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OSCILLATION OF MEROMORPHIC SOLUTIONS TO LINEAR DIFFERENTIAL EQUATIONS WITH COEFFICIENTS OF $[p, q]$-ORDER

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Abstract. We study the relationship between "small functions" and the derivative of solutions to the higher order linear differential equation

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_0 f = 0, \quad (k \geq 2)$$

Here $A_j(z)$ $(j = 0, 1, \dots, k-1)$ are entire functions or meromorphic functions of $[p, q]$-order.

1. Introduction and statement of main results

The study of oscillation theory for linear differential equations in the complex plane $\mathbb{C}$ was started by Bank and Laine [2, 3]. After their well-known work, many important results have been obtained, see for example [19, 20].

We assume that the reader knows the standard notations and the fundamental results of the Nevanlinna value distribution theory of meromorphic functions [12, 15]. In addition, we use $\sigma(f)$, $\lambda(f)$ and $\overline{\lambda}(f)$ to denote the order, the exponent of convergence of the zero-sequence, and the exponent of convergence of the nonzero zero sequence of a meromorphic function $f(z)$, respectively. We also denote by $\tau(f)$ the type of an entire function $f(z)$ with $0 < \sigma(f) = \sigma < +\infty$ (see [15]).

We use $mE = \int_E dt$ and $m_l E = \int_E \frac{dt}{t}$ to denote the linear measure and the logarithmic measure of a set $E \subset [1, +\infty)$, respectively. We denote by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f))$, as $r \to +\infty$, possibly outside of a set with finite linear measure. A meromorphic function $\psi(z)$ is called a small function with respect to f if $T(r, \psi) = S(r, f)$.

For results on the growth of solutions to equations of the form

$$f'' + A(z)f' + B(z)f = 0, \tag{1.1}$$

with $A(z)$ and $B(z)(\not\equiv 0)$ are entire functions, the reader is referred to [1, 7, 8, 11, 14].

In 1996, Kwon [18] investigated the hyper-order of the solutions of (1.1) and obtained the following result.

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Theorem 1.1 ([18])**.** *Let $A(z)$ and $B(z)$ be entire functions such that $\sigma(A) < \sigma(B)$ or $\sigma(B) < \sigma(A) < 1/2$. Then every solution $f \not\equiv 0$ of (1.1) satisfies $\sigma_2(f) \geq \max\{\sigma(A), \sigma(B)\}$.*

In 2006, Chen and Shon [9] investigated the zeros concerning small functions and fixed points of solutions of second order linear differential equations and obtained the following results.

Theorem 1.2 ([9])**.** *Let $A_j(z) \not\equiv 0$ $(j = 1, 2)$ be entire functions with $\sigma(A_j) < 1$, suppose that a, b are complex numbers that satisfy $ab \neq 0$ and $\arg a \neq \arg b$ or $a = cb$ $(0 < c < 1)$. If $\varphi(z) \not\equiv 0$ is an entire function of finite order, then every non-trivial solution f of equation*

$$f'' + A_1(z)e^{az}f' + A_2(z)e^{bz}f = 0$$

satisfies $\overline{\lambda}(f - \varphi) = \overline{\lambda}(f' - \varphi) = \overline{\lambda}(f'' - \varphi) = \infty$.

Theorem 1.3. [9] *Let $A_1(z) \not\equiv 0$, $\varphi(z) \not\equiv 0$, $Q(z)$ be entire functions with $\sigma(A_1) < 1$, $1 < \sigma(Q) < \infty$ and $\sigma(\varphi) < \infty$, then every non-trivial solution f of equation*

$$f'' + A_1(z)e^{az}f' + Q(z)f = 0$$

satisfies $\overline{\lambda}(f - \varphi) = \overline{\lambda}(f' - \varphi) = \overline{\lambda}(f'' - \varphi) = \infty$, where $a \neq 0$ is a complex number.

In 2012, Wu and Chen [24] investigate the problem on the fixed-points of solutions of some second order differential equation with transcendental entire function coefficients and obtained the following theorems.

Theorem 1.4 ([24, Theorem 1])**.** *Let $A_j(z) \not\equiv 0 (j = 0, 1)$ be entire functions, $P(z)$ be a polynomial satisfying $\sigma(A_1) < \deg P(z)$ and $0 < \sigma(A_0) < 1/2$, and let $\varphi(z)(\not\equiv 0)$ be an entire function of finite order. Then every non-trivial solution f of equation*

$$f'' + A_1(z)e^{P(z)}f' + A_0(z)f = 0$$

satisfies $\overline{\lambda}(f - \varphi) = \infty$.

Theorem 1.5 ([24, Theorem 2])**.** *Under the assumptions of Theorem 1.4, every non-trivial solution f of the equation*

$$f'' + A_1(z)e^{P(z)}f' + A_0(z)f = 0$$

satisfies

(i) $\overline{\lambda}(f - z) = \overline{\lambda}(f' - z) = \overline{\lambda}(f'' - z) = \sigma(f) = \infty$;
(ii) *$g(z)$ has infinitely many fixed points and $\overline{\lambda}(g - z) = \infty$, where $g(z) = d_0f(z) + d_1f'(z) + d_2f''(z)$, $d_0d_2 \neq 0$.*

An interesting subject arises naturally about the problems of the zeros concerning small function and fixed points of solutions of differential equations

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_0f = 0, \quad (k \geq 2) \tag{1.2}$$

where $A_j(z)$ $(j = 0, 1, \ldots, k-1)$ are entire functions.

In 2000s, Belaïdi [4], Belaïdi and El Farissi [6] (see also [10, 22, 23]) investigated the fixed points and the relationship between small functions and differential polynomials of solutions of (1.2) and obtained some results which improve Theorem 1.3.

Recently, the growth of solutions of higher order linear differential equation with meromorphic coefficients of $[p, q]$-order was studied and some results were obtained in [5, 21].

In this article, we study the zeros of small functions and the fixed points of solutions to equation (1.2) with entire or meromorphic coefficients of $[p, q]$-order and obtain some results that extend the work of Chen and Belaïdi.

Before stating our theorems, we introduce the concepts of entire functions of $[p, q]$-order (see [16, 17, 21]). Juneja and co-authors [16, 17] introduced the concept of entire functions of $[p, q]$-order, and studied some of their properties for p, q integers satisfying $p > q \geq 1$.

Definition 1.6. If $f(z)$ is a transcendental entire function, the $[p, q]$-order of $f(z)$ is defined by

$$\sigma_{[p,q]}(f) = \limsup_{r\to\infty} \frac{\log_{p+1} M(r, f)}{\log_q r} = \limsup_{r\to\infty} \frac{\log_p T(r, f)}{\log_q r},$$

where p, q are two integers and $p > q \geq 1$.

