Electronic Journal of Differential Equations, Vol. 2014 (2014), No. 72, pp. 1–6.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

BOUNDARY DIFFERENTIABILITY FOR INHOMOGENEOUS INFINITY LAPLACE EQUATIONS

GUANGHAO HONG

Abstract. We study the boundary regularity of the solutions to inhomogeneous infinity Laplace equations. We prove that if $u \in C(\bar{\Omega})$ is a viscosity solution to $\Delta_\infty u := \sum_{i,j=1}^n u_{x_i} u_{x_j} u_{x_i x_j} = f$ with $f \in C(\Omega) \cap L^\infty(\Omega)$ and for $x_0 \in \partial\Omega$ both $\partial\Omega$ and $g := u|_{\partial\Omega}$ are differentiable at x_0, then u is differentiable at x_0.

1. Introduction

Infinity Laplace equation $\Delta_\infty u = 0$ arose as the Euler equation of L^∞ variational problem of $|\nabla u|$, or equivalently, absolutely minimizing Lipschitz extension (AML) problem. This problem was initially studied by Aronsson [1] at the classical solutions level from 1960's. In 1993, Jensen [7] proved that a function $u(x) \in C(\Omega)$ is an AML:

$$\text{for any } V \subset\subset \Omega,\ Lip(u, V) = Lip(u, \partial V)$$

if and only if $u(x)$ is a viscosity solution to $\Delta_\infty u = 0$. Moreover, for any bounded domain $\Omega \subset \mathbb{R}^n$ and $g \in C(\partial\Omega)$, the Dirichlet problem:

$$\Delta_\infty u = 0 \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega \tag{1.1}$$

has an unique viscosity solution. Such an solution is called an infinity harmonic function.

In 2001, Crandall, Evans and Gariepy [3] proved that a function $u(x) \in C(\Omega)$ is an infinity harmonic function if and only if u satisfies the following *comparison with cone property*: for any $V \subset\subset \Omega$ and and $c(x) = a + b|x - x_0|$,

$$u(x) \le c(x) \text{ on } \partial\{V\backslash\{x_0\}\} \Rightarrow u(x) \le c(x) \text{ in } V,$$
$$u(x) \ge c(x) \text{ on } \partial\{V\backslash\{x_0\}\} \Rightarrow u(x) \ge c(x) \text{ in } V.$$

This comparison property turns out to be a very useful tool in the study of many aspects of this equation. Especially, it implies the following conclusions as a direct result [3].

Lemma 1.1. *Let $u(x) \in C(\Omega)$ satisfy comparison with cone property, $x_0 \in \Omega$, $0 < r < \operatorname{dist}(x_0, \partial\Omega)$. Then*

2000 *Mathematics Subject Classification.* 35J25, 35J70, 49N60.
Key words and phrases. Boundary regularity; infinity Laplacian; comparison principle; monotonicity.

Submitted December 17, 2013. Published March 16, 2014.

(1) *the slope functions*

$$S_r^+(x_0) := \max_{x\in\partial B(x_0,r)} \frac{u(x)-u(x_0)}{r}, \quad S_r^-(x_0) := \max_{x\in\partial B(x_0,r)} \frac{u(x_0)-u(x)}{r}$$

are non-negative and non-decreasing as a function of r *for fixed* x_0*. So the limits* $S^\pm(x_0) := \lim_{r\to 0} S_r^\pm(x_0)$ *exist.*

(2) $S^+(x_0) = S^-(x_0) := S(x_0)$.

(3) $S(x)$ *is upper-semicontinuous, i.e.,* $\limsup_{y\to x} S(y) \le S(x)$ *for all* $x \in \Omega$*.*

The lemma implies locally Lipschitz continuity of u immediately. Crandall and Evans [2] applied this lemma to prove that at any interior point x_0, a blow-up limit

