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PULLBACK ATTRACTOR FOR NON-AUTONOMOUS p-LAPLACIAN EQUATIONS WITH DYNAMIC FLUX BOUNDARY CONDITIONS

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Abstract. This article studies the long-time asymptotic behavior of solutions for the non-autonomous p-Laplacian equation

$$u_t - \Delta_p u + |u|^{p-2}u + f(u) = g(x,t)$$

with dynamic flux boundary conditions

$$u_t + |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} + f(u) = 0$$

in a n-dimensional bounded smooth domain Ω under some suitable assumptions. We prove the existence of a pullback attractor in $\big(W^{1,p}(\Omega)\cap L^q(\Omega)\big)\times L^q(\Gamma)$ by asymptotic a priori estimate.

1. Introduction

We are concerned with the existence of a pullback attractor in $\big(W^{1,p}(\Omega)\cap L^q(\Omega)\big)\times L^q(\Gamma)$ for the process $\{U(t,\tau)\}_{t\geq\tau}$ associated with solutions of the following non-autonomous p-Laplacian equation

$$u_t - \Delta_p u + |u|^{p-2}u + f(u) = g(x,t), \quad (x,t)\in\Omega\times\mathbb{R}_\tau. \tag{1.1}$$

This equation is subject to the dynamic flux boundary condition

$$u_t + |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} + f(u) = 0, \qquad (x,t)\in\Gamma\times\mathbb{R}_\tau \tag{1.2}$$

and the initial conditions

$$u(x,\tau) = u_\tau(x), \quad x\in\Omega, \tag{1.3}$$

$$u(x,\tau) = \theta_\tau(x), \quad x\in\Gamma, \tag{1.4}$$

where $\Omega\subset\mathbb{R}^n$ $(n\geq 3)$ is a bounded domain with smooth boundary Γ, ν denotes the outer unit normal on Γ, $p\geq 2$, $\mathbb{R}_\tau = [\tau,+\infty)$, the nonlinearity f and the external force g satisfy some conditions, specified later.

To study problem (1.1)-(1.4), we assume the following conditions:

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(H1) the function $f \in C^1(\mathbb{R}, \mathbb{R})$ and satisfies

$$f'(u) \geq -l \tag{1.5}$$

for some $l \geq 0$, and

$$c_1|u|^q - k \leq f(u)u \leq c_2|u|^q + k, \tag{1.6}$$

where $c_i > 0$ $(i = 1, 2)$, $q \geq 2$, $k > 0$.

(H2) The external force $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous, g belongs to $H^1_{\rm loc}(\mathbb{R}, L^2(\Omega))$, and satisfies

$$\int_{-\infty}^t e^{c_1 s}\|g(s)\|^2_{L^2(\Omega)}\, ds + \int_{-\infty}^t e^{c_1 s}\|g_t(s)\|^2_{L^2(\Omega)}\, ds < \infty \tag{1.7}$$

for all $t \in \mathbb{R}$.

Dynamic boundary conditions are very natural in many mathematical models such as heat transfer in a solid in contact with a moving fluid, thermoelasticity, diffusion phenomena, heat transfer in two medium, problems in fluid dynamics (see [1, 2, 3, 6, 7, 14, 22, 23, 28, 29]). The understanding of the asymptotic behavior of dynamical systems is one of the most important problems of modern mathematical physics. One way to treat this problem for a dissipative system is to analyze the existence and structure of its attractor. Generally speaking, the attractor has a very complicated geometry which reflects the complexity of the long-time behavior of the system. There are many authors who have considered the long-time behavior of solutions for the problems of dynamic boundary conditions. For example, the authors considered the existence of global attractors, respectively, in $L^2(\bar{\Omega}, d\mu)$, $L^q(\bar{\Omega}, d\mu)$ and $\big(H^1(\Omega) \cap L^q(\Omega)\big) \times L^q(\Gamma)$ for the reaction-diffusion equation with dynamic flux boundary conditions in [14]. The existence of uniform attractors in $L^2(\bar{\Omega}, d\mu)$ and $\big(H^1(\Omega) \cap L^q(\Omega)\big) \times L^q(\Gamma)$ for the reaction-diffusion equation with dynamic flux boundary conditions was proved in [28]. In [27], the authors proved the existence of global attractors for the autonomous p-Laplacian equation with dynamic flux boundary conditions in $L^2(\bar{\Omega}, d\mu)$, $L^q(\bar{\Omega}, d\mu)$ by the Sobolev compactness embedding theorem and the existence of a global attractor for the autonomous p-Laplacian equation with dynamic flux boundary conditions in $\big(W^{1,p}(\Omega) \cap L^q(\Omega)\big) \times L^q(\Gamma)$ by asymptotical a priori estimate. Recently, the existence of uniform attractors in $L^2(\bar{\Omega}, d\mu)$ and $\big(W^{1,p}(\Omega) \cap L^q(\Omega)\big) \times L^q(\Gamma)$ for the non-autonomous p-Laplacian equation with dynamic flux boundary conditions was obtained in [18].