Remark 1.7. For sufficiently large $r \in [1, \infty)$, we define $\log_{i+1} r = \log_i(\log r)$ $(i \in \mathbb{N})$ and $\exp_{i+1} r = \exp(\exp_i r)$ $(i \in \mathbb{N})$ and $\exp_0 r = r = \log_0 r$, $\exp_{-1} r = \log r$.

Definition 1.8. The $[p, q]$-type of an entire function f of $[p, q]$-order σ $(0 < \sigma < \infty)$ is defined by

$$\tau_{[p,q]} = \tau_{[p,q]}(f) = \limsup_{r\to\infty} \frac{\log_p M(r, f)}{(\log_{q-1} r)^\sigma}.$$

And the $[p, q]$ exponent of convergence of the zero sequence of f is defined by

$$\lambda_{[p,q]} = \lambda_{[p,q]}(f) = \limsup_{r\to\infty} \frac{\log_p n(r, \frac{1}{f})}{\log_q r} = \limsup_{r\to\infty} \frac{\log_p N(r, \frac{1}{f})}{\log_q r},$$

and the $[p, q]$ exponent of convergence of the distinct zero sequence of f is defined by

$$\overline{\lambda}_{[p,q]} = \overline{\lambda}_{[p,q]}(f) = \limsup_{r\to\infty} \frac{\log_p \overline{n}(r, \frac{1}{f})}{\log_q r} = \limsup_{r\to\infty} \frac{\log_p \overline{N}(r, \frac{1}{f})}{\log_q r}.$$

Let $\varphi(z)$ be an entire function with $\sigma_{[p,q]}(\varphi) < \sigma_{[p,q]}(f)$, the $[p, q]$ exponent of convergence of zeros and distinct zeros of $f(z) - \varphi(z)$ are defined to be

$$\lambda_{[p,q]}(f - \varphi) = \limsup_{r\to\infty} \frac{\log_p N(r, \frac{1}{f-\varphi})}{\log_q r}, \overline{\lambda}_{[p,q]}(f - \varphi) = \limsup_{r\to\infty} \frac{\log_p N(r, \frac{1}{f-\varphi})}{\log_q r},$$

especially if $\varphi(z) = z$, we use $\lambda_{[p,q]}(f - z)$ and $\overline{\lambda}_{[p,q]}(f - z)$ to denote the $[p, q]$ exponent of convergence of fixed points and distinct fixed points of $f(z)$, respectively.

Next we state our main results.

Theorem 1.9. *It $A_j(z)$ $(j = 0, 1, \dots, k-1)$ are entire functions and satisfy one of the following two conditions:*

(i) $\max\{\sigma_{[p,q]}(A_j) : j = 1, 2, \dots, k-1\} < \sigma_{[p,q]}(A_0) < \infty$;
(ii) $\max\{\sigma_{[p,q]}(A_j) : j = 1, 2, \dots, k-1\} \leq \sigma_{[p,q]}(A_0) < \infty$ *and* $\max\{\tau_{[p,q]}\ (A_j) | \sigma_{[p,q]}(A_j) = \sigma_{[p,q]}(A_0) > 0\} = \tau_1 < \tau_{[p,q]}(A_0) = \tau$,

then for every solution $f \not\equiv 0$ of (1.2) *and for any entire function $\varphi(z) \not\equiv 0$ satisfying $\sigma_{[p+1,q]}(\varphi) < \sigma_{[p,q]}(A_0)$. Moreover*

$$\begin{aligned}\overline{\lambda}_{[p+1,q]}(f-\varphi) &= \overline{\lambda}_{[p+1,q]}(f'-\varphi) = \overline{\lambda}_{[p+1,q]}(f''-\varphi)\\ &= \overline{\lambda}_{[p+1,q]}(f^{(i)}-\varphi) = \sigma_{[p+1,q]}(f)\\ &= \sigma(A_0), \quad (i \in \mathbb{N}).\end{aligned}$$

Throughout this paper we assume that A_0 does not vanish identically.

Theorem 1.10. *If $A_j(z)$, $j = 0, 1, \dots, k-1$ are meromorphic functions satisfying $\max\{\sigma_{[p,q]}\ (A_j) : j = 1, 2, \dots, k-1\} < \sigma_{[p,q]}(A_0)$ and $\delta(\infty, A_0) > 0$, then for every meromorphic solution $f \not\equiv 0$ of* (1.2) *and for any meromorphic function $\varphi(z) \not\equiv 0$ satisfying $\sigma_{[p+1,q]}(\varphi) < \sigma_{[p,q]}(A_0)$, we have*

$$\overline{\lambda}_{[p+1,q]}(f^{(i)}-\varphi) = \lambda_{[p+1,q]}(f^{(i)}-\varphi) \geq \sigma_{[p,q]}(A_0)(i = 0, 1, \dots),$$

where $f^{(0)} = f$.

Example 1.11. For the equation

$$f'' + \frac{e^{2z}+e^z-1}{1-e^z}f' + \frac{-e^{2z}}{1-e^z}f = 0, \tag{1.3}$$

we can easily see that this equation has a solution $f(z) = e^{e^z} + e^z$. The functions $\frac{e^{2z}+e^z-1}{1-e^z}$, $\frac{-e^{2z}}{1-e^z}$ are meromorphic and satisfy $\sigma(\frac{e^{2z}+e^z-1}{1-e^z}) = \sigma(\frac{-e^{2z}}{1-e^z}) = 1$ and $\delta\big(\infty, \frac{-e^{2z}}{1-e^z}\big) = \frac{1}{2}$. Taking $\varphi(z) = e^z$, then $\sigma_{[2,1]}(\varphi) < \sigma_{[1,1]}(\frac{-e^{2z}}{1-e^z})$. Thus, we get that $\overline{\lambda}_{[2,1]}(f'-\varphi) = \overline{\lambda}_{[2,1]}(e^{e^z}e^z) = 0 \neq 1 = \sigma_{[1,1]}(\frac{-e^{2z}}{1-e^z})$.

For $p > q \geq 1$, we have the following example.