$$v(x) = \lim_{r_j\to 0} \frac{u(x_0+r_j x)-u(x_0)}{r_j}$$

of an infinity harmonic function u must be a linear function, i.e., $v(x) = a\cdot x$ for some $a \in \mathbb{R}^n$ with $|a| = S(x_0)$. The sketch of their proof is the following. Firstly, (3) of Lemma 1.1 implies $Lip(v,\mathbb{R}^n) \le S(x_0)$. Secondly, for any $R > 0$ fixed, for every j there exists a maximal direction $e_j \in \mathbb{R}^n$ with $|e_j| = 1$ such that $u(x_0 + Rr_j e_j) = \max_{x\in\partial B_{Rr_j}(x_0)} u(x)$. The sequence $\{e_j\}$ must have an accumulating point say e^+, then $v(Re^+) = Re^+$. For all R, we will have the same e^+. By considering the minimum directions we will get an e^- and moreover $e^- = -e^+$. So v is tight on the line te^+, $t \in (-\infty,\infty)$. Finally, a Lipschitz function on $\mathbb{R}^n$ that is tight on a line must be linear. However, this result does not imply the differentiability of u in general since for different sequences r_j one may get different linear functions v although they must have same slope $S(x_0)$. Ten years later, by using much deeper pde techniques Evans and Smart [4] proved that the blow-up limits are unique and accomplished the proof of interior differentiability. The continuously differentiability is still left open as the most prominent problem in this field although in 2 dimension C^1 and $C^{1,\alpha}$ regularity was achieved by Savin [10] and Evans-Savin [4] respectively.

Boundary regularity for infinity harmonic function was initially studied by Wang and Yu [11]. They proved the following result.

Theorem 1.2. *For* $n \ge 2$*, let* $\Omega \subset R^n$ *be a bounded domain with* $\partial\Omega \in C^1$ *and* $g \in C^1(R^n)$*. Assume that* $u \in C(\bar\Omega)$ *is the viscosity solution of the infinity Laplace equation* (1.1)*. Then* u *is differentiable on the boundary, i.e., for any* $x_0 \in \partial\Omega$*, there exists* $Du(x_0) \in R^n$ *such that*

$$u(x) = u(x_0) + Du(x_0)\cdot(x-x_0) + o(|x-x_0|), \quad \forall x \in \bar\Omega.$$

The boundary differentiability is much easier than interior differentiability. They defined the slope functions near and on the boundary by

$$S_r^+(x) = \sup_{y\in\partial(B(x,r)\cap\Omega)\setminus\{x\}} \frac{u(y)-u(x)}{|y-x|} \text{ and } S_r^-(x) = \sup_{y\in\partial(B(x,r)\cap\Omega)\setminus\{x\}} \frac{u(x)-u(y)}{|y-x|}$$

for $x \in \bar\Omega$ and $r > 0$ small. $S_r^\pm(x)$ are still monotone and have limits $S^\pm(x)$. But $S^+(x) \ne S^-(x)$ in general if $x \in \partial\Omega$. Denote $S(x) := \max\{S^+(x), S^-(x)\}$. $S(x)$ is upper-semicontinuous $\forall x \in \bar\Omega$ with the assumption that both $\partial\Omega$ and g are C^1. They applied a similar argument as in [2] and proved that any blow-up limit of u at a boundary point x_0 is a linear function $v(x) = e\cdot x$ with $|e| = S(x_0)$ on the half space $\mathbb{R}^n_+ = \{x_n > 0\}$. But this time it is very easy to prove the uniqueness of blow-up limits since the tangential part of e is already given by the boundary data. So $e = (\sqrt{S(x_0)^2 - |D_T g(x_0)|^2}, D_T g(x_0))$ or $e = (-\sqrt{S(x_0)^2 - |D_T g(x_0)|^2}, D_T g(x_0))$. The former happens when $S(x_0) = S^+(x_0)$ and the latter happens when $S(x_0) = S^-(x_0)$.

It is not natural to put C^1 assumption on the boundary conditions in order to prove merely differentiability of a solution. In a recent work [6] we improved Wang-Yu's Theorem to the following sharp version.

Theorem 1.3. *Let $\Omega \subset R^n$ be a domain and $u \in C(\bar{\Omega})$ is an infinity harmonic function in Ω. Assume that for $x_0 \in \partial\Omega$, $\partial\Omega$ and $g := u|_{\partial\Omega}$ are differentiable at x_0. Then u is differentiable at x_0.*

Under this weaker assumption, it is not true that $S(x)$ is upper-semicontinuous at x_0. However we managed to show that $\limsup_{x\to x_0} S(x) \le S(x_0)$ if $x \to x_0$ in a non-tangentially way. This is enough to imply $Lip(v, \mathbb{R}^n_+) \le S(x_0)$.

