\|_{L^q(Q)} + \gamma_2^{1-\frac{1}{q}} \|u_0\|_{B^{1-2/q}_{q,q,B,\sigma}(\Omega)}\Big).$$

Proof. We omit the proof. It can be found in [15, Proposition 2.1] for periodic boundary conditions or in [13, Proposition 2.1] for homogeneous Dirichlet boundary conditions. The only generalization consists of including perfect slip boundary conditions. L^q theory result for classical Stokes system with perfect slip boundary conditions is needed, but it is shown in Lemma 3.10. □

We also use the L^q theory for stationary variant of the system (3.14). For symmetric coefficient matrix $\mathbb{M} \in L^\infty(\Omega)$ fulfilling for all $B \in \mathbb{R}^{2\times 2}$ and $x \in \Omega$, $\gamma_1|B|^2 \le \mathbb{M}(x) : B \otimes B \le \gamma_2 |B|^2$, $0 < \gamma_1 \le \gamma_2$ we sutdy the problem

$$\int_\Omega \mathbb{M} : Du \otimes D\varphi \,\mathrm{d}x = \int_\Omega G : D\varphi \,\mathrm{d}x \quad \forall \varphi \in W^{1,q}_\sigma(\Omega). \tag{3.15}$$

Lemma 3.12. *Let Ω be a bounded non-axisymmetric $\mathcal{C}^{2,1}$ domain. Then there are constants $K, L > 0$ such that if $q \in [2, 2 + L\frac{\gamma_1}{\gamma_2})$ and $G \in L^q(\Omega)$, then the unique weak solution of* (3.15) *satisfies*

$$\|Du\|_{L^q(\Omega)} \le \frac{K}{\gamma_1}\|G\|_{L^q(\Omega)}.$$

Proof. See [15, Lemma 2.6] for no slip boundary conditions. For perfect slip boundary conditions we would proceed analogically. □

4. Proof of the main results for the quadratic potential

In this section we prove Theorem 1.2 for $p = 2$.

Step 1. In this step we obtain a priori estimates from the existence theory. For $f \in W^{1,2}(I, W^{-1,2}_\sigma(\Omega))$ with $f(0) \in L^2(\Omega)$ and $u_0 \in W^{2,2}(\Omega) \cap W^{1,2}_\sigma(\Omega)$ we know the existence of a unique weak solution of (1.1) with (1.2) fulfilling

$$\begin{gathered} u \in L^\infty(I, L^2(\Omega)) \cap L^2(I, W^{1,2}_\sigma(\Omega)), \\ \partial_t u \in L^\infty(I, L^2(\Omega)) \cap L^2(I, W^{1,2}_\sigma(\Omega)), \quad \pi \in L^2(I, L^2(\Omega)). \end{gathered} \tag{4.1}$$

It can be shown using Galerkin approximation. Let $\{\omega^k\}_{k=1}^\infty$ be the orthogonal basis of $L^2_\sigma(\Omega)$ and $W^{1,2}_\sigma(\Omega)$ consisting of eigenvectors of the Stokes operator with perfect slip boundary conditions. Such basis can be easily constructed provided Ω is non-circular domain. Set $H^n = \operatorname{span}\{\omega_1, \dots, \omega^N\}$ and define the continuous projection $P^N : L^2_\sigma(\Omega) \to H^N$ as follows:

$$P^N u = \sum_{k=1}^N (u, \omega^k)\omega^k.$$

Define $u^N(t,x) = \sum_{k=1}^N c^N_k(t)\omega^k$ where $c^N_k(t)$ solves the Galerkin system

$$\begin{gathered} \langle \partial_t u^N(t), \omega^k\rangle + \int_\Omega \mathcal{S}(Du^N) : D(\omega^k)\,\mathrm{d}x + \int_\Omega (u^n \otimes u^n) : \nabla \omega^k \,\mathrm{d}x = \langle f, w^k\rangle, \\ u^N(0) = u^N_0 = P^N u_0, \quad 1 \le k \le N. \end{gathered} \tag{4.2}$$

After multiplying the Galerkin system (4.2) by $c^N_k(t)$, summing up, using Gronwall's and Korn's inequalities we derive the following a priori estimate,

$$\sup_{t\in I} \|u^N(t)\|_2^2 + \int_I \|u^N(\tau)\|_{1,2}^2 \,\mathrm{d}\tau \le C.$$

Further we apply the time derivative to (4.2), multiply it by $\partial_t c_k^N(t)$ and sum up. Unlike the previous apriori estimates, before using Gronwall's inequality, the boundedness of $\|\partial_t u^N(0)\|_2^2$ needs to be shown. This can be done easily, since $P^N : W^{2,2}(\Omega) \cap W_\sigma^{1,2}(\Omega) \to H^N$ is bounded uniformly with respect to N (c. f. [19, Lemma 4.26]), we can use (4.2). Thus, after Gronwall's inequality we have

$$\sup_{t\in I} \|\partial_t u^N(t)\|_2^2 + \int_I \|\partial_t u^N(\tau)\|_{1,2}^2 \,\mathrm{d}\tau \le C.$$

Passing to the limit with $N \to \infty$ (where we use the Aubin-Lions' lemma to obtain the strong convergence of u^N in $L^2(I, L^4(\Omega))$ and Minty's trick to identify the limit of $\mathcal{S}(Du^N)$ with $\mathcal{S}(Du)$) we get the first two relations in (4.1).

Since $\partial_t u$, $\operatorname{div} \mathcal{S}(Du)$, $\operatorname{div}(u \otimes u)$ and f lie in $L^2(I, W_\sigma^{-1,2}(\Omega))$, we can reconstruct the pressure π at almost every time level via De Rham's theorem and Nečas' theorem on negative norms and obtain $\pi \in L^2(\Omega)$ for almost every $t \in I$.