Example 1.12. Consider the equation

$$f'' + A_1 f' + A_0 f = 0,$$

where

$$A_1 = -\frac{1+e^z-2e^{e^z}+e^{2e^z}-2e^{e^z}e^z+e^{3e^z}e^z}{(1-e^{e^z})^2}, \quad A_0 = \frac{e^{3e^z}e^{2z}-e^{2e^z}e^{2z}}{(1-e^{e^z})^2}.$$

Obviously, A_0, A_1 are meromorphic functions, $\sigma_{[2,1]}(A_1) = \sigma_{[2,1]}(A_0) = 1$ and $\delta(\infty, A_0) > 0$. By calculating, the equation (1.12) has a solution $f(z) = e^{e^{e^z}} + e^{e^z}$. Taking $\varphi(z) = e^{e^z}e^z$, then $\sigma_{[3,1]}(\varphi) < \sigma_{[2,1]}(A_0)$. Thus, we can get that $\overline{\lambda}_{[3,1]}(f'-\varphi) = \overline{\lambda}_{[3,1]}(e^{e^{e^z}}e^{e^z}e^z) = 0 \neq 1 = \sigma_{[2,1]}(A_0)$.

From Theorems 1.9 and 1.10, we obtain the following corollaries.

Corollary 1.13. *Under the assumptions of Theorem 1.9, if $\varphi(z) = z$, for every solution $f \not\equiv 0$ of* (1.2)*, we have*

$$\begin{aligned}\overline{\lambda}_{[p+1,q]}(f-z) &= \overline{\lambda}_{[p+1,q]}(f'-z) = \overline{\lambda}_{[p+1,q]}(f''-z)\\ &= \overline{\lambda}_{[p+1,q]}(f^{(i)}-z) = \sigma_{[p+1,q]}(f)\\ &= \sigma_{[p,q]}(A_0), \quad (i \in \mathbb{N}).\end{aligned}$$

Corollary 1.14. *Under the assumptions of Theorem 1.10, if $\varphi(z) = z$, for every meromorphic solution $f \not\equiv 0$ of* (1.2)*, we have $\overline{\lambda}_{[p+1,q]}(f^{(i)}-z) = \lambda_{[p+1,q]}(f^{(i)}-z) \geq \sigma_{[p+1,q]}(A_0), (i = 0, 1, \dots)$, where $f^{(0)} = f$.*

2. Preliminary results

To prove our theorems, we require the following lemmas.

Lemma 2.1 ([25, Lemma 2.1]). *Assume $f \not\equiv 0$ is a solution of* (1.2)*, set $g = f - \varphi$, then g satisfies the equation*

$$g^{(k)} + A_{k-1} g^{(k-1)} + \dots + A_0 g = -[\varphi^{(k)} + A_{k-1}\varphi^{(k-1)} + \dots + A_0\varphi]. \tag{2.1}$$

Lemma 2.2 ([25, Lemma 2.2]). *Assume $f \not\equiv 0$ is a solution of equation* (1.2)*, set $g_1 = f' - \varphi$, then g_1 satisfies the equation*

$$g_1^{(k)} + U_{k-1}^1 g_1^{(k-1)} + \dots + U_0^1 g_1 = -[\varphi^{(k)} + U_{k-1}^1\varphi^{(k-1)} + \dots + U_0^1\varphi], \tag{2.2}$$

where $U_j^1 = A'_{j+1} + A_j - \frac{A'_0}{A_0} A_{j+1}$, $j = 0, 1, 2, \dots, k-1$ and $A_k \equiv 1$.

Lemma 2.3 ([25, Lemma 2.5]). *Assume $f \not\equiv 0$ is a solution of equation* (1.2)*, set $g_i = f^{(i)} - \varphi$, then g_i satisfies the equation*

$$g_i^{(k)} + U_{k-1}^i g_i^{(k-1)} + \dots + U_0^i g_i = -[\varphi^{(k)} + U_{k-1}^i\varphi^{(k-1)} + \dots + U_0^i\varphi], \tag{2.3}$$

where $U_j^i = {U_{j+1}^{i-1}}' + U_j^{i-1} - \frac{{U_0^{i-1}}'}{U_0^{i-1}} U_{j+1}^{i-1}$, $j = 0, 1, 2, \dots, k-1$, $U_k^{i-1} \equiv 1$ and $i \in \mathbb{N}$.

Lemma 2.4 ([21, Lemma 3.9]). *Let $f(z)$ be an entire function of $[p, q]$-order, then $\sigma_{[p,q]}(f) = \sigma_{[p,q]}(f')$.*

Lemma 2.5 ([21, Lemma 3.10]). *Let $f(z)$ be an entire function of $[p, q]$-order satisfying $\sigma_{[p,q]}(f) = \sigma_2$, then there exists a set $E \subset [1, +\infty)$ with infinite logarithmic measure such that for all $r \in E$, we have*

$$\lim_{r\to\infty} \frac{\log_p T(r, f)}{\log_q r} = \sigma_2, \quad r \in E.$$

Lemma 2.6. *Let $A_0(z), A_1(z), \dots, A_{k-1}(z)$ be entire functions with $[p, q]$-order and satisfy $\max\{\sigma_{[p,q]}(A_j) : j = 1, 2, \dots, k-1\} = \sigma_1 < \sigma_{[p,q]}(A_0) < \infty$, and set*

$$U_j^1 = A'_{j+1} + A_j - \frac{A'_0}{A_0} A_{j+1}$$

and

$$U_j^i = {U_{j+1}^{i-1}}' + U_j^{i-1} - \frac{{U_0^{i-1}}'}{U_0^{i-1}} U_{j+1}^{i-1},$$

where $j = 0, 1, 2, \dots, k-1$, $A_k \equiv 1$, $U_k^{i-1} \equiv 1$ and $i \in \mathbb{N}$. Then there exists a set E with infinite logarithmic measure such that for $r \in E$, we have

$$\begin{aligned} \sigma_{[p,q]}(A_0) &= \lim_{r\to\infty} \frac{\log_p m(r, U_0^i)}{\log_q r} \\ &> \limsup_{r\to\infty} \frac{\max_{1\le j\le k-1}\{\log_p m(r, U_j^i)\}}{\log_q r} = \sigma_1. \end{aligned} \tag{2.4}$$

Proof. We will use the inductive method to prove it.