Non-autonomous equations appear in many applications in natural sciences, so they are of great importance and interest. The long-time behavior of solutions for the non-autonomous equations has been studied extensively in recent years (see [8, 9, 10, 11, 16, 17, 19, 24, 28]). For instance, the existence of a pullback attractor in $L^2(\Omega)$ was studied in [12]. The authors obtained the existence of a pullback attractor in $H^1_0(\Omega)$ in [25]. The existence of a pullback attractor in $H^1_0(\Omega)$ was considered in [20]. The authors proved the existence of a pullback attractor in $L^p(\Omega)$ for a reaction-diffusion equation in [21] under the assumption

$$\|g(s)\|^2_2 \leq M e^{\alpha|s|}$$

for all $s \in \mathbb{R}$ and $0 \leq \alpha < \lambda_1$, where λ_1 is the first eigenvalue of $-\Delta$ with Dirichlet boundary condition. In [29], the authors used a new type of uniform Gronwall inequality and proved the existence of a pullback attractor in $L^{r_1}(\Omega) \times L^{r_2}(\Gamma)$ for the equation

$$\begin{gathered} u_t - \Delta_p u + |u|^{p-2}u + f(u) = h(t), \quad (x,t) \in \Omega \times \mathbb{R}_\tau, \\ u_t + |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} + g(u) = 0, \quad (x,t) \in \Gamma \times \mathbb{R}_\tau, \\ u(x,\tau) = u_0(x), \quad x \in \bar{\Omega} \end{gathered}$$

under the assumptions that f, g satisfy the polynomial growth condition with orders r_1, r_2 and $\|h(t)\|_{L^2(\Omega)}$ satisfies some weak assumption

$$\int_{-\infty}^{t} e^{\theta s}\|h(s)\|^2_{L^2(\Omega)}\, ds < \infty$$

for all $t \in \mathbb{R}$, where θ is some positive constant. By using their main result, we can get the following result.

Corollary 1.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary Γ, let f and g satisfy* (H1)–(H2). *Then the process $\{U(t,\tau)\}_{t\geq\tau}$ corresponding to* (1.1)-(1.4) *has a pullback $\mathcal{D}$-attractor $\mathcal{A}_q$ in $L^q(\bar{\Omega}, d\mu)$, which is pullback $\mathcal{D}$-attracting in the topology of $L^q(\bar{\Omega}, d\mu)$-norm.*

The study of non-autonomous dynamical systems is an important subject, it is necessary to study the existence of pullback attractors for the non-autonomous p-Laplacian equation with dynamic flux boundary conditions. Nevertheless, there are few results about the existence of a pullback attractor in $(W^{1,p}(\Omega) \cap L^q(\Omega)) \times L^q(\Gamma)$ for the non-autonomous p-Laplacian equation with dynamic flux boundary conditions. The main difficulty is that in our case of the equation with p-Laplacian operator for $p > 2$, we cannot use $-\Delta u_2$ as the test function to verify pullback $\mathcal{D}$-condition, which increases the difficulty in getting an appropriate form of compactness. To overcome this difficulty, we combine the idea of norm-to-weak process with asymptotic a priori estimates to prove the existence of a pullback attractor for the non-autonomous p-Laplacian equation with dynamic flux boundary conditions in $\big(W^{1,p}(\Omega) \cap L^q(\Omega)\big) \times L^q(\Gamma)$.

The main purpose of this paper is to study the existence of a pullback attractor in $(W^{1,p}(\Omega) \cap L^q(\Omega)) \times L^q(\Gamma)$ for the non-autonomous p-Laplacian evolutionary equation (1.1)-(1.4) under quite general assumptions (1.5)-(1.7). Here, we state our main result as follows.

Theorem 1.2. *Assume that* (H1)–(H2) *hold. Then the process $\{U(t,\tau)\}_{t\geq\tau}$ corresponding to problem* (1.1)-(1.4) *has a pullback $\mathcal{D}$-attractor $\mathcal{A}$ in $(W^{1,p}(\Omega) \cap L^q(\Omega)) \times L^q(\Gamma)$.*

This article is organized as follows: In the next section, we give some notation and lemmas used in the sequel. Section 3 is devoted to proving the existence of a pullback absorbing set in $\big(L^2(\Omega) \cap W^{1,p}(\Omega) \cap L^q(\Omega)\big) \times \big(L^2(\Gamma) \cap L^q(\Gamma)\big)$ and the existence of a pullback attractor in $\big(L^q(\Omega) \cap W^{1,p}(\Omega)\big) \times L^q(\Gamma)$.

Throughout this paper, let C be a positive constant, which may be different from line to line (and even in the same line), we denote the trace of u by v.