The inhomogeneous infinity Laplace Equation $\Delta_\infty u = f$ was studied by Lu and Wang [9]. They proved existence and uniqueness of a viscosity solution of the Dirichlet problem

$$\Delta_\infty u = f \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega \tag{1.2}$$

under the conditions that $\Omega \subset \mathbb{R}^n$ is bounded, $f \in C(\Omega)$ with $\inf_\Omega f > 0$ or $\sup_\Omega f < 0$ and $g \in C(\partial\Omega)$. They also proved some comparison principles and stability results. Lindgren [8] investigated the interior regularity of viscosity solutions of (1.2). He proved that the blow-ups are linear if $f \in C(\Omega) \cap L^\infty(\Omega)$ and u is differentiable if $f \in C^1(\Omega) \cap L^\infty(\Omega)$. For inhomogeneous equation (1.2), the slope functions $S^\pm_r(x)$ is not monotone anymore, but so is $S^\pm_r(x) + r$ [8, Corollary 1]. Hence the limits $S^\pm(x) := \lim_{r\to 0} S^\pm_r(x)$ still exist and the arguments in [2] and [4] work.

In this paper, we combine the techniques used in [6, 8, 11] to prove boundary differentiability for inhomogeneous infinity Laplace equation.

Theorem 1.4. *Let $\Omega \subset R^n$ and $u \in C(\bar{\Omega})$ is a viscosity solution of the inhomogeneous infinity Laplace equation* (1.2)*. Assume that $f \in C(\Omega) \cap L^\infty(\Omega)$ and for $x_0 \in \partial\Omega$, both $\partial\Omega$ and g are differentiable at x_0. Then u is differentiable at x_0.*

2. Proof of Theorem 1.4

Without lost of generality, we may assume that $x_0 = 0$ and the tangential plane of $\partial\Omega$ at 0 is $\{x = (x', x_n) \in R^n : x_n = 0\}$. Denote $B(x, r) := \{y \in R^n : |y - x| < r\}$ for $x \in R^n$, $B(r) := B(0, r)$, $\hat{B}(x', r) := \{y' \in R^{n-1} : |y' - x'| < r\}$ for $x' \in R^{n-1}$ and $\hat{B}(r) := \hat{B}(0, r)$. We assume for some $0 < r_0 < 1$,

$$\Omega \cap B(r_0) = \{x \in B(r_0) : x_n > f(x')\},$$

where $f \in C(\hat{B}(r_0))$ is differentiable at 0 with $f(0) = Df(0) = 0$. Denote $\hat{g}(x') = g(x', f(x'))$ for $x' \in \hat{B}(r_0)$, then $\hat{g}(x') \in C(\hat{B}(r_0))$ is differentiable at 0.

We will prove the following easier conclusion first and then apply it to prove Theorem 1.4.

Proposition 2.1. *Assume that u, f, Ω and g satisfy the conditions in Theorem 1.4. We assume additionally $\hat{g}(x') \in C^1(\hat{B}(r_0))$. Then u is differentiable at* 0.

For $x \in \bar{\Omega} \cap B_{r_0/2}$ and $0 < r < r_0/2$, we define

$$S_r^+(x) = \sup_{y \in \partial(B(x,r)\cap\Omega)\setminus\{x\}} \frac{u(y)-u(x)}{|y-x|}$$

and

$$S_r^-(x) = \sup_{y \in \partial(B(x,r)\cap\Omega)\setminus\{x\}} \frac{u(x)-u(y)}{|y-x|}.$$

We make the following two assumptions on the solution u of (1.2) as in [8]:

(A1) $S_r^{\pm}(x) \geq 1$ for all x and r;
(A2) $-\frac{3}{4} \leq f \leq -\frac{1}{4}$.

This is not a restriction since we can define

$$\tilde{u}(x_1,\dots,x_{n+2}) = \frac{u(x_1,\dots,x_n)}{4^{\frac13}\|f\|_{L^\infty(\Omega)}^{\frac13}} + x_{n+1} - \frac{3^{\frac43}}{2^{\frac13}\cdot 4}|x_{n+2}|^{\frac34}$$

to make $\tilde{u}$ satisfy the assumptions. Any regularity result (up to $C^{1,\frac13}$) on $\tilde{u}$ in general dimension also holds for u.