Step 2. We improve the regularity in space. If we additionally assume $f \in L^\infty(I, L^2(\Omega))$ we are able to show that

$$u \in L^\infty(I, W^{2,2}(\Omega)), \quad \pi \in L^\infty(I, W^{1,2}(\Omega)). \tag{4.3}$$

From Step 1 we know that $\partial_t u$ is regular enough in order to move it to the right hand side of $(1.1)_1$. At almost every time level $t \in I$ we can use the stationary theory. Boundary regularity in tangent direction is based on the difference quotient technique. In normal direction near the boundary the main tools are the operator curl and Nečas' theorem on negative norms. See for example [20, Section 3] for homogeneous Dirichlet boundary conditions. The information about the pressure comes from the fact that the right hand side of $\nabla \pi = f + \operatorname{div} \mathcal{S} - \operatorname{div}(u \otimes u) - \partial_t u$ is in $L^2(\Omega)$ for a. a. $t \in I$. Adding the assumption $\int_\Omega \pi \,\mathrm{d}x = 0$ we get by Poincaré inequality the existence of $\pi \in W^{1,2}(\Omega)$ at almost every time level $t \in I$ together with a bound independent of t.

Step 3. We improve the regularity in time using L^q theory for Stokes system. If we moreover suppose that $f \in L^{q_1}(I, W_\sigma^{-1,q_1'}(\Omega))$ for some $q_1 > 2$ and $u_0 \in W^{2+\beta,2}(\Omega)$ for $\beta \in (0, 1/4)$ we are able to prove the existence of $q_2 > 2$ such that the unique weak solution satisfies for all $q \in (2, q_2)$

$$\partial_t u \in L^q(I, W_\sigma^{1,q}(\Omega)) \cap BUC(I, B_{q,q,B,\sigma}^{1-2/q}(\Omega)). \tag{4.4}$$

Denoting $w := \partial_t u$ and $\tau := \partial_t \pi$ in the sense of distributions, we observe from (1.1) that (w, τ) solves

$$\int_I \langle \partial_t w, \varphi \rangle \,\mathrm{d}t + \int_Q \partial_{Du}^2 \Phi(|Du|) : Dw \otimes D\varphi \,\mathrm{d}x \,\mathrm{d}t = \int_I \langle \partial_t (f - (u \cdot \nabla) u), \varphi \rangle \,\mathrm{d}t, \tag{4.5}$$

for all $\varphi \in L^q(I, W_\sigma^{1,q}(\Omega))$. It is easy to see that $\partial_t(u \cdot \nabla u) \in L^s(I, W^{-1,s}(\Omega))$ for all $s \in [1, 4]$.

To obtain (4.4) as a result of application of Lemma 3.11 for the system (4.5) we need to ensure that $\|\partial_t u(0)\|_{B_{q,q,B,\sigma}^{1-2/q}(\Omega)}$ is bounded. Let $\beta \in (0, 1/4)$ and $\varphi \in W^{-\beta,2}(\Omega)$ with $\|\varphi\|_{W^{-\beta,2}(\Omega)} \le 1$ be arbitrary. We recall that the Helmholtz projection P enjoys the continuity properties as mentioned in Remark 3.3. Thus,

$$\begin{aligned} |\langle \partial_t u(0), \varphi\rangle| &= |\langle \partial_t u(0), P\varphi\rangle| \\ &\le |\langle \operatorname{div} S(Du_0) + (u_0\cdot\nabla)u_0 - f(0), P\varphi\rangle| \\ &\le C(\|u_0\|_{W^{2+\beta,2}(\Omega)} + \|u_0\|^2_{W^{2,2}(\Omega)} + \|f(0)\|_{W^{\beta,2}(\Omega)}) \le C. \end{aligned} \tag{4.6}$$

Since $W^{\beta,2}(\Omega) \hookrightarrow B^{1-2/q}_{q,q}(\Omega)$ if q is close enough to 2 we obtain $\|\partial_t u(0)\|_{B^{1-2/q}_{q,q,B,\sigma}(\Omega)} \le C$ for all $q \in (2, q_2)$ where q_2 is sufficiently close to 2.

Step 4. We show that $u \in L^\infty(I, W^{2,q}(\Omega))$ due to the stationary theory. The previous step shows us that $\partial_t u \in L^\infty(I, L^q(\Omega))$ for some $q > 2$. Therefore we are able to move $\partial_t u$ to the right hand side of $(1.1)_1$ and apply the result [14, Theorem 3] for $p = 2$ which tells us that there exists a positive ε, such that $u \in W^{2,2+\varepsilon}(\Omega)$ and $\pi \in W^{1,2+\varepsilon}(\Omega)$ for (1.1) with perfect slip boundary conditions.

Step 5. We improve the regularity of π in time. There exists a $q > 2$ such that for all $s \in (0, \frac{1}{2})$

$$\pi \in W^{s,q}(I, L^q(\Omega)).$$

We closely follow the proof of [13, Lemma 3.4]. For a function $g(t)$ defined on the time interval I and $(t_1, t_2) \subset I$ set $\delta^t g := g(t_2) - g(t_1)$. The idea of the proof is based on subtracting the equation $(1.1)_1$ in the time t_2 from the same equation in time t_1 which leads to

$$\int_\Omega \delta^t \pi \operatorname{div}\varphi \,\mathrm{d}x = \int_\Omega [\delta^t(\partial_t u - f)\varphi - \delta^t(u\otimes u - \mathcal{S}(Du))D\varphi]\,\mathrm{d}x, \tag{4.7}$$