First, when $i = 1$, it follows that $U_j^1 = A'_{j+1} + A_j - \frac{A'_0}{A_0} A_{j+1}$ for $j = 0, 1, 2, \dots, k-1$ and $A_k \equiv 1$. When $j = 0$, that is, $U_0^1 = A'_1 + A_0 - \frac{A'_0}{A_0} A_1$. Then, we have

$$m(r, U_0^1) \le m(r, A_1) + m(r, A_0) + m\big(r, \frac{A'_1}{A_1}\big) + m\big(r, \frac{A'_0}{A_0}\big) + O(1). \tag{2.5}$$

From $A_0 = -A_1' + U_0^1 + \frac{A_0'}{A_0} A_1$, we have

$$m(r, A_0) \le m(r, A_1) + m(r, U_0^1) + m\big(r, \frac{A_1'}{A_1}\big) + m\big(r, \frac{A_0'}{A_0}\big) + O(1). \tag{2.6}$$

When $j \neq 0$, from the definitions of $U_j^1 (j = 1 \dots, k)$, we have

$$\begin{aligned} m\left(r, U_j^1\right) &\le m(r, A_{j+1}) + m(r, A_j) + m\big(r, \frac{A_0'}{A_0}\big) \\ &\quad + m\big(r, \frac{A_{j+1}'}{A_{j+1}}\big) + O(1), \quad j = 1, 2, \dots, k-1. \end{aligned} \tag{2.7}$$

Since $A_0(z), \dots, A_{k-1}(z)$ are entire functions with $\max\{\sigma_{[p,q]}(A_j) : j = 1, 2, \dots, k-1\} < \sigma_{[p,q]}(A_0) < \infty$ and (2.7), we have

$$\begin{aligned} &\max_{1\le j\le k-1} \{m(r, U_j^1)t\} \\ &\le \max_{1\le j\le k-1} \{m(r, A_j) + o(m(r, A_0) + O(\log(rT(r,f)))\} + O(1), \end{aligned} \tag{2.8}$$

holds for all $r \in E_1 - E_2$ (where E_1 is a set of infinite logarithmic measure and E_2 is a set of finite linear measure). From (2.5), (2.6), (2.8) and Lemma 2.5, there exists a set $E \subset [1, +\infty)$ with infinite logarithmic measure such that

$$\begin{aligned} \sigma_{[p,q]}(A_0) &= \lim_{r\to\infty} \frac{\log_p m(r, U_0^1)}{\log_q r} \\ &> \sigma_1 = \limsup_{r\to\infty} \frac{\max_{1\le j\le k-1}\{\log_p m(r, A_j)\}}{\log_q r} \\ &\ge \limsup_{r\to\infty} \frac{\max_{1\le j\le k-1}\{\log_p m(r, U_j^1)\}}{\log_q r}, \ r \in E. \end{aligned} \tag{2.9}$$

Now, suppose that (2.4) holds for $i \le n (n \in \mathbb{N})$, thus, there exists a set E with infinite logarithmic measure such that

$$\begin{aligned} \sigma_{[p,q]}(A_0) &= \lim_{r\to\infty} \frac{\log_p m(r, U_0^n)}{\log_q r} \\ &> \limsup_{r\to\infty} \frac{\max_{1\le j\le k-1}\{\log_p m(r, U_j^n)\}}{\log_q r} = \sigma_1. \end{aligned} \tag{2.10}$$

Next, we prove that (2.4) holds for $i = n+1$. From the assumptions of this lemma, we have $U_j^{n+1} = {U_{j+1}^n}' + U_j^n - \frac{{U_0^n}'}{U_0^n} U_{j+1}^n$, $(j = 0, 1, 2, \dots, k-1)$ and $U_k^n \equiv 1$ for $i = n+1$. Thus, when $j = 0$, it follows that $U_0^{n+1} = {U_1^n}' + U_0^n - \frac{{U_0^n}'}{U_0^n} U_1^n$. Then, we have

$$m\left(r, U_0^{n+1}\right) \le m\left(r, U_0^n\right) + m(r, U_1^n) + m(r, \frac{{U_0^n}'}{U_0^n}) + m(r, \frac{{U_1^n}'}{U_1^n}) + O(1). \tag{2.11}$$

And since $U_0^n = -{U_1^n}' + U_0^{n+1} + \frac{{U_0^n}'}{U_0^n} U_1^n$, we have

$$m(r, U_0^n) \le m(r, U_0^{n+1}) + m(r, U_1^n) + m(r, \frac{{U_0^n}'}{U_0^n}) + m(r, \frac{{U_1^n}'}{U_1^n}) + O(1). \tag{2.12}$$

When $j \neq 0$, it follows from the definitions of U_j^{n+1} $(j = 1, 2, \dots, k-1)$ and $U_k^n \equiv 1$ that

$$m(r, U_j^{n+1}) \le m(r, U_{j+1}^n) + m(r, U_j^n) + m(r, \frac{{U_{j+1}^n}'}{U_{j+1}^n}) + m(r, \frac{{U_0^n}'}{U_0^n}) + O(1). \tag{2.13}$$

From (2.10)–(2.13), there exists a set E with infinite logarithmic measure such that

$$\begin{aligned} \lim_{r\to\infty} \frac{\log_p m\left(r, U_0^{n+1}\right)}{\log_q r} &= \lim_{r\to\infty} \frac{\log_p m\left(r, U_0^{n}\right)}{\log_q r} = \sigma_{[p,q]}(A_0) \\ &> \sigma_1 = \limsup_{r\to\infty} \frac{\max_{1\le j\le k-1}\{\log_p m\left(r, U_j^n\right)\}}{\log_q r} \\ &= \limsup_{r\to\infty} \frac{\max_{1\le j\le k-1}\{\log_p m(r, U_j^{n+1})\}}{\log_q r}, \quad r \in E. \end{aligned} \tag{2.14}$$

Thus, the proof is complete. □

Lemma 2.7. *Let $H_j(z)$ $(j = 0, 1, \dots, k-1)$ be meromorphic functions of finite $[p, q]$-order. If*

$$\limsup_{r\to\infty} \frac{\max_{1\le j\le k-1}\{\log_p m(r, H_j)\}}{\log_q r} = \beta_1$$

and there exists a set E_1 with infinite logarithmic measure such that

$$\lim_{r\to\infty} \frac{\log_p m(r, H_0)}{\log_q r} = \beta_2 > \beta_1$$

holds for all $r \in E_1$, then every meromorphic solution $f \not\equiv 0$ of

$$f^{(k)} + H_{k-1} f^{(k-1)} + \dots + H_1 f' + H_0 f = 0 \tag{2.15}$$

satisfies $\sigma_{[p+1,q]}(f) \ge \beta_2$.