Lemma 2.2. *Under the assumptions and notation above, $S_r^{\pm}(x)+r$ is non-decreasing for all $x \in \bar{\Omega}\cap B_{r_0/2}$ and $0 < r < \frac{r_0}{2}$. So the limit $S^{\pm}(x) := \lim_{r\to 0} S_r^{\pm}(x)+r$ exist.*

Proof. Fix a point $x \in \bar{\Omega} \cap B_{r_0/2}$ and $0 < r < \frac{r_0}{2}$. Define

$$\phi(y) := u(x) + S_r^+(x) r^{\frac{r}{S_r^+(x)}} \cdot |y-x|^{1-\frac{r}{S_r^+(x)}}.$$

Direct computation shows that

$$\Delta_\infty \phi(y) = S_r^+(x)^3 r^{\frac{3r}{S_r^+(x)}} \Big(-\frac{r}{S_r^+(x)}\Big)|y-x|^{-\frac{3r}{S_r^+(x)}-1} \leq -S_r^+(x)^2 \leq -1 < f$$

when $y \in B_r(x)\cap\Omega\setminus\{x\}$. And $\phi(y) \geq u(y)$ on $\partial(B(x,r)\cap\Omega)\cup\{x\}$. So $\phi(y) \geq u(y)$ in $B(x,r)\cap\Omega$ from the comparison principle [9, Theorem 3].

For $0 < \rho < r$, let $y \in \partial(B(x,\rho)\cap\Omega)\setminus\{x\}$. If $y \in \partial\Omega \cap B(x,\rho)\setminus\{x\}$ then $y \in \partial\Omega\cap B(x,r)\setminus\{x\}$, so $\frac{u(y)-u(x)}{|y-x|} \leq S_r^+(x)$. If $y \in \partial B(x,\rho)\cap\Omega$, then

$$\frac{u(y)-u(x)}{|y-x|} \leq \frac{\phi(y)-u(x)}{\rho} = S_r^+(x)\Big(\frac{r}{\rho}\Big)^{\frac{r}{S_r^+(x)}}.$$

Hence, $S_\rho^+(x) \leq S_r^+(x)(\frac{r}{\rho})^{\frac{r}{S_r^+(x)}}$. Therefore,

$$\liminf_{\rho\to r} \frac{S_r^+(x)-S_\rho^+(x)}{r-\rho} \geq \liminf_{\rho\to r} \frac{S_r^+(x)(1-(\frac{r}{\rho})^{\frac{r}{S_r^+(x)}})}{r-\rho} = -1.$$

The same argument applies to $S_r^-(x)$. □

Define $S(x) := max\{S^+(x), S^-(x)\}$. we prove that $S(x)$ is upper-semicontinuous at 0 under the conditions of Proposition 1.

Lemma 2.3. *For any $\epsilon > 0$, there exists $r(\epsilon,u) > 0$, such that*

$$\sup_{x\in\bar{\Omega}\cap B(r)} S(x) \leq S(0)+\epsilon.$$

Proof. For $\epsilon > 0$, since $\hat{g}(x') \in C^1(\hat{B}(r_0))$ and $|D\hat{g}(0)| \le S(0)$, there exists $r_1 > 0$ such that

$$\sup_{x \ne y \in \partial\Omega \cap B(r_1)} \frac{|u(x) - u(y)|}{|x - y|} \le \sup_{x \ne y \in \partial\Omega \cap B(r_1)} \frac{|\hat{g}(x') - \hat{g}(y')|}{|x' - y'|} \le S(0) + \frac{\epsilon}{3}. \tag{2.1}$$

Since $\lim_{r \to 0} S_r(0) = S(0)$, there exists $0 < r_2 \le \min(r_1/2, \frac{\epsilon}{3})$, such that

$$S_{r_2}(0) \le S(0) + \frac{\epsilon}{4}.$$

From the continuity of u, there exists $0 < r_3 \ll r_2$, such that

$$\sup_{y \in \partial B(x, r_2) \cap \Omega} \frac{|u(y) - u(x)|}{r_2} \le S(0) + \frac{\epsilon}{3} \quad \text{for} \quad x \in \bar{\Omega} \cap B(r_3). \tag{2.2}$$

From (2.1) and (2.2), we have

$$S_{r_2}(x) = \sup_{y \in \partial(B(x, r_2) \cap \Omega) \setminus \{x\}} \frac{|u(y) - u(x)|}{|y - x|} \le S(0) + \frac{\epsilon}{3} \quad \text{for} \quad x \in \partial\Omega \cap B(r_3).$$