which holds for all $\varphi \in W^{1,2}(\Omega)$ with $\varphi\cdot\nu = 0$ on $\partial\Omega$. From (4.3) and (4.4) one may easily show the existence of $q > 2$ and $s \in (0, 1/2)$ such that $u \in W^{s,q}(I, W^{1,q}(\Omega))$ and $\partial_t u \in W^{s,q}(I, L^q(\Omega))$. Together with the assumptions on the right hand side f we can notice that (4.7) holds also for all $\varphi \in W^{1,q'}(\Omega)$ with $\varphi = 0$ at $\partial\Omega$. Consider the problem

$$\begin{gathered} \operatorname{div}\varphi^t = \delta^t\pi|\delta^t\pi|^{q-2} - \frac{1}{|\Omega|}\int_\Omega \delta^t\pi|\delta^t\pi|^{q-2}\,\mathrm{d}x \quad \text{in } \Omega, \\ \varphi^t = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{4.8}$$

The right hand side of (4.8) has zero mean value over Ω and belongs to $L^{q'}(\Omega)$ due to (4.3), therefore Bogovskiĭ's Lemma (for the formulation and proof c.f. [6, Lemma 3.3]) guaranties the existence of φ^t satisfying the estimate $\|\varphi^t\|_{1,q'} \le C\|\delta^t\pi\|_q^{q-1}$. Taking φ^t as a test function in (4.7) leads to

$$\|\delta^t\pi\|_q^q \le \varepsilon\|\delta^t\pi\|_q^q + C_\varepsilon(\|\delta^t\partial_t u\|_q^q + \|\delta^t f\|_{-1,q}^q + \|\delta^t\nabla u\|_q^q). \tag{4.9}$$

Dividing (4.9) by $|t_2 - t_1|^{1+sq}$ and integrating twice over I gives

$$\|\pi\|^q_{W^{s,q}(I,L^q(\Omega))} = \int_I\int_I \frac{\|\delta^t\pi\|_q^q}{|t_2 - t_1|^{1+sq}}\,\mathrm{d}t_1\,\mathrm{d}t_2 \le C,$$

which completes the proof.

Step 6. We summarize the result of this section and uses imbedding theorems to complete the proof. Up to now we have shown

$$u \in L^\infty(I, W^{2,q}(\Omega)) \cap W^{1,q}(I, L^q(\Omega)), \quad \pi \in L^\infty(I, W^{1,q}(\Omega)) \cap W^{s,q}(I, L^q(\Omega)).$$

As we are in two dimensions, $q > 2$, $s \in (\frac{1}{q}, \frac{1}{2})$, following imbeddings hold

$$L^\infty(I, W^{1,q}(\Omega)) \hookrightarrow L^\infty(I, \mathcal{C}^{0,1-\frac{2}{q}}(\overline{\Omega})), \tag{4.10}$$

$$W^{1,q}(I, L^q(\Omega)) \hookrightarrow \mathcal{C}^{1-\frac{1}{q}}(\overline{I}, L^q(\Omega)), \tag{4.11}$$

$$W^{s,q}(I, L^q(\Omega)) \hookrightarrow \mathcal{C}^{s-\frac{1}{q}}(\overline{I}, L^q(\Omega)). \tag{4.12}$$

Now we are ready to apply the following lemma.

Lemma 4.1 ([13, Lemma 2.6]). *Let $\Omega \subset \mathbb{R}^2$ be a bounded $\mathcal{C}^2$ domain. Let $f \in L^\infty(I, \mathcal{C}^{0,\alpha}(\overline{\Omega}))$ and $f \in \mathcal{C}^{0,\beta}(\overline{I}, L^s(\Omega))$ for some $\alpha, \beta \in (0,1)$ and $s > 1$. Then $f \in \mathcal{C}^{0,\gamma}(\overline{Q})$ with $\gamma = \min\{\alpha, \frac{\alpha\beta s}{\alpha s+2}\}$.*

Using (4.10) and (4.11) together with Lemma 4.1 we obtain $\nabla u \in \mathcal{C}^{0,\alpha}(\overline{Q})$ for certain $\alpha > 0$. (4.10), (4.12) with Lemma 4.1 gives us $\pi \in \mathcal{C}^{0,\alpha}(\overline{Q})$ for some $\alpha > 0$, which concludes the proof of main results for $p = 2$.

5. Proof of the main results for the super-quadratic potential

In this section we prove Theorem 1.2 for $p > 2$. The proof consists of several steps.

Step 1. We introduces quadratic approximations. In a similar way as in [18] we are concerned with the regularized problem

$$\begin{gathered} \partial_t u^\varepsilon - \operatorname{div} \mathcal{S}^\varepsilon(Du^\varepsilon) + (u^\varepsilon \cdot \nabla)u^\varepsilon + \nabla\pi^\varepsilon = f, \quad \operatorname{div} u^\varepsilon = 0 \quad \text{in } Q, \\ u^\varepsilon(0,\cdot) = u_0 \quad \text{in } \Omega, \end{gathered} \tag{5.1}$$

where we consider quadratic approximation $\mathcal{S}^\varepsilon$ of $\mathcal{S}$ defined for $\varepsilon \in (0,1)$ by the truncation of the viscosity μ from above,

$$\mu^\varepsilon(|Du^\varepsilon|) := \min\Big\{\mu(|Du|), \frac{1}{\varepsilon}\Big\}, \quad \mathcal{S}^\varepsilon(Du^\varepsilon) := \mu^\varepsilon(|Du^\varepsilon|)Du^\varepsilon. \tag{5.2}$$

Scalar potential Φ^ε to $\mathcal{S}^\varepsilon(Du^\varepsilon)$ can be constructed in the following way

$$\Phi^\varepsilon(s) := \int_0^s \mu^\varepsilon(t)t \, \mathrm{d}t$$

and satisfies growth conditions (1.3) for $p = 2$, i.e. there exists $C_1 > 0$ and $C(\varepsilon)$ such that for all $A, B \in \mathbb{R}^{2\times 2}_{\text{sym}}$

$$C_1|B|^2 \le \partial_A^2 \Phi^\varepsilon(|A|) : B \otimes B \le C(\varepsilon)|B|^2. \tag{5.3}$$

The approximation (5.2) guarantees that for a fixed $\varepsilon \in (0,1)$ the results of the previous section holds for u^ε and π^ε solving (5.1) equipped with the perfect slip boundary conditions.