2. Preliminaries

To study (1.1)-(1.4), we recall the Sobolev space $W^{1,p}(\Omega)$ defined as the closure of $C^\infty(\Omega) \cap W^{1,p}(\Omega)$ in the norm

$$\|u\|_{1,p} = \Big(\int_\Omega |\nabla u|^p + |u|^p \, dx \Big)^{1/p}$$

and denote by X^* the dual space of X. We also define the Lebesgue spaces as follows

$$L^r(\Gamma) = \{v : \|v\|_{L^r(\Gamma)} < \infty\},$$

where

$$\|v\|_{L^r(\Gamma)} = \Big(\int_\Gamma |v|^r \, dS \Big)^{1/r}$$

for $r \in [1, \infty)$. Moreover, we have

$$L^s(\Omega) \oplus L^s(\Gamma) = L^s(\bar{\Omega}, d\mu), \quad s \in [1, \infty),$$
$$\|U\|_{L^s(\bar{\Omega}, d\mu)} = \Big(\int_\Omega |u|^s \, dx \Big)^{1/s} + \Big(\int_\Gamma |v|^s \, dS \Big)^{1/s}$$

for any $U = \begin{pmatrix} u \\ v \end{pmatrix} \in L^s(\bar{\Omega}, d\mu)$, where the measure $d\mu = dx|_\Omega \oplus dS|_\Gamma$ on $\bar{\Omega}$ is defined for any measurable set $A \subset \bar{\Omega}$ by $\mu(A) = |A \cap \Omega| + S(A \cap \Gamma)$. In general, any vector $\theta \in L^s(\bar{\Omega}, d\mu)$ will be of the form $\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$ with $\theta_1 \in L^s(\Omega, dx)$ and $\theta_2 \in L^s(\Gamma, dS)$, and there need not be any connection between θ_1 and θ_2.

Remark 2.1 ([15])**.** $C(\bar{\Omega})$ is a dense subspace of $L^2(\bar{\Omega}, d\mu)$ and a closed subspace of $L^\infty(\bar{\Omega}, d\mu)$.

Next, we recall briefly some lemmas used to prove the existence of pullback absorbing sets for (1.1)-(1.4) under some suitable assumptions.

Lemma 2.2 ([5])**.** *Let $x, y \in \mathbb{R}^n$ and let $\langle \cdot, \cdot \rangle$ be the standard scalar product in $\mathbb{R}^n$. Then for any $p \ge 2$, there exist two positive constants C_1, C_2 which depend on p such that*

$$\langle |x|^{p-2}x - |y|^{p-2}y, x - y \rangle \ge C_1 |x - y|^p,$$
$$\big| |x|^{p-2}x - |y|^{p-2}y \big| \le C_2 (|x| + |y|)^{p-2} |x - y|.$$

3. Existence of pullback attractors

In this section, we prove the existence of pullback attractors of solutions for problem (1.1)-(1.4).

3.1. Well-posedness of solutions for problem (1.1)-(1.4).

In this subsection, we give the well-posedness of solutions for problem (1.1)-(1.4) which can be obtained by the Faedo-Galerkin method (see [26]). Here, we only state it as follows.

Theorem 3.1. *Under the assumptions* (H1)–(H2)*, for any initial data $(u_\tau, \theta_\tau) \in L^2(\bar{\Omega}, d\mu)$, there exists a unique weak solution $u(x, t) \in C(\mathbb{R}_\tau; L^2(\bar{\Omega}, d\mu))$ of problem* (1.1)-(1.4) *and the mapping*

$$(u_\tau, \theta_\tau) \to (u(t), v(t))$$

is continuous on $L^2(\bar{\Omega}, d\mu)$.

By Theorem 3.1, we can define a family of continuous processes $\{U(t,\tau): -\infty < \tau \leq t < \infty\}$ in $L^2(\bar{\Omega}, d\mu)$ as follows: for all $t \geq \tau$,

$$U(t,\tau)(u_\tau, \theta_\tau) = (u(t), v(t)) := (u(t;\tau,(u_\tau,\theta_\tau)), v(t;\tau,(u_\tau,\theta_\tau))),$$

where $u(t)$ is the solution of problem (1.1)-(1.4) with initial data $(u(\tau), v(\tau)) = (u_\tau, \theta_\tau) \in L^2(\bar{\Omega}, d\mu)$. That is, a family of mappings $U(t,\tau) : L^2(\bar{\Omega}, d\mu) \to L^2(\bar{\Omega}, d\mu)$ satisfies

$$U(\tau,\tau) = id \quad \text{(identity)},$$
$$U(t,\tau) = U(t,r)U(r,\tau) \quad \text{for all } \tau \leq r \leq t.$$

3.2. **Existence of a pullback absorbing set.** In this subsection, we recall some basic definitions and abstract results about pullback attractors.

Definition 3.2 ([20, 28])**.** Let X be a Banach space. A process $\{U(t,\tau)\}_{t\geq\tau}$ is said to be norm-to-weak continuous on X, if for any t, $\tau \in \mathbb{R}$ with $t \geq \tau$ and for every sequence $x_n \in X$, from the condition $x_n \to x$ strongly in X, it follows that $U(t,\tau)x_n \to U(t,\tau)x$ weakly in X.