Proof. Assume that $f(z) \not\equiv 0$ is a meromorphic solution of (2.15). From (2.15), we have

$$m(r, H_0) \le m(r, \frac{f^{(k)}}{f}) + \dots + m(r, \frac{f'}{f}) + \sum_{j=1}^{k-1} m(r, H_j) + O(1). \tag{2.16}$$

By the logarithmic derivative lemma and (2.16), we have

$$m(r, H_0) \le O\{\log r T(r, f)\} + \sum_{j=1}^{k-1} m(r, H_j), \quad r \notin E_2, \tag{2.17}$$

where $E_2 \subset [1, +\infty)$ is a set with finite linear measure. From the assumptions of Lemma 2.7, there exists a set E_1 with infinite logarithmic measure such that for all $|z| = r \in E_1 - E_2$, we have

$$\exp_p\{(\beta_2 - \varepsilon)\log_q r\} \le O\{\log r T(r, f)\} + (k-1)\exp_p\{(\beta_1 + \varepsilon)\log_q r\}, \tag{2.18}$$

where $0 < 2\varepsilon < \beta_2 - \beta_1$. From (20), we have $\sigma_{[p+1,q]}(f) \ge \beta_2$. □

Lemma 2.8 ([13])**.** *Let $f(z)$ be a transcendental meromorphic function and $\alpha > 1$ be a given constant. Then for any given $\varepsilon > 0$, there exists a set $E_7 \subset [1, \infty)$ that has finite logarithmic measure and a constant $M > 0$ that depends only on α*

and $(m,n)(m,n\in\{0,\dots,k\}$ with $m<n)$ such that for all z satisfying $|z|=r\notin[0,1]\cup E_7$, we have

$$\Big|\frac{f^{(n)}(z)}{f^{(m)}(z)}\Big|\le M\Big(\frac{T(\alpha r,f)}{r}(\log^{\alpha} r)\log T(\alpha r,f)\Big)^{n-m}.$$

Lemma 2.9 ([21, Lemma 3.13])**.** *Let $f(z)$ be an entire function of $[p,q]$-order satisfying $\sigma_{[p,q]}(f)=\sigma$, $\tau_{[p,q]}(f)=\tau$, $0<\sigma<\infty$, $0<\tau<\infty$, then for any given $\beta<\tau$, there exists a set $E_4\subset[1,+\infty)$ that has infinite logarithmic measure such that for all $r\in E_4$, we have*

$$\log_p M(r,f)>\beta(\log_{q-1} r)^{\sigma}.$$

Lemma 2.10. *Let $A_0(z),A_1(z),\dots,A_{k-1}(z)$ be entire functions with finite $[p,q]$-order and satisfy $\max\{\sigma_{[p,q]}(A_j):j=1,2,\dots,k-1\}\le\sigma_{[p,q]}(A_0)=\sigma_2<\infty$ and $\max\{\tau_{[p,q]}(A_j)\ |\sigma_{[p,q]}\ (A_j)=\sigma_{[p,q]}(A_0)>0\}=\tau_1<\tau_{[p,q]}(A_0)=\tau$, and let U_j^1,U_j^i be as stated in Lemma 2.6. Then for any given $\varepsilon(0<2\varepsilon<\tau-\tau_1)$, there exists a set E_5 with infinite logarithmic measure such that*

$$|U_j^i|\le\exp_p\{(\tau_1+\varepsilon)(\log_{q-1} r)^{\sigma_2}\},\quad |U_0^i|\ge\exp_p\{(\tau-\varepsilon)(\log_{q-1} r)^{\sigma_2}\},\tag{2.19}$$

where $i\in\mathbb{N}$ and $j=1,2,\dots,k-1$.

Proof. We will use the induction method for this proof.

(i) First, we prove that $U_j^i(j=0,1,\dots,k-1)$ satisfy (2.19) when $i=1$. From the definition $U_j^1=A'_{j+1}+A_j-\frac{A'_0}{A_0}A_{j+1}(j\ne0)$ and $U_0^1=A'_1+A_0-\frac{A'_0}{A_0}A_1$, we have

$$|U_0^1|\ge-|A_1|\left(|\frac{A'_1}{A_1}|+|\frac{A'_0}{A_0}|\right)+|A_0|\tag{2.20}$$

and

$$|U_j^1|\le|A_{j+1}|\big(|\frac{A'_{j+1}}{A_{j+1}}|+|\frac{A'_0}{A_0}|\big)+|A_j|,\quad j=1,2,\dots,k-1;\ A_k\equiv1.\tag{2.21}$$

From Lemma 2.8, Lemma 2.9 and (2.20)–(2.21), for any $\varepsilon(0<4\varepsilon<\tau-\tau_1)$, there exists a set E_5 with infinite logarithmic measure such that

$$\begin{aligned}|U_0^1|&\ge-2M\exp_p\{(\tau_1+\frac{\varepsilon}{8})(\log_{q-1} r)^{\sigma_2}\}(T(2r,A_0))^2\\&\quad+\exp_p\{(\tau-\frac{\varepsilon}{4})(\log_{q-1} r)^{\sigma_2}\}\\&\ge-2M\exp_p\{(\tau_1+\frac{\varepsilon}{8})(\log_{q-1} r)^{\sigma_2}\}\big(\exp_p\{(\sigma_2+\frac{\varepsilon}{8})(\log_q 2r)\}\big)^2\\&\quad+\exp_p\{\Big(\tau-\frac{\varepsilon}{4}\Big)(\log_{q-1} r)^{\sigma_2}\}\\&\ge\exp_p\{\Big(\tau-\frac{\varepsilon}{2}\Big)(\log_{q-1} r)^{\sigma_2}\}\end{aligned}\tag{2.22}$$

and

$$\begin{aligned}|U_j^1| &\leq 2M \exp_p\{(\tau_1+\frac{\varepsilon}{4})(\log_{q-1} r)^{\sigma_2}\}(T(2r,A_0))^2\\ &\quad+\exp_p\{(\tau_1+\frac{\varepsilon}{4})(\log_{q-1} r)^{\sigma_2}\}\\ &\leq 2M\exp_p\{(\tau_1+\frac{\varepsilon}{4})(\log_{q-1} r)^{\sigma_2}\}\big(\exp_p\{(\sigma_2+\frac{\varepsilon}{8})(\log_q 2r)\}\big)^2\\ &\quad+\exp_p\{(\tau_1+\frac{\varepsilon}{4})(\log_{q-1} r)^{\sigma_2}\}\\ &\leq \exp_p\{(\tau_1+\frac{\varepsilon}{2})(\log_{q-1} r)^{\sigma_2}\},\quad j\neq 0,\end{aligned}\tag{2.23}$$

where $M>0$ is a constant, not necessarily the same at each occurrence.