Step 2. We present growth conditions dependent on ε. Due to the results of the previous section we are able to use techniques which enable us to gain uniform estimates with respect to ε. At first we need a growth estimates of Φ^ε with precise dependence on ε. In other words, the constant $C(\varepsilon)$ in the estimate (5.3) needs to be specified. To this purpose we define the function ϑ_ε by $\vartheta_\varepsilon(s) := \min\{(1+s^2)^{\frac{1}{2}}, \frac{1}{\varepsilon}\}$. Now, there exist constants $0 < C_3 \le C_4$ such that for all $\varepsilon \in (0,1)$ and $A, B \in \mathbb{R}^{2\times 2}_{\text{sym}}$

$$C_3\vartheta_\varepsilon(|A|)^{p-2}|B|^2 \le \partial_A^2 \Phi^\varepsilon(|A|) : B \otimes B \le C_4\vartheta_\varepsilon(|A|)^{p-2}|B|^2. \tag{5.4}$$

As a corollary of (5.4), the following estimates can be derived (see [20, Lemma 2.22] for the proof.)

$$C\vartheta_\varepsilon(|A|)^{p-2}|A|^2 \le \mathcal{S}^\varepsilon(A) : A, \tag{5.5}$$

$$C|\mathcal{S}^\varepsilon(A)| \le \vartheta_\varepsilon(|A|)^{p-2}|A|. \tag{5.6}$$

The lower estimate in (5.5) can be done independent of ε, since (5.3) holds:

$$C_5|A|^2 \le \mathcal{S}^\varepsilon(A) : A. \tag{5.7}$$

At this point we would like to emphasize that from now all constants in following steps are independent of ε.

Step 3. We provide $L^\infty(I, L^2(\Omega)) \cap L^2(I, W^{1,2}(\Omega))$ estimates of u^ε and $\partial_t u^\varepsilon$. We recall estimates from the previous section which hold also for the approximated problem since the lower bound in (5.7) is independent on ε.

$$\|u^\varepsilon\|_{L^\infty(I,L^2(\Omega))} + \|\nabla u^\varepsilon\|_{L^2(Q)} \le C, \tag{5.8}$$

$$\|\partial_t u^\varepsilon\|^2_{L^\infty(I,L^2(\Omega))} + \|\nabla \partial_t u^\varepsilon\|_{L^2(Q)} \le C. \tag{5.9}$$

The relation (5.8) is an a priori estimate obtained by taking solution as a test function (at the level of Galerkin approximation). Roughly speaking, the estimate (5.9) is performed by taking time derivative of the equation (5.1) and testing by time derivative of u^ε. More precisely, it is not applied directly to the equation (5.1), but still to the Galerkin system. To estimate the time derivative of the Galerkin approximation of u^ε at the time $t = 0$ we proceed in the same way like in (4.6).

Note that (5.8) and (5.9) give $u^\varepsilon \in L^\infty(I, W^{1,2}(\Omega))$,

$$\begin{aligned}\|\nabla u^\varepsilon(s,\cdot)\|_2^2 - \|\nabla u^\varepsilon(0,\cdot)\|_2^2 &= \int_\Omega \int_0^s \partial_t |\nabla u^\varepsilon(t,\cdot)|^2 \,\mathrm{d}t\,\mathrm{d}x \\ &\le 2\|\nabla u^\varepsilon\|_{L^2(Q)} \|\partial_t \nabla u^\varepsilon\|_{L^2(Q)} \le C.\end{aligned}$$

Step 4. We escribe the boundary $\partial\Omega$. To discuss boundary regularity in following steps, we need a suitable description of the boundary $\partial\Omega$. Let us denote $x = (x_1, x_2)$. We suppose that $\Omega \in \mathcal{C}^3$, therefore there exists $c_0 > 0$ such that for all $a_0 > 0$ there exists n_0 points $P \in \partial\Omega$, $r > 0$ and open smooth set $\Omega_0 \subset\subset \Omega$ that we have

$$\Omega \subset \Omega_0 \cup \bigcup_P B_r(P)$$

and for each point $P \in \partial\Omega$ there exists local system of coordinates for which $P = 0$ and the boundary $\partial\Omega$ is locally described by $\mathcal{C}^3$ mapping a_P that for $x_1 \in (-3r, 3r)$ fulfils

$$\begin{gathered} x \in \partial\Omega \Leftrightarrow x_2 = a_P(x_1), \quad B_{3r}(P) \cap \Omega = \{x \in B_r(P) \,\text{and}\, x_2 > a_P(x_1)\} =: \Omega^P_{3r}, \\ \partial_1 a_P(0) = 0, \quad |\partial_1 a_P(x_1)| \le a_0, \quad |\partial_1^2 a_P(x_1)| + |\partial_1^3 a_P(x_1)| \le c_0. \end{gathered}$$

Point P can be divided into k groups such that in each group Ω^P_{3r} are disjoint and k depends only on dimension n. Let the cut-off function $\xi_P(x) \in \mathcal{C}^\infty(B_{3r}(P))$ and reaches values

$$\xi_P(x) \begin{cases} = 1 & x \in B_r(P), \\ \in (0,1) & x \in B_{2r}(P) \setminus B_r(P), \\ = 0 & x \in \mathbb{R}^2 \setminus B_{2r}(P). \end{cases}$$