Lemma 3.3 ([20, 28])**.** *Let X and Y be two Banach spaces, and let X^* and Y^* be the dual spaces of X and Y, respectively. If X is dense in Y, the injection $i : X \to Y$ is continuous and its adjoint $i^* : Y^* \to X^*$ is dense. In addition, assume that $\{U(t,\tau)\}_{t\geq\tau}$ is a continuous or weak continuous process on Y. Then $\{U(t,\tau)\}_{t\geq\tau}$ is a norm-to-weak continuous process on X if and only if $\{U(t,\tau)\}_{t\geq\tau}$ maps compact sets of X into bounded sets of X for any t, $\tau \in \mathbb{R}$, $t \geq \tau$.*

Let $\mathcal{D}$ be a nonempty class of families $\hat{D} = \{D(t) : t \in \mathbb{R}\}$ of nonempty subsets of X.

Definition 3.4 ([11])**.** The process $\{U(t,\tau)\}_{t\geq\tau}$ is said to be pullback $\mathcal{D}$-asymptotically compact, if for any $t \in \mathbb{R}$ and any $\hat{D} \in \mathcal{D}$, any sequence $\tau_n \to -\infty$ and any sequence $x_n \in D(\tau_n)$, the sequence $\{U(t,\tau_n)x_n\}_{n=1}^\infty$ is relatively compact in X.

Definition 3.5 ([28])**.** A family $\hat{\mathcal{A}} = \{A(t) : t \in \mathbb{R}\}$ of nonempty subsets of X is said to be a pullback $\mathcal{D}$-attractor for the process $\{U(t,\tau)\}_{t\geq\tau}$ in X, if

(i) $A(t)$ is compact in X for any $t \in \mathbb{R}$,
(ii) $\hat{\mathcal{A}}$ is invariant, i.e., $U(t,\tau)A(\tau) = A(t)$ for any $\tau \leq t$,
(iii) $\hat{\mathcal{A}}$ is pullback $\mathcal{D}$-attracting, i.e.,

$$\lim_{\tau\to-\infty} \operatorname{dist}(U(t,\tau)D(\tau), A(t)) = 0$$

for any $t \in \mathbb{R}$ and any $\hat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}$.

Such a family $\hat{\mathcal{A}}$ is called minimal if $A(t) \subset C(t)$ for any family $\hat{C} = \{C(t) : t \in \mathbb{R}\}$ of closed subsets of X such that $\lim_{\tau\to-\infty} \operatorname{dist}(U(t,\tau)B(\tau), C(t)) = 0$ for any $\hat{B} = \{B(t) : t \in \mathbb{R}\} \in \mathcal{D}$.

Definition 3.6 ([11, 28])**.** It is said that $\hat{B} \in \mathcal{D}$ is pullback $\mathcal{D}$-absorbing for the process $\{U(t,\tau)\}_{t\geq\tau}$, if for any $\hat{D} \in \mathcal{D}$ and any $t \in \mathbb{R}$, there exists a $\tau_0(t,\hat{D}) \leq t$ such that $U(t,\tau)D(\tau) \subset B(t)$ for any $\tau \leq \tau_0(t,\hat{D})$.

Lemma 3.7 ([11, 20, 28])**.** *Let $\{U(t,\tau)\}_{t\geq\tau}$ be a process in X satisfying the following conditions:*

(1) *$\{U(t,\tau)\}_{t\geq\tau}$ be norm-to-weak continuous in X.*

(2) *There exists a family* $\hat{B}$ *of pullback* $\mathcal{D}$*-absorbing sets* $\{B(t) : t \in \mathbb{R}\}$ *in* X.
(3) $\{U(t,\tau)\}_{t\geq\tau}$ *is pullback* $\mathcal{D}$*-asymptotically compact.*

Then there exists a minimal pullback $\mathcal{D}$*-attractor* $\hat{\mathcal{A}} = \{A(t) : t \in \mathbb{R}\}$ *in* X *given by*

$$A(t) = \cap_{s\leq t}\overline{\cup_{\tau\leq s} U(t,\tau)B(\tau)}.$$

Lemma 3.8 ([28]). *Suppose that*

$$y'(s) + \delta y(s) \leq b(s)$$

for some $\delta > 0$, $t_0 \in \mathbb{R}$ *and for any* $s \geq t_0$, *where the functions* y, y', b *are assumed to be locally integrable and* y, b *are nonnegative on the interval* $t < s < t + r$ *for some* $t \geq t_0$. *Then*

$$y(t + r) \leq e^{-\frac{\delta r}{2}} \frac{2}{r} \int_t^{t+\frac{r}{2}} y(s)\, ds + e^{-\delta(t+r)} \int_t^{t+r} e^{\delta s} b(s)\, ds$$

for all $t \geq t_0$.