From Lemma 2.2, we have

$$\frac{|u(y) - u(x)|}{|y - x|} \le S_{r_2}(x) + r_2 \le S(0) + \frac{2\epsilon}{3} \tag{2.3}$$

for $x \in \partial\Omega \cap B(r_3)$ and $y \in \Omega \cap B(r_3)$. From the continuity of u again, there exists $0 < r_4 \le r_3/2$, such that

$$\sup_{y \in \partial B(x, r_3/2) \cap \Omega} \frac{|u(y) - u(x)|}{r_3/2} \le S(0) + \frac{2\epsilon}{3} \quad \text{for } x \in \bar{\Omega} \cap B(r_4). \tag{2.4}$$

From (2.3) and (2.4) and Lemma 2.2, we have

$$S(x) \le S_{r_3/2}(x) + \frac{r_3}{2} = \sup_{y \in \partial(B(x, r_3/2) \cap \Omega) \setminus \{x\}} \frac{|u(y) - u(x)|}{|y - x|} + \frac{r_3}{2} \le S(0) + \epsilon$$

for $x \in \bar{\Omega} \cap B(r_4)$. Finally we choose $r(\epsilon, u) = r_4$. □

The rest of the proof of Proposition 1 is the same as that in [11]. We have described the idea in the introduction and refer the readers to [11] for the details.

Now we prove the non-tangentially upper-semicontinuity of $S(x)$ at 0 under the conditions of Theorem 1.4 and assumptions (A1) and (A2).

Lemma 2.4. *Given any $0 < \theta \ll 1$, we have that for all $0 < \epsilon < \frac{1}{8}$, there exists $r(\epsilon, \theta, u) > 0$, such that*

$$\sup_{x \in \bar{\Omega} \cap B(r) \cap \{x_n \ge \theta |x'|\}} S(x) \le S(0) + \epsilon.$$

The proof of Lemma 2.4 is essentially same as the proof of [6, Lemma 2] for the homogeneous equation. Several places need minor modification, but this can be easily justified. So we omit the proof and refer the readers to [6].

With the result in Lemma 2.4 the rest of the proof of Theorem 1.4 follows the same way as in the homogeneous equation case.

References

[1] Aronsson, G.: *Extension of functions satisfying Lipschitz conditions*, Arkiv für Mate. **6** (1967), 551-561.

[2] Crandall, M. G.; Evans, L.C.: *A remark on infinity harmonic functions*, Proceedings of the USA-Chile Workshop on Nonlinear Analysis (Vina del Mar-Valparaiso, 2000). Electron. J. Differ. Equ. Conf. **6** (2001), 123-129.

[3] Crandall, M. G.; Evans, L. C.; Gariepy, R. F.: *Optimal Lipschitz extensions and the infinity Laplacian*, Calc. Var. Partial Differ. Equ. **13** (2001), No.2, 123-139.

[4] Evans, L. C.; Savin, O.: *$C^{1,\alpha}$ regularity for infinity harmonic functions in two dimensions*, Calc. Var. Partial Differ. Equ., **32** (2008), No.3, 325-347.

[5] Evans, L. C.; Smart, C. K.: *Everywhere differentiability of infinity harmonic functions*, Calc. Var. Partial Differ. Equ., **42** (2011), No.1-2, 289-299.

[6] Hong, G.: *Boundary differentiability of infinity harmonic functions*, Nonl. Anal.: TMA, **93** (2013), 15-20.

[7] Jensen, R.: *Uniqueness of Lipschitz extensions minimizing the sup-norm of the gradient*, Arch. Rat. Mech. Anal., **123** (1993), 51-74.

[8] Lindgren, E.: *On the regularity of solutions of the inhomogeneous infinity Laplace equation*, preprint.

[9] Lu, G.; Wang, P.: *Inhomogeneous infinity Laplace equation*, Advances in Mathematics, **217**(4) (2008), 1838-1868.

[10] Savin, O.: *C^1 regularity for infinity harmonic functions in two dimensions*, Arch. Ration. Mech. Anal., **176** (2005), No.3, 351-361.

[11] Wang, C. Y.; Yu, Y. F.: *C^1-boundary regularity of planar infinity harmonic functions*, Math. Res. Let., **19** (2012) No. 4, 823-835.

Guanghao Hong

School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an, 710049, China

E-mail address: `ghhongmath@gmail.com`