Next, we assume that we work in the coordinate system corresponding to P. Particularly, $P = 0$. Let us fix P and drop for simplicity the index P. The tangent vector and the outer normal vector to $\partial\Omega$ are defined as

$$\tau = \big(1, \partial_1 a(x_1)\big), \quad \nu = \big(\partial_1 a(x_1), -1\big),$$

tangent and normal derivatives as

$$\partial_\tau = \partial_1 + \partial_1 a(x_1)\partial_2, \quad \partial_\nu = -\partial_2 + \partial_1 a(x_1)\partial_1.$$

Step 5. We show that $u^\varepsilon \in L^\infty(I, W^{2,2}(\Omega))$ uniformly in $\varepsilon \in (0,1)$. From Step 3 we obtained that $\partial_t u^\varepsilon \in L^\infty(I, L^2(\Omega))$, therefore we can fix $t \in I$, move $\partial_t u^\varepsilon$ to the right hand side of (5.1) and at almost every time level consider the stationary problem

$$\begin{gathered} -\operatorname{div} \mathcal{S}^\varepsilon(Du^\varepsilon) + (u^\varepsilon \cdot \nabla)u^\varepsilon + \nabla\pi^\varepsilon = h, \quad \operatorname{div} u^\varepsilon = 0 \quad \text{in } \Omega, \\ u^\varepsilon \cdot \nu = 0, \quad [\mathcal{S}^\varepsilon(Du^\varepsilon)\nu] \cdot \tau = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{5.10}$$

where $h := f - \partial_t u^\varepsilon \in L^2(\Omega)$. Previous section provides $u^\varepsilon \in W^{2,2}(\Omega)$, $\mathcal{S}^\varepsilon(Du^\varepsilon) \in W^{1,2}(\Omega)$ and $\pi^\varepsilon \in W^{1,2}(\Omega)$. Thus we can multiply (5.10) by a suitable test function which is at least in $L^2(\Omega)$ and integrate over Ω. We focus only on the boundary regularity and work in the local system of coordinates. Following [18, Lemma 4.2, Remark 4.9] we choose as a test function $\varphi = (\varphi_1, \varphi_2)$,

$$\begin{gathered} \varphi = (\partial_2[\Theta - \partial_\tau(u^\varepsilon \cdot \nu)\xi^2], \partial_1[-\Theta + \partial_\tau(u^\varepsilon \cdot \nu)\xi^2]), \\ \Theta := \partial_\nu(u^\varepsilon \cdot \tau)\xi^2 - u^\varepsilon \cdot (\partial_\nu \tau + \partial_\tau \nu)\xi^2. \end{gathered}$$

This test function is constructed to get rid of the pressure π^ε and to obtain optimal information from the elliptic term. These most difficult estimates, in which we extract from $-\int_\Omega \operatorname{div} \mathcal{S}^\varepsilon(Du^\varepsilon) \cdot \varphi \, dx$ boundedness of the term $\int_\Omega \mu^\varepsilon(|Du^\varepsilon|)|\nabla^2 u^\varepsilon|^2 \, dx$, are done in [18, Proof of Theorem 1.7], therefore we omit the calculations. It remains to estimate the convective term and the right hand side of (5.10). After long, but elementary calculations we are able to show that

$$\Big|\int_\Omega (u^\varepsilon \cdot \nabla)u^\varepsilon \cdot \varphi \, dx\Big| \le C \int_\Omega (|u^\varepsilon||\nabla u^\varepsilon|^2 + |u^\varepsilon|^2|\nabla u^\varepsilon|) \, dx, \tag{5.11}$$

where we used the divergence-free constraint and the properties of the test function φ. Using Hölder and Young inequalities, $\|\cdot\|_4^2 \le C\|\cdot\|_{1,2}\|\cdot\|_2$ and the information $u^\varepsilon \in L^\infty(I, W^{1,2}(\Omega))$ we continue estimating (5.11):

$$C(\|u^\varepsilon\|_2\|\nabla u^\varepsilon\|_4^2 + \|u^\varepsilon\|_4^2\|\nabla u^\varepsilon\|_2) \le \varepsilon\|\nabla^2 u^\varepsilon\|_2^2 + C\|u\|_{1,2}^2 + C\|\nabla u^\varepsilon\|_2^2\|u^\varepsilon\|_2^2.$$

The last estimate is easy.

$$\Big|\int_\Omega h \cdot \varphi \, dx\Big| \le \int_\Omega |h|(|\nabla^2 u^\varepsilon| + |\nabla u^\varepsilon| + |u^\varepsilon|) \, dx \le C\|h\|_2^2 + \varepsilon\|\nabla^2 u^\varepsilon\|_2^2 + C\|u\|_{1,2}^2.$$

Since $\mu^\varepsilon(|Du^\varepsilon|) > 1$ and $\varepsilon > 0$ can be chosen arbitrarily small, we obtain

$$\|\nabla^2 u^\varepsilon\|_2^2 \le \int_\Omega \mu^\varepsilon(|Du^\varepsilon|)|\nabla^2 u^\varepsilon|^2 \, dx \le C, \tag{5.12}$$

where C does not depend on ε and $t \in I$, therefore we have

$$u^\varepsilon \in L^\infty(I, W^{2,2}(\Omega)). \tag{5.13}$$

Step 6. We improve the information about $\partial_t u^\varepsilon$. In the same spirit as in Step 3 from the previous section we denote $w := \partial_t u$ and $\tau := \partial_t \pi$ in the sense of distributions, which solves (4.5) where Φ is replaced by Φ^ε. The right hand side of (4.5) is bounded uniformly with respect to $\varepsilon \in (0,1)$ in $L^{q_0}(I, W_\sigma^{-1,q_0'}(\Omega))$ for some $q_0 > 2$, since from (5.8), (5.9) and (5.13) we have $\partial_t[(u^\varepsilon \cdot \nabla)u^\varepsilon] \in L^s(I, W^{-1,s}(\Omega))$ for all $s \in [1,4]$.