In the following, let $\mathcal{D}$ be the class of all families $\{D(t) : t \in \mathbb{R}\}$ of nonempty subsets of $L^2(\bar{\Omega}, d\mu)$ such that

$$\lim_{t\to-\infty} e^{c_1 t}[D(t)] = 0,$$

where $[D(t)] = \sup\{\|(u, v)\|_{L^2(\bar{\Omega},d\mu)} : (u, v) \in D(t)\}$. We prove the existence of a pullback absorbing set for the process $\{U(t,\tau)\}_{t\geq\tau}$ corresponding to problem (1.1)-(1.4).

Theorem 3.9. *Under assumptions* (H1)–(H2). *Let* $\{U(t,\tau)\}_{t\geq\tau}$ *be a process associated with problem* (1.1)-(1.4). *Then there exists a pullback* $\mathcal{D}$*-absorbing set in* $\big(L^2(\Omega) \cap W^{1,p}(\Omega) \cap L^q(\Omega)\big) \times \big(L^2(\Gamma) \cap L^q(\Gamma)\big)$.

Proof. Taking the inner product of (1.1) with u, we deduce that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\|u\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Gamma)}\Big) + \|u\|^p_{W^{1,p}} + \int_\Omega f(u)u\, dx + \int_\Gamma f(v)v\, dS \\ &= \int_\Omega g(t)u\, dx. \end{aligned} \tag{3.1}$$

By (1.6), Hölder inequality and Young inequality, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\|u\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Gamma)}\Big) + \|u\|^p_{W^{1,p}(\Omega)} + c_1\|u\|^q_{L^q(\Omega)} + c_1\|v\|^q_{L^q(\Gamma)} \\ &\leq \frac{1}{2}\|g(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|u\|^2_{L^2(\Omega)} + k|\Omega| + k|\Gamma| \\ &\leq \frac{1}{2}\|g(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|u\|^2_{L^2(\Omega)} + \frac{1}{2}\|v\|^2_{L^2(\Gamma)} + k|\Omega| + k|\Gamma|. \end{aligned}$$

Therefore,

$$\begin{aligned} &\frac{d}{dt}\Big(\|u\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Gamma)}\Big) + 2\|u\|^p_{W^{1,p}(\Omega)} + 2c_1\|u\|^q_{L^q(\Omega)} + 2c_1\|v\|^q_{L^q(\Gamma)} \\ &\leq \|g(t)\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Gamma)} + 2k|\Omega| + 2k|\Gamma|. \end{aligned} \tag{3.2}$$

It follows from (3.2) that

$$\begin{aligned} &\frac{d}{dt}\Big(\|u\|_{L^2(\Omega)}^2+\|v\|_{L^2(\Gamma)}^2\Big)+c_1\Big(\|u\|_{L^2(\Omega)}^2+\|v\|_{L^2(\Gamma)}^2\Big)\\ &\quad+2\|u\|_{W^{1,p}(\Omega)}^p+c_1\|u\|_{L^q(\Omega)}^q+c_1\|v\|_{L^q(\Gamma)}^q\\ &\le\|g(t)\|_{L^2(\Omega)}^2+C. \end{aligned}\tag{3.3}$$

From the classical Gronwall inequality, we find that

$$\begin{aligned} &\|u(t)\|_{L^2(\Omega)}^2+\|v(t)\|_{L^2(\Gamma)}^2\\ &\le\Big(\|u_\tau\|_{L^2(\Omega)}^2+\|\theta_\tau\|_{L^2(\Gamma)}^2\Big)e^{c_1(\tau-t)}+e^{-c_1t}\int_{-\infty}^t e^{c_1s}\|g(s)\|_{L^2(\Omega)}^2\,ds+C, \end{aligned}\tag{3.4}$$

which implies

$$\|u(t)\|_{L^2(\Omega)}^2+\|v(t)\|_{L^2(\Gamma)}^2\le\mathcal{C}_0\Big(e^{-c_1t}\int_{-\infty}^t e^{c_1s}\|g(s)\|_{L^2(\Omega)}^2\,ds+1\Big)\tag{3.5}$$

uniformly with respect to all initial conditions $(u_\tau,v_\tau)\in D(\tau)$ for $\tau\le\tau_0(t,\hat{D})$, where $\mathcal{C}_0$ is a positive constant.

Let $F(s)=\int_0^s f(\theta)\,d\theta$, we deduce from (1.6) that there exist three positive constants α_1, α_2, β such that

$$\alpha_1|u|^q-\beta\le F(u)\le\alpha_2|u|^q+\beta,$$

$$\alpha_1|u|_{L^q(\Omega)}^q-\beta|\Omega|\le\int_\Omega F(u)\,dx\le\alpha_2|u|_{L^q(\Omega)}^q+\beta|\Omega|,\tag{3.6}$$

$$\alpha_1|v|_{L^q(\Gamma)}^q-\beta|\Gamma|\le\int_\Gamma F(v)\,dS\le\alpha_2|v|_{L^q(\Gamma)}^q+\beta|\Gamma|.\tag{3.7}$$