(ii) Next, we show that U_j^i $(j=0,1,2,\dots,k-1)$ satisfy (2.19) when $i=2$. From $U_0^2={U_1^1}'+U_0^1-\frac{{U_0^1}'}{U_0^1}U_1^1$ and $U_j^2={U_{j+1}^1}'+U_j^1-\frac{{U_0^1}'}{U_0^1}U_{j+1}^1$ $(j=0,1,\dots,k-1)$ and $U_k^1\equiv 1$, we have

$$|U_0^2|\geq |U_0^1|-|U_1^1|\Big(|\frac{{U_1^1}'}{U_1^1}|+|\frac{{U_0^1}'}{U_0^1}|\Big)\tag{2.24}$$

and

$$|U_j^2|\leq |U_j^1|+|U_{j+1}^1|\Big(|\frac{{U_{j+1}^1}'}{U_{j+1}^1}|+|\frac{{U_0^1}'}{U_0^1}|\Big),\quad j=1,2,\dots,k-1.\tag{2.25}$$

By the conclusions in (i), Lemma 2.8 and Lemma 2.9, (2.22)–(2.23), for all $|z|=r\in E_5$, we have

$$\begin{aligned}|U_0^2| &\geq -2M\exp_p\big\{(\tau_1+\frac{\varepsilon}{2})(\log_{q-1} r)^{\sigma_2}\big\}\big(\exp_p\big\{(\sigma_2+\frac{\varepsilon}{8})(\log_q 2r)\big\}\big)^2\\ &\quad+\exp_p\{\Big(\tau-\frac{\varepsilon}{2}\Big)(\log_{q-1} r)^{\sigma_2}\}\\ &\geq \exp_p\{(\tau-\varepsilon)(\log_{q-1} r)^{\sigma_2}\}\end{aligned}\tag{2.26}$$

and

$$\begin{aligned}|U_j^2| &\leq 2M\exp_p\{(\tau_1+\frac{\varepsilon}{2})(\log_{q-1} r)^{\sigma_2}\}\exp_p\{(\sigma_2+\frac{\varepsilon}{8})(\log_q 2r)\}\\ &\quad+\exp_p\{(\tau_1+\frac{\varepsilon}{2})(\log_{q-1} r)^{\sigma_2}\}\\ &\leq \exp_p\{(\tau_1+\varepsilon)(\log_{q-1} r)^{\sigma_2}\},\quad j\neq 0.\end{aligned}\tag{2.27}$$

(iii) Now, suppose that (2.19) holds for $i\leq n(n\in\mathbb{N})$. Thus, for any given $\varepsilon(0<4\varepsilon<\tau-\tau_1)$, there exists a set E_5 with infinite logarithmic measure such that

$$|U_j^i|\leq \exp_p\{(\tau_1+\varepsilon)(\log_{q-1} r)^{\sigma_2}\},|U_0^i|\geq \exp_p\{(\tau-\varepsilon)(\log_{q-1} r)^{\sigma_2}\},\tag{2.28}$$

where $i\leq n$ and $j=1,2,\dots,k-1$. From $U_0^{n+1}={U_1^n}'+U_0^n-\frac{{U_0^n}'}{U_0^n}U_1^n$ and $U_j^{n+1}={U_{j+1}^n}'+U_j^n-\frac{{U_0^n}'}{U_0^n}U_{j+1}^n$ $(j=0,1,\dots,k-1)$ and $U_k^n\equiv 1$, we have

$$|U_0^{n+1}|\geq |U_0^n|-|U_1^n|\Big(|\frac{{U_1^n}'}{U_1^n}|+|\frac{{U_0^n}'}{U_0^n}|\Big)\tag{2.29}$$

and

$$|U_j^{n+1}|\leq |U_j^n|+|U_{j+1}^n|\Big(|\frac{{U_{j+1}^n}'}{U_{j+1}^n}|+|\frac{{U_0^n}'}{U_0^n}|\Big),\quad j=1,2,\dots,k-1.\tag{2.30}$$

Then, from Lemma 2.8, Lemma 2.9 and (2.28)–(2.30), for all $|z| = r \in E_5$, we have

$$\begin{aligned} |U_j^{n+1}| &\le 2M \exp_p\{(\tau_1+\varepsilon)(\log_{q-1} r)^{\sigma_2}\}\big(\exp_p\{(\sigma_2+\frac{\varepsilon}{8})(\log_q 2r)\}\big)^2 \\ &\quad + \exp_p\{(\tau_1+\varepsilon)(\log_{q-1} r)^{\sigma_2}\} \\ &\le \exp_p\{(\tau_1+2\varepsilon)(\log_{q-1} r)^{\sigma_2}\}, \quad j \ne 0, \end{aligned} \tag{2.31}$$

and

$$\begin{aligned} |U_0^{n+1}| &\ge -2M \exp\{(\tau_1+\varepsilon)(\log_{q-1} r)^{\sigma_2}\}(\exp_p\{(\sigma_2+\frac{\varepsilon}{8})(\log_q 2r)\})^2 \\ &\quad + \exp_p\{(\tau-\varepsilon)(\log_{q-1} r)^{\sigma_2}\} \\ &\ge \exp_p\{(\tau-2\varepsilon)(\log_{q-1} r)^{\sigma_2}\}. \end{aligned} \tag{2.32}$$

Thus, the proof is complete. □

Lemma 2.11. *Let $B_j(z)$ $(j = 0, 1, \dots, k-1)$ be meromorphic functions such that $\max\{\sigma_{[p,q]}(B_j) : j = 1, 2, \dots, k-1\} = \sigma_4 < \sigma_{[p,q]}(B_0) = \sigma_3$ and $\delta := \delta(\infty, B_0) = \underline{\lim}_{r\to\infty} \frac{m(r,B_0)}{T(r,B_0)} > 0$. Then every meromorphic solution $f \not\equiv 0$ of equation*

$$f^{(k)} + B_{k-1}f^{(k-1)} + \dots + B_1 f' + B_0 f = 0 \tag{2.33}$$

satisfies $\sigma_{[p+1,q]}(f) \ge \sigma_3$.