Set $V_\varepsilon := \sup_Q |\vartheta_\varepsilon(|Du^\varepsilon|)|$. From (5.4) we have for all $t \in I$, $x \in \Omega$, for all $\varepsilon \in (0,1)$ and $A, B \in \mathbb{R}^{2\times 2}_{\rm sym}$

$$c|B|^2 \le \partial_A^2 \Phi^\varepsilon(|A|) : B \otimes B \le C V_\varepsilon^{p-2}(|A|)|B|^2.$$

From Lemma 3.11 we have the existence of positive constants K and L such that for all $q \in (2, q_2]$, where $q_2 := 2 + L/V_\varepsilon^{p-2}$ holds

$$\begin{aligned} &\|\nabla w\|_{L^q(Q)} + V_\varepsilon^{\frac{2-p}{q}} \|w\|_{BUC(I, B^{1-2/q}_{q,q,B,\sigma}(\Omega))} \\ &\le K\Big(\|f\|_{L^q(I,W^{-1,q'}(\Omega))} + V_\varepsilon^{(p-2)(1-1/q)} \|\partial_t u_0\|_{B^{1-2/q}_{q,q,B,\sigma}(\Omega)}\Big). \end{aligned} \tag{5.14}$$

Without loss of generality we may assume that $q_2 < q_0$. Thus, after estimating last norm on the right hand side of (5.14) in the same way like in Step 3 in Section 3 we have

$$\|\partial_t u^\varepsilon\|_{BUC(I, B^{1-2/q}_{q,q,B,\sigma}(\Omega))} \le C\Big(V_\varepsilon^{\frac{p-2}{q}} + V_\varepsilon^{p-2}\Big) \le C V_\varepsilon^{p-2}.$$

Step 7. We improve the information about $\nabla^2 u^\varepsilon$. In this step we obtain better space regularity. Up to now we have $\vartheta_\varepsilon \in L^\infty(I, W^{1,2}(\Omega))$. We are going to show that $\vartheta_\varepsilon \in L^\infty(I, W^{1,q}(\Omega))$ for some $q > 2$.

We omit estimates of $\nabla^2 u^\varepsilon$ in the interior of Ω and we focus on estimates near the boundary. We start with the tangential direction. Localizing the problem, we work in Ω^P_{3r}, where the boundary is locally described by the $\mathcal{C}^3$ mapping a_p. For simplicity we drop the index P.

We multiply (5.10) by $-\partial_\tau \varphi\xi$, integrate over Ω_{3r} and after similar steps as in [18, Lemma 4.6] we derive the identity

$$\begin{aligned} &\int_{\Omega_{3r}} \partial_\tau \mathcal{S}^\varepsilon(Du^\varepsilon) : D\varphi\xi \,\mathrm{d}x \\ &= -\int_{\Omega_{3r}} h \cdot \partial_\tau(\varphi\xi)\,\mathrm{d}x + \int_\Omega (u^\varepsilon \cdot \nabla)u^\varepsilon \partial_\tau \varphi\xi \,\mathrm{d}x \\ &\quad + \int_{\Omega_{3r}} \mathcal{S}^\varepsilon(Du^\varepsilon) : \Big[\partial_\tau\varphi \otimes \nabla\xi - \nabla\varphi\partial_\tau\xi + (\partial_1^2 a, 0) \otimes \partial_2\varphi\xi + \nabla\Big(\varphi \cdot \partial_\tau \nu \frac{\nu}{|\nu|^2}\xi\Big)\Big]\,\mathrm{d}x \\ &\quad + \int_{\Omega_{3r}} \operatorname{div} \mathcal{S}^\varepsilon(Du^\varepsilon) \cdot [(\varphi \cdot \partial_\tau\nu)\frac{\nu}{|\nu|^2}\xi - \varphi\partial_\tau\xi]\,\mathrm{d}x \\ &\quad + \int_{\Omega_{3r}} \partial_1^2 a[h_2 + (\operatorname{div}\mathcal{S}^\varepsilon(Du^\varepsilon))_2 - (u^\varepsilon \cdot \nabla u^\varepsilon)_2]\varphi_1\xi\,\mathrm{d}x \\ &\quad + \int_{\Omega_{3r}} [h_1 + \partial_1 a h_2 + (\operatorname{div}\mathcal{S}^\varepsilon(Du^\varepsilon))_1 + \partial_1 a(\operatorname{div}\mathcal{S}^\varepsilon(Du^\varepsilon))_2 + (u^\varepsilon \cdot \nabla u^\varepsilon)_1 \\ &\quad + \partial_1 a(u^\varepsilon \cdot \nabla u^\varepsilon)_2]\varphi\nabla\xi\,\mathrm{d}x \end{aligned} \tag{5.15}$$

for all $\varphi \in W^{1,q'}_\sigma(\Omega)$, $\operatorname{supp}\varphi \subset \overline{\Omega_{3r}}$. Terms on the right hand side of (5.15) comes at first from the fact that we add and subtract some lower order terms in order to let the boundary term vanish while integrating by parts. Second, tangent derivative does not commute with the gradient and we use $\nabla\partial_\tau\varphi = \partial_\tau\nabla\varphi + (\partial_1^2 a, 0)\otimes\partial_2\varphi$. Third, we use (5.10) and replace $\partial_2\pi^\varepsilon$ by $h_2 + (\operatorname{div}\mathcal{S}^\varepsilon(Du^\varepsilon))_2 + (u^\varepsilon\cdot\nabla u^\varepsilon)_2$ and similarly for $\partial_\tau\pi^\varepsilon$.