Integrating (3.3) from t to $t+1$ and combining (3.4) with (3.6)-(3.7), we obtain

$$\begin{aligned} &2\int_t^{t+1}\|u(s)\|_{W^{1,p}(\Omega)}^p\,ds+\frac{c_1}{\alpha_2}\int_t^{t+1}\int_\Omega F(u(s))\,dx\,ds+\frac{c_1}{\alpha_2}\int_t^{t+1}\int_\Gamma F(v(s))\,dS\,ds\\ &\le\mathcal{C}_0\Big(e^{-c_1t}\int_{-\infty}^t e^{c_1s}\|g(s)\|_{L^2(\Omega)}^2\,ds+1\Big)+\int_t^{t+1}\|g(s)\|_{L^2(\Omega)}^2\,ds+C\\ &\le\mathcal{C}_1\Big(e^{-c_1t}\int_{-\infty}^t e^{c_1s}\|g(s)\|_{L^2(\Omega)}^2\,ds+1\Big) \end{aligned}$$

uniformly with respect to all initial conditions $(u_\tau,v_\tau)\in D(\tau)$ for $\tau\le\tau_0(t,\hat{D})$, where $\mathcal{C}_1$ is a positive constant.

Taking the inner product of (1.1) with u_t, we obtain

$$\begin{aligned} &\|u_t\|_{L^2(\Omega)}^2+\|v_t\|_{L^2(\Gamma)}^2+\frac{d}{dt}\Big(\frac{1}{p}\|u\|_{W^{1,p}(\Omega)}^p+\int_\Omega F(u)\,dx+\int_\Gamma F(v)\,dS\Big)\\ &=\int_\Omega g(x,t)u_t\,dx\\ &\le\frac{1}{2}\|g(t)\|_{L^2(\Omega)}^2+\frac{1}{2}\|u_t\|_{L^2(\Omega)}^2, \end{aligned}$$

which implies

$$\begin{aligned} &\|u_t\|_{L^2(\Omega)}^2+\|v_t\|_{L^2(\Gamma)}^2+\frac{d}{dt}\Big(\frac{2}{p}\|u\|_{W^{1,p}(\Omega)}^p+2\int_\Omega F(u)\,dx+2\int_\Gamma F(v)\,dS\Big)\\ &\le\|g(t)\|_{L^2(\Omega)}^2. \end{aligned}\tag{3.8}$$

It follows from the uniform Gronwall inequality that

$$\begin{aligned} &\|u(t+1)\|_{W^{1,p}(\Omega)}^p + \int_\Omega F(u(t+1))\,dx + \int_\Gamma F(v(t+1))\,dS \\ &\le \mathcal{C}_2\Big(e^{-c_1 t}\int_{-\infty}^t e^{c_1 s}\|g(s)\|_{L^2(\Omega)}^2\,ds + 1\Big) \end{aligned} \tag{3.9}$$

uniformly with respect to all initial conditions $(u_\tau, v_\tau) \in D(\tau)$ for $\tau \le \tau_0(t, \hat{D})$, where $\mathcal{C}_2$ is a positive constant.

We infer from (3.6)-(3.7) and (3.9) that

$$\begin{aligned} &\|u(t+1)\|_{W^{1,p}(\Omega)}^p + \|u(t+1)\|_{L^q(\Omega)}^q + \|v(t+1)\|_{L^q(\Gamma)}^q \\ &\le \mathcal{C}_3\Big(e^{-c_1 t}\int_{-\infty}^t e^{c_1 s}\|g(s)\|_{L^2(\Omega)}^2\,ds + 1\Big) \end{aligned} \tag{3.10}$$

uniformly with respect to all initial conditions $(u_\tau, v_\tau) \in D(\tau)$ for $\tau \le \tau_0(t, \hat{D})$, where $\mathcal{C}_3$ is a positive constant. □

Since $W^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$ and $W^{1,p}(\Omega) \hookrightarrow L^2(\Gamma)$ is compact, we obtain the following result.

Theorem 3.10. *Under the assumptions* (H1)–(H2)*, the process* $\{U(t,\tau)\}_{t\ge\tau}$ *corresponding to problem* (1.1)-(1.4) *has a pullback* $\mathcal{D}$*-attractor* $\mathcal{A}_2$ *in* $L^2(\bar{\Omega}, d\mu)$*, which is compact, connected and invariant.*

3.3. **Existence of a pullback attractor in** $(W^{1,p}(\Omega) \cap L^q(\Omega)) \times L^q(\Gamma)$**.** From Lemma 3.3 and Theorem 3.9, we know that the process $\{U(t,\tau)\}_{t\ge\tau}$ corresponding to problem (1.1)-(1.4) is norm-to-weak continuous in $(W^{1,p}(\Omega) \cap L^q(\Omega)) \times L^q(\Gamma)$. In this subsection, we prove the existence of a pullback $\mathcal{D}$-attractor in $(W^{1,p}(\Omega) \cap L^q(\Omega)) \times L^q(\Gamma)$ by verifying asymptotic a priori estimates.