Proof. Let $f \not\equiv 0$ be a meromorphic solution of equation (2.33). Then from (2.33), we have

$$\begin{aligned} m(r, B_0) &\le m(r, \frac{f^{(k)}}{f}) + m(r, \frac{f^{(k-1)}}{f}) + \dots + m(r, \frac{f'}{f}) + \sum_{j=1}^{k-1} m(r, B_j) + O(1) \\ &\le O\{\log rT(r,f)\} + \sum_{j=1}^{k-1} T(r, B_j), \quad r \notin E_6, \end{aligned} \tag{2.34}$$

where $E_6 \subset [1, +\infty)$ is a set with finite linear measure. By Lemma 2.5, there exists a set E with infinite logarithmic measure such that for all $|z| = r \in E$, we have

$$\lim_{r\to\infty} \frac{\log_p T(r, B_0)}{\log_q r} = \sigma_3. \tag{2.35}$$

Since $\delta := \delta(\infty, B_0) > 0$, then for any given $\varepsilon(0 < 2\varepsilon < \min\{\delta, \sigma_3 - \sigma_4\})$ and for all $r \in E$, by (37), we have

$$m(r, B_0) \ge (\delta - \varepsilon) \exp_p\{(\sigma_3 - \varepsilon)\log_q r\}. \tag{2.36}$$

From (2.34) and (2.36), we have

$$(\delta-\varepsilon) \exp_p\{(\sigma_3-\varepsilon)\log_q r\} \le O\{\log rT(r,f)\} + (k-1)\exp_p\{(\sigma_4+\varepsilon)\log_q r\}, \tag{2.37}$$

where $r \in E - E_6$. From (2.37), we obtain $\sigma_{[p+1,q]}(f) \ge \sigma_3 = \sigma_{[p,q]}(B_0)$. □

Lemma 2.12. *Let $B_j(z)$, $j = 0, 1, \dots, k-1$ be meromorphic functions of finite $[p, q]$ order. If there exist positive constants $\sigma_5, \beta_3, \beta_4 (0 < \beta_3 < \beta_4)$ and a set E_8 with infinite logarithmic measure such that*

$$\max\{|B_j(z)| : j = 1, 2, \dots, k-1\} \le \exp_p\{\beta_3(\log_{q-1} r)^{\sigma_5}\},$$

and

$$|B_0(z)| \ge \exp_p\{\beta_4(\log_{q-1} r)^{\sigma_5}\}$$

hold for all $|z| = r \in E_8$*, then every meromorphic solution* $f \not\equiv 0$ *of* (2.33) *satisfies* $\sigma_{[p+1,q]}(f) \geq \sigma_5$*.*

Proof. Suppose that $f \not\equiv 0$ is a meromorphic function of (2.33). Then it follows that

$$|B_0(z)| \leq |\frac{f^{(k)}}{f}| + \sum_{j=1}^{k-1} |B_j(z)||\frac{f^{(j)}}{f}|. \tag{2.38}$$

By Lemma 2.8, there exists a set E_7 with finite logarithmic measure such that for all $|z| = r \notin E_7$, we have

$$|\frac{f^{(j)}}{f}| \leq M[T(2r,f)]^{2j}, \quad j = 1, 2, \dots, k. \tag{2.39}$$

By (2.38), (2.39) and the assumptions of Lemma 2.12, for all $|z| = r \in E_8 - E_7$, we have

$$\exp_p\{\beta_4(\log_{q-1} r)^{\sigma_5}\} \leq Mk[T(2r,f)]^{2k} \exp_p\{\beta_3(\log_{q-1} r)^{\sigma_5}\}. \tag{2.40}$$

Since $0 < \beta_3 < \beta_4$ and by (2.40), we have $\sigma_{[p+1,q]}(f) \geq \sigma_5$. □

Lemma 2.13 ([21, Lemma 3.12]). *Let* $A_0, A_1, \dots, A_{k-1}$*,* $F \not\equiv 0$ *be meromorphic functions, if* f *is a meromorphic solution of the equation*

$$f^{(k)} + A_{k-1}f^{(k-1)} + \dots + A_0 f = F,$$

satisfying $\max\{\sigma_{[p,q]}(F), \sigma_{[p,q]}(A_j); j = 0, 1, \dots, k-1\} < \sigma_{[p,q]}(f)$*, then we have* $\sigma_{[p,q]}(f) = \lambda_{[p,q]}(f) = \overline{\lambda}_{[p,q]}(f)$*.*

Lemma 2.14 ([21, Theorem 2.3]). *Let* $A_j(z)$ $(j = 0, 1, \dots, k-1)$ *be entire functions satisfying* $\max\{\sigma_{[p,q]}(A_j) : j = 1, 2, \dots, k-1\} \leq \sigma_{[p,q]}(A_0) < \infty$ *and*

$$\max\{\tau_{[p,q]}(A_j) | \sigma_{[p,q]}(A_j) = \sigma_{[p,q]}(A_0) > 0\} < \tau_{[p,q]}(A_0).$$

Then every nontrivial solution f *of* (1.2) *satisfies* $\sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0)$*.*

3. Proofs of Theorems

Proof of Theorem 1.9. We will consider two cases as follows.

Case 1. Suppose that $\max\{\sigma_{[p,q]}(A_j) : j = 1, 2, \dots, k-1\} < \sigma_{[p,q]}(A_0) < \infty$.

(i) First, we prove that $\overline{\lambda}_{[p+1,q]}(f - \varphi) = \sigma_{[p+1,q]}(f)$. Assume that f is a nontrivial solution of (1.2), from [21, Theorem 2.2], we have $\sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0)$. Set $g = f - \varphi$. Since $\sigma_{[p+1,q]}(\varphi) < \sigma_{[p,q]}(A_0)$, then $\sigma_{[p+1,q]}(g) = \sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0)$ and $\overline{\lambda}_{[p+1,q]}(g) = \overline{\lambda}_{[p+1,q]}(f - \varphi)$. By Lemma 2.1, we get that g satisfies the equation (2.1). Set $F = \varphi^{(k)} + A_{k-1}\varphi^{(k-1)} + \dots + A_0\varphi$. If $F \equiv 0$, then from [21], we have $\sigma_{[p+1,q]}(\varphi) = \sigma_{[p,q]}(A_0)$, a contradiction. Then $F \not\equiv 0$. From Lemma 2.4 and assumption of Case 1, we have

$$\sigma_{[p+1,q]}(F) \leq \max\{\sigma_{[p+1,q]}(\varphi), \sigma_{[p+1,q]}(A_0)\} = \max\{\sigma_{[p+1,q]}(\varphi), 0\}.$$

Since $\sigma_{[p+1,q]}(\varphi) < \sigma_{[p,q]}(A_0)$, we have

$$\max\{\sigma_{[p+1,q]}(F), \sigma_{[p+1,q]}(A_j) : j = 0, 1, 2, \dots, k-1\} < \sigma_{[p+1,q]}(f).$$

By Lemma 2.13, we have $\overline{\lambda}_{[p+1,q]}(g) = \lambda_{[p+1,q]}(g) = \sigma_{[p+1,q]}(g) = \sigma_{[p,q]}(A_0)$. Thus, we have

$$\overline{\lambda}_{[p+1,q]}(f - \varphi) = \lambda_{[p+1,q]}(f - \varphi) = \sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0).$$