We denote $w := \partial_\tau u^\varepsilon\xi - (0, \partial_1^2 a u_1^\varepsilon)\xi + z$, where z is the solution of

$$\operatorname{div} z = -\partial_\tau u^\varepsilon\cdot\nabla\xi - \partial_1^2 a u_1^\varepsilon\partial_2\xi \quad \text{in } \Omega_{3r}, \tag{5.16}$$

$$z = 0 \quad \text{on } \partial\Omega_{3r}. \tag{5.17}$$

The right hand side of (5.16) was obtained from $\operatorname{div}\big(-\partial_\tau u^\varepsilon\xi + (0,\partial_1^2 a u_1^\varepsilon)\xi\big)$ using the fact that $\operatorname{div} u^\varepsilon = 0$. The role of z is to ensure that $\operatorname{div} w = 0$. On $\partial\Omega$ it holds $w\cdot\nu = 0$ since

$$w\cdot\nu = [\partial_\tau u^\varepsilon\cdot\nu + \partial_1^2 a u_1^\varepsilon]\xi + z\cdot\nu = \partial_\tau(u^\varepsilon\cdot\nu)\xi = 0.$$

Thus, the compatibility condition holds

$$\begin{aligned}\int_{\partial\Omega} z\cdot\nu \,\mathrm{d}\sigma &= \int_\Omega \operatorname{div} z\,\mathrm{d}x = \int_\Omega \operatorname{div}(-\partial_\tau u^\varepsilon\xi + (0,\partial_1^2 a u_1^\varepsilon)\xi)\,\mathrm{d}x\\ &= -\int_{\partial\Omega}\partial_\tau(u^\varepsilon\cdot\nu)\xi\,\mathrm{d}\sigma = 0\end{aligned}$$

and z solving (5.16) and (5.17) exists by Bogovskiĭ's Lemma and enjoys the estimate $\|z\|_{1,q} \le C\|\nabla u^\varepsilon\|_q$ for some $C > 0$.

Using the definition of w, from (5.15) we obtain

$$\int_\Omega \partial_{Du^\varepsilon}\mathcal{S}^\varepsilon(Du^\varepsilon) : Dw\otimes D\varphi\,\mathrm{d}x = \langle g,\varphi\rangle \quad \forall\varphi\in W^{1,q'}_\sigma(\Omega),$$

with

$$\begin{aligned}\langle g,\varphi\rangle = \text{RHS of } (5.15) + \int_\Omega &\partial_{Du^\varepsilon}\mathcal{S}^\varepsilon(Du^\varepsilon) : [Dz + \partial_\tau u^\varepsilon\otimes\nabla\xi\\ &+ (\partial_1^2 a, 0)\otimes_S \partial_2 u^\varepsilon\xi - D\big((0,\partial_1^2 a,0)\xi\big)]D\varphi\,\mathrm{d}x.\end{aligned}$$

Due to the assumption on f and results from Step 4 we have $\|g\|_{-1,q_2'} \le CV_\varepsilon^{p-2}$ and after application of Lemma 3.12 we obtain

$$\|\nabla\partial_\tau u^\varepsilon\xi\|_{L^q(\Omega)} \le CV_\varepsilon^{p-2}. \tag{5.18}$$

We recall that q depends on ε by the relation $q\in(2, 2+L/V_\varepsilon^{p-2}]$. In order to control whole $\nabla^2 u^\varepsilon$ we need an estimate of type (5.18) in the normal direction which is locally x_2. Since $\partial_2^2 u_2^\varepsilon$ can be expressed from the condition $\operatorname{div} u^\varepsilon = 0$, we focus on $\partial_2^2 u_1^\varepsilon$. Following [15, Theorem 3.19] we can extract the desired estimate from the equation (5.10) after employment of the operator curl. Let us shorten $\mathcal{S}^\varepsilon(Du^\varepsilon)$ to $\mathcal{S}^\varepsilon$ and $\vartheta_\varepsilon(|Du^\varepsilon|)$ to ϑ_ε. Denoting $G := \partial_2\mathcal{S}^\varepsilon_{12}$ we have due to (5.6) and (5.4),

$$\begin{gathered}\|\xi G\|_{-1,q} \le \|\mathcal{S}^\varepsilon_{12}\|_q \le \|{\vartheta_\varepsilon}^{p-2}Du^\varepsilon\|_q,\\ \|\partial_1(\xi G)\|_{-1,q} \le C\|{\vartheta_\varepsilon}^{p-2}Du^\varepsilon\|_q + C'\|{\vartheta_\varepsilon}^{p-2}\partial_1\nabla u^\varepsilon\|_q.\end{gathered}$$

From (5.10) after applying curl we have

$$\begin{aligned}\|\partial_2(\xi G)\|_{-1,q} &\le C(\|\partial_1(\mathcal{S}^\varepsilon_{21} + \mathcal{S}^\varepsilon_{22} - \mathcal{S}^\varepsilon_{11})\|_q + \|f\|_q + \|u^\varepsilon\cdot\nabla u^\varepsilon\|_q + \|\partial_t u^\varepsilon\|_q)\\ &\le C\big\{\|{\vartheta_\varepsilon}^{p-2}Du^\varepsilon\|_q + \|{\vartheta_\varepsilon}^{p-2}\partial_1\nabla u^\varepsilon\|_q + V_\varepsilon^{p-2} + 1\big\} := H.\end{aligned}$$