Next, we give an auxiliary theorem to prove the pullback $\mathcal{D}$-asymptotical compactness of the process $\{U(t,\tau)\}_{t\ge\tau}$ in $(W^{1,p}(\Omega) \cap L^q(\Omega)) \times L^q(\Gamma)$.

Theorem 3.11. *Under assumptions* (H1)–(H2)*, for any* $\hat{D} \in \mathcal{D}$ *and* $t \in \mathbb{R}$*, there exists a family of positive constants* $\{\rho(t) : t \in \mathbb{R}\}$ *and* $\tau_1(t, \hat{D}) \le t$ *such that*

$$\|u_t(t)\|_{L^2(\Omega)}^2 + \|v_t(t)\|_{L^2(\Gamma)}^2 \le \rho(t)$$

for any $(u_\tau, \theta_\tau) \in D(t)$ *and* $\tau \le \tau_1(t, \hat{D})$*, where*

$$(u_t(s), v_t(s)) = \frac{d}{dt}\left(U(t,\tau)(u_\tau, \theta_\tau)\right)\big|_{t=s}$$

and $\rho(t)$ *is a positive constant which is independent of the initial data.*

Proof. Differentiating (1.1) and (1.2) with respect to t, and denoting by $\zeta = u_t$, $\eta = v_t$, we obtain

$$\begin{aligned} &\zeta_t - \operatorname{div}(|\nabla u|^{p-2}\nabla\zeta) - (p-2)\operatorname{div}\big(|\nabla u|^{p-4}(\nabla u\cdot\nabla\zeta)\nabla u\big) \\ &+ (p-1)|u|^{p-2}\zeta + f'(u)\zeta = \frac{dg}{dt}, \end{aligned} \tag{3.11}$$

$$\eta_t + (p-2)|\nabla v|^{p-4}(\nabla v\cdot\nabla\eta)\frac{\partial v}{\partial\nu} + |\nabla v|^{p-2}\frac{\partial\eta}{\partial\nu} + f'(v)\eta = 0, \tag{3.12}$$

where "·" denotes the dot product in $\mathbb{R}^n$.

Multiplying (3.11) by ζ and integrating over Ω, and combining (1.5) with (3.12), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|\zeta\|^2_{L^2(\Omega)}+\|\eta\|^2_{L^2(\Gamma)}\Big)+\int_\Omega |\nabla u|^{p-2}|\nabla\zeta|^2\,dx\\
&+(p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u\cdot\nabla\zeta)^2\,dx+(p-1)\int_\Omega |u|^{p-2}|\zeta|^2\,dx\\
&\le l\Big(\|\zeta\|^2_{L^2(\Omega)}+\|\eta\|^2_{L^2(\Gamma)}\Big)+\|\frac{dg}{dt}(t)\|_{L^2(\Omega)}\|\zeta\|_{L^2(\Omega)}.
\end{aligned}
$$

Integrating (3.8) from t to $t+1$ and using (3.9), we find that

$$
\begin{aligned}
&\int_t^{t+1}\|\zeta(s)\|^2_{L^2(\Omega)}\,ds+\int_t^{t+1}\|\eta(s)\|^2_{L^2(\Gamma)}\,ds\\
&\le \mathcal{C}_4(e^{-c_1t}\int_{-\infty}^{t+1}e^{c_1s}\|g(s)\|^2_{L^2(\Omega)}\,ds+1)
\end{aligned}
$$

uniformly with respect to all initial conditions $(u_\tau,v_\tau)\in D(\tau)$ for $\tau\le\tau_0(t,\hat{D})$, where $\mathcal{C}_4$ is a positive constant.

Therefore, we deduce from the uniform Gronwall inequality that

$$
\begin{aligned}
&\|u_t(t+2)\|^2_{L^2(\Omega)}+\|v_t(t+2)\|^2_{L^2(\Gamma)}\\
&\le \mathcal{C}_5\Big(e^{-c_1t}\int_{-\infty}^{t+1}e^{c_1s}\|g(s)\|^2_{L^2(\Omega)}\,ds+1+\int_{t-1}^{t}\|\frac{dg}{dt}(t)\|^2_{L^2(\Omega)}\,ds\Big),
\end{aligned}
$$

uniformly with respect to all initial conditions $(u_\tau,v_\tau)\in D(\tau)$ for $\tau\le\tau_0(t,\hat{D})$, where $\mathcal{C}_5$ is a positive constant. □

Next, we prove the process $\{U(t,\tau)\}_{t\ge\tau}$ is pullback $\mathcal{D}$-asymptotically compact in $(W^{1,p}(\Omega)\cap L^q(\Omega))\times L^q(\Gamma)$.

Theorem 3.12. *Assume that f and g satisfy conditions* (H1)–(H2). *Then the process $\{U(t,\tau)\}_{t\ge\tau}$ corresponding to problem* (1.1)-(1.4) *is pullback $\mathcal{D}$-asymptotically compact in* $(W^{1,p}(\Omega)\cap L^q(\Omega))\times L^q(\Gamma)$.