(ii) Second, we prove that $\overline{\lambda}_{[p+1,q]}(f'-\varphi) = \sigma_{[p+1,q]}(f)$. Set $g_1 = f' - \varphi$, then $\sigma_{[p+1,q]}(g_1) = \sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0)$. From Lemma 2.2, we get that g_1 satisfies the equation (2.2). Set $F_1 = \varphi^{(k)} + U^1_{k-1}\varphi^{(k-1)} + \dots + U^1_0\varphi$, where $U^1_j(j = 0, 1, \dots, k-1)$ are stated as in Lemma 2.2. If $F_1 \equiv 0$, from Lemma 2.6 and Lemma 2.7, we have $\sigma_{[p+1,q]}(\varphi) \geq \sigma_{[p,q]}(A_0)$, a contradiction with $\sigma_{[p+1,q]}(\varphi) < \sigma_{[p,q]}(A_0)$. Hence $F_1 \not\equiv 0$. From the definition of $U^1_j(j = 0, 1, \dots, k-1)$, we have $\sigma_{[p+1,q]}(U^1_j) \leq \sigma_{[p+1,q]}(A_j)$ $j = 0, 1, \dots, k-1$. Thus, we can get $\sigma_{[p+1,q]}(F_1) \leq \max\{\sigma_{[p+1,q]}(\varphi), \sigma_{[p+1,q]}(U^1_j) : j = 0, 1, \dots, k-1\}$. Since $\sigma_{[p+1,q]}(\varphi) < \sigma_{[p,q]}(A_0)$, we have $\max\{\sigma_{[p+1,q]}(F_1), \sigma_{[p+1,q]}(U^1_j) : j = 0, 1, \dots, k-1\} < \sigma_{[p,q]}(A_0) = \sigma_{[p+1,q]}(g_1)$. By Lemma 2.13, we obtain

$$\overline{\lambda}_{[p+1,q]}(f'-\varphi) = \lambda_{[p+1,q]}(f'-\varphi) = \sigma_{[p+1,q]}(f).$$

(iii) We will prove that $\overline{\lambda}_{[p+1,q]}(f^{(i)}-\varphi) = \sigma_{[p+1,q]}(f), (i > 1, i \in \mathbb{N})$. Set $g_i = f^{(i)} - \varphi$, then $\sigma_{[p+1,q]}(g_i) = \sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0)$. From Lemma 2.3, we have g_i satisfies equation (2.3). Set $F_i = \varphi^{(k)} + U^i_{k-1}\varphi^{(k-1)} + \dots + U^i_0\varphi$, where $U^i_j(j = 0, 1, \dots, k-1; i \in \mathbb{N})$ are stated as in Lemma 2.3. If $F_i \equiv 0$, from Lemma 2.6 and Lemma 2.7, we have $\sigma_{[p+1,q]}(\varphi) \geq \sigma_{[p,q]}(A_0)$, a contradiction with $\sigma_{[p+1,q]}(\varphi) < \sigma_{[p,q]}(A_0)$. Hence $F_i \not\equiv 0$. By using the same argument as in Case 1(ii), we can get

$$\overline{\lambda}_{[p+1,q]}(f^{(i)}-\varphi) = \lambda_{[p+1,q]}(f^{(i)}-\varphi) = \sigma_{[p+1,q]}(f).$$

Case 2. Suppose that $\max\{\sigma_{[p,q}(A_j) : j = 1, 2, \dots, k-1\} \leq \sigma_{[p,q]}(A_0) < \infty$ and $\max\{\tau_{[p,q]}(A_j) | \sigma_{[p,q]}\ (A_j) = \sigma_{[p,q]}(A_0) > 0\} < \tau_{[p,q]}(A_0)$.

(i) We first prove that $\overline{\lambda}_{[p+1,q]}(f-\varphi) = \sigma_{[p+1,q]}(f)$. Since f is a nontrivial solution of (1.2), by Lemma 2.14, we have $\sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0) > 0$. Set $g = f - \varphi$. Since $\varphi \not\equiv 0$ is an entire function satisfying $\sigma_{[p+1,q]}(\varphi) < \sigma_{[p,q]}(A_0)$, then we have $\sigma_{[p+1,q]}(g) = \sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0)$ and $\overline{\lambda}_{[p+1,q]}(g) = \overline{\lambda}_{[p+1,q]}(f-\varphi)$. From Lemma 2.1, we get that g satisfies equation (2.1). We will affirm $F \not\equiv 0$. If $F \equiv 0$, by Lemma 2.14, we get $\sigma_{[p+1,q]}(\varphi) = \sigma_{[p,q]}(A_0)$, a contradiction. Hence $F \not\equiv 0$. From the assumptions of Case 2, we get

$$\max\{\sigma_{[p+1,q]}(F), \sigma_{[p+1,q]}(A_j) : j = 0, 1, \dots, k-1\} < \sigma_{[p+1,q]}(g) = \sigma_{[p,q]}(A_0).$$

From Lemma 2.13, we have

$$\overline{\lambda}_{[p+1,q]}(f-\varphi) = \lambda_{[p+1,q]}(f-\varphi) = \sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0).$$

(ii) Now we prove that $\overline{\lambda}_{[p+1,q]}(f'-\varphi) = \sigma_{[p+1,q]}(f)$. Let $g_1 = f' - \varphi$. Since $\sigma_{[p+1,q]}(\varphi) < \sigma_{[p,q]}(A_0)$, we have $\sigma_{[p+1,q]}(g_1) = \sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0)$. By Lemma 2.2, we get that g_1 satisfies equation (2.2). If $F_1 \equiv 0$, from Lemma 2.10 and Lemma 2.12, we have $\sigma_{[p+1,q]}(\varphi) \geq \sigma_{[p,q]}(A_0)$. Then we can get a contradiction with $\sigma_{[p+1,q]}(\varphi) < \sigma_{[p,q]}(A_0)$. Therefore, we have $F_1 \not\equiv 0$. By (2.2) and Lemma 2.13, we have

$$\overline{\lambda}_{[p+1,q]}(f'-\varphi) = \lambda_{[p+1,q]}(f'-\varphi) = \sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0).$$

Similar to the arguments as in Case 1 (iii) and by using Lemmas 2.3, 2.10 and 2.12, we obtain

$$\overline{\lambda}_{[p+1,q]}(f^{(i)}-\varphi) = \lambda_{[p+1,q]}(f^{(i)}-\varphi) = \sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0), \quad (i \in \mathbb{N}).$$

Thus, the proof is complete. ◻

Proof of Theorem 1.10. According to the conditions of Theorem 1.2, we can easily obtain the conclusions by using the similar argument as in Theorem 1.9 and Lemma 2.11. □

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