Nečas' theorem on negative norms gives

$$\|\xi G\|_q \le C(\|\xi G\|_{-1,q} + \|\nabla(\xi G)\|_{-1,q}) \le H.$$

From the definition of G and symmetry of Du we obtain

$$\partial_{12}\mathcal{S}^\varepsilon_{12}\partial_2 Du^\varepsilon_{12} = \frac{G}{2} - \frac{1}{2}\partial_{11}\mathcal{S}^\varepsilon_{12}\partial_2 Du^\varepsilon_{11} - \frac{1}{2}\partial_{22}\mathcal{S}^\varepsilon_{12}\partial_2 Du^\varepsilon_{22}.$$

Using $\partial_{12}\mathcal{S}^\varepsilon_{12} \ge C{\vartheta_\varepsilon}^{p-2}$ and the condition $\operatorname{div} u^\varepsilon = 0$ we get that $\|\xi {\vartheta_\varepsilon}^{p-2}\partial_2^2 u_1^\varepsilon\|_q \le H$. Thus,

$$\|\xi\vartheta_\varepsilon^{p-2}\nabla^2 u^\varepsilon\|_q \le C\|\xi G\|_q + \|\xi\vartheta_\varepsilon^{p-2}\nabla\partial_\tau u^\varepsilon\|_q + \tilde{C}\sup_{x_1\in(-3r,3r)}|\partial_1 a|\|\xi\vartheta_\varepsilon^{p-2}\nabla^2 u^\varepsilon\|_q, \tag{5.19}$$

where $\tilde{C}$ is absolute constant. Since we can choose r sufficiently small in order to $\tilde{C}\max_{P\in\partial\Omega}\sup_{x_1\in(-3r,3r)}|\partial_1 a| \le 1/2$, the last term (5.19) can be absorbed into the left hand side. We have

$$\|\xi\vartheta_\varepsilon^{p-2}\nabla^2 u^\varepsilon\|_{q_2} \le CV_\varepsilon^{p-2}V_\varepsilon^{p-2}. \tag{5.20}$$

From (5.12) the boundedness of the term $\int_\Omega \mu^\varepsilon(|Du^\varepsilon|)|\nabla^2 u^\varepsilon|^2\,\mathrm{d}x$ is obtained, in other words $\|\vartheta_\varepsilon^{\frac{p-2}{2}}\nabla^2 u^\varepsilon\|_2 \le C$. Interpolation of this result with (5.20) gives for $q\in(2,q_2)$

$$\|\vartheta_\varepsilon^{\frac{p-2}{2}}\nabla^2 u^\varepsilon\|_q \le CV_\varepsilon^{\beta 2(p-2)}, \tag{5.21}$$

where $1/q = \beta/q_2 + (1-\beta)/2$. Since it holds

$$\|\vartheta_\varepsilon^{p/2}\|_{1,q} \le \|\vartheta_\varepsilon^{p/2}\|_q + \|\vartheta_\varepsilon^{\frac{p-2}{2}}\nabla^2 u^\varepsilon\|_q,$$

we want to use the following lemma for $f = \vartheta_\varepsilon^{p/2}$.

Lemma 5.1. *Let $\Omega\subset\mathbb{R}^2$ be a bounded $\mathcal{C}^2$ domain and $f\in W^{1,q}(\Omega)$ for some $q>2$. Then $f\in\mathcal{C}(\overline{\Omega})$ and there is $C>0$ independent of q such that*

$$\sup_\Omega |f| \le C\Big(\frac{q-1}{q-2}\Big)^{1-1/q}\|f\|_{1,q}. \tag{5.22}$$

Proof. It follows from the proof of [26, Theorem 2.4.1]. The result holds also for $\Omega\subset\mathbb{R}^n$, with $q>n$ and $q-n$ instead of $q-2$ in the denominator of (5.22). ☐

Because $\frac{q-1}{q-2} \le CV^{p-2}$, we obtain

$$V_\varepsilon^{p/2} \le CV^{(p-2)(1-\frac{1}{q})}V_\varepsilon^{\beta 2(p-2)}. \tag{5.23}$$

Note that $(p-2)(1-1/q)\to p/2-1$ as $q\to 2$ and the exponent containing the interpolation parameter β can be made arbitrarily small, therefore we can rewrite (5.23) as $V_\varepsilon \le \hat{C}$. This together with (5.21) gives

$$\sup_{t\in I}\|\nabla^2 u^\varepsilon\|_q \le C.$$

Step 8. We pass from the regularized problem to the original problem. In the previous step we showed $V_\varepsilon \le \hat{C}$, where $V_\varepsilon = \sup_Q|\vartheta_\varepsilon(|Du^\varepsilon|)|$. Since $\vartheta_\varepsilon(s) = \min\{(1+s^2)^{1/2}, \frac{1}{\varepsilon}\} \le \frac{1}{\varepsilon}$, it is sufficient to choose ε in order to have $\hat{C}\le\frac{1}{\varepsilon}$. Thus, $u^\varepsilon = u$ is the solution of the original problem (1.1) and it holds that $\sup_Q(1+|Du|^2)^{1/2} \le C$ which leads to $\sup_{t\in I}\|\nabla^2 u\|_q \le C$.

Since we passed from the regularized problem to the original one, the regularity of pressure π which we proved in Section 4 for quadratic potential holds also for the super-quadratic case. $\square$

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Jakub Tichý

Charles University in Prague, Department of Mathematical Analysis, Sokolovská 83, 186 75 Praha 8, Czech Republic.

Czech Technical University in Prague, Faculty of Information Technology, Thákurova 9, 160 00, Praha 6, Czech Republic

E-mail address: `jakub.tichy@fit.cvut.cz`