Proof. Let $B_0=\{B(t):t\in\mathbb{R}\}$ be a pullback $\mathcal{D}$-absorbing set in $(W^{1,p}(\Omega)\cap L^q(\Omega))\times L^q(\Gamma)$ obtained in Theorem 3.9, then we need only to show that for any $t\in\mathbb{R}$, any $\tau_n\to-\infty$ and $(u_{\tau_n},v_{\tau_n})\in B(\tau_n)$, $\{(u_n(\tau_n),v_n(\tau_n))\}_{n=0}^\infty$ is pre-compact in $(W^{1,p}(\Omega)\cap L^q(\Omega))\times L^q(\Gamma)$, where

$$
(u_n(\tau_n),v_n(\tau_n))=(u(t;\tau_n,(u_{\tau_n},v_{\tau_n})),v(t;\tau_n,(u_{\tau_n},v_{\tau_n})))=U(t,\tau_n)(u_{\tau_n},v_{\tau_n}).
$$

Note that for Corollary 1.1, it remains to prove that for any $(u_{\tau_n},v_{\tau_n})\in B(\tau_n)$ and $\tau_n\to-\infty$, $\{u_n(\tau_n)\}_{n=0}^\infty$ is pre-compact in $W^{1,p}(\Omega)$.

From Theorem 3.10 and Corollary 1.1, we know that $\{(u_n(\tau_n),v_n(\tau_n))\}_{n=0}^\infty$ is pre-compact in $L^2(\bar{\Omega},d\mu)$ and $L^q(\bar{\Omega},d\mu)$. Without loss of generality, we assume that $\{(u_n(\tau_n),v_n(\tau_n))\}_{n=0}^\infty$ is a Cauchy sequence in $L^2(\bar{\Omega},d\mu)$ and $L^q(\bar{\Omega},d\mu)$.

In the following, we prove that $\{u_n(\tau_n)\}_{n=0}^\infty$ is a Cauchy sequence in $W^{1,p}(\Omega)$. Then, by simply calculations, we deduce from Lemma 2.2 that

$$
\begin{aligned}
&\|u_{n_k}(\tau_{n_k})-u_{n_j}(\tau_{n_j})\|^p_{W^{1,p}(\Omega)}\\
&\le(-\frac{d}{dt}u_{n_k}(\tau_{n_k})-f(u_{n_k}(\tau_{n_k}))+\frac{d}{dt}u_{n_j}(\tau_{n_j})+f(u_{n_j}(\tau_{n_j})),u_{n_k}(\tau_{n_k})-u_{n_j}(\tau_{n_j}))
\end{aligned}
$$

$$
\begin{aligned}
&+ (-\frac{d}{dt}v_{n_k}(\tau_{n_k}) - f(v_{n_k}(\tau_{n_k})) + \frac{d}{dt}v_{n_j}(\tau_{n_j}) + f(v_{n_j}(\tau_{n_j})), v_{n_k}(\tau_{n_k}) - v_{n_j}(\tau_{n_j}))\\
&= I_1 + I_2.
\end{aligned}
$$

We now estimate separately the two terms I_1 and I_2. By simply calculations and Hölder's inequality, we deduce that

$$
\begin{aligned}
I_1 \le{}& \|\frac{d}{dt}u_{n_k}(\tau_{n_k}) - \frac{d}{dt}u_{n_j}(\tau_{n_j})\|_{L^2(\Omega)}\|u_{n_k}(\tau_{n_k}) - u_{n_j}(\tau_{n_j})\|_{L^2(\Omega)}\\
&+ C(1 + \|u_{n_k}(\tau_{n_k})\|^{q-1}_{L^q(\Omega)} + \|u_{n_j}(\tau_{n_j})\|^{q-1}_{L^q(\Omega)})\|u_{n_k}(\tau_{n_k}) - u_{n_j}(\tau_{n_j})\|_{L^q(\Omega)}
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 \le{}& \|\frac{d}{dt}u_{n_k}(\tau_{n_k}) - \frac{d}{dt}u_{n_j}(\tau_{n_j})\|_{L^2(\Gamma)}\|u_{n_k}(\tau_{n_k}) - u_{n_j}(\tau_{n_j})\|_{L^2(\Gamma)}\\
&+ C(1 + \|u_{n_k}(\tau_{n_k})\|^{q-1}_{L^q(\Gamma)} + \|u_{n_j}(\tau_{n_j})\|^{q-1}_{L^q(\Gamma)})\|u_{n_k}(\tau_{n_k}) - u_{n_j}(\tau_{n_j})\|_{L^q(\Gamma)}.
\end{aligned}
$$

Combining Theorem 3.10, Corollary 1.1 with Theorem 3.11, yields Theorem 3.12 immediately. □

From Lemma 3.7 and Theorems 3.9, 3.12, we immediately obtain Theorem 1.2.

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