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SIGN-CHANGING SOLUTIONS OF A FOURTH-ORDER ELLIPTIC EQUATION WITH SUPERCRITICAL EXPONENT

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Abstract. In this article we study the nonlinear elliptic problem involving nearly critical exponent

$$\Delta^2 u = |u|^{8/(n-4)+\varepsilon} u \quad \text{in } \Omega,$$
$$\Delta u = u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$ with $n \geq 5$, and ε is a positive parameter. We show that, for ε small, there is no sign-changing solution with low energy which blow up at exactly two points. Moreover, we prove that this problem has no bubble-tower sign-changing solutions.

1. Introduction and statement of results

In this article, we consider the semi-linear elliptic problem with supercritical nonlinearity

$$\begin{gathered} \Delta^2 u = |u|^{p-1+\varepsilon} u \quad \text{in } \Omega, \\ \Delta u = u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 5$, ε is a positive real parameter and $p+1 = \frac{2n}{n-4}$ is the critical Sobolev exponent for the embedding of $H^2(\Omega) \cap H^1_0(\Omega)$ into $L^{p+1}(\Omega)$.

Problem (1.1) is related to the limiting problem (when $\varepsilon = 0$) which exhibits a lack of compactness. In fact, van Der Vorst [25, 26] (see also [19]) showed that (1.1) with $\varepsilon = 0$ has no positive solutions if Ω is a starshaped domain. Whereas Ebobisse and Ould Ahmedou [13] proved that (1.1) with $\varepsilon = 0$ has a positive solution provided that some homology group of Ω is non trivial. This topological condition is sufficient, but not necessary, as examples of contractible domains Ω on which a positive solution exists as shown in [14] (see also [15]). Note that some problems of type (1.1) were studied in case of Riemannian manifolds, see for example [17] and [20].

In view of this qualitative change of the situation for (1.1) with $\varepsilon = 0$, it is interesting to study the problem (1.1) with $\varepsilon < 0$ and $\varepsilon > 0$ and to understand what happens to the solutions of (1.1) (if they exist) as $\varepsilon \to 0$.

Observe that, when $\varepsilon < 0$, the existence of solutions of (1.1) has been proved in [5, 7] (see [3, 4, 9] for the Laplacian case) for each $\varepsilon \in (1-p, 0)$. For the positive

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solutions, Chou and Geng [10] made the first study, when Ω is a convex domain. They gave the asymptotic behavior of the low energy positive solution. They used the method of moving planes to show that the blow-up point is away from the boundary of the domain. The process is standard if the domain is convex. We note that, for non convex regions, this method still works in the Laplacian case through the applications of Kelvin transformations, see [16] (since the problem is invariant under these transformations). However, the Navier boundary conditions are not invariant under the Kelvin transformation of the biharmonic operator. But the method of moving planes also works for convex domains, see [10]. To remove the convexity assumption, Ben Ayed and El Mehdi [5] used another method based on some ideas introduced by Bahri in [1]. This result is the analogous one to the one in [24] and [16] where the Laplacian operator was studied.

Concerning the supercritical case, $\varepsilon > 0$, the problem (1.1) becomes more delicate since we lose the Sobolev embedding which is an important point to overcome. We recall that, when the biharmonic operator in (1.1) is replaced by the Laplacian one, there are many works devoted to the study of the positive solutions of the counterpart of (1.1). It was proved in [6] that (1.1) has no positive solution which blows up at a single point. This result shows that the situation is different from the subcritical case. However, Del Pino et al [11] (see also [18]) gave an existence result for two blow up points, provided that Ω satisfies some geometrical conditions. In sharp contrast to this, very little study has been made concerning the sign-changing solutions, see [8].

It is well known that problem (1.1) (with $\varepsilon < 0$) has always a positive least energy solution u_ε which is obtained by solving the variational problem

$$\inf J(u) \quad \text{where } J(u) := \frac{\int_\Omega |\Delta u|^2}{\big(\int_\Omega |u|^{p+1+\varepsilon}\big)^{2/(p+1+\varepsilon)}} \quad u \in H^2(\Omega) \cap H_0^1(\Omega), \quad u \not\equiv 0.$$

Removing the assumption of the positivity of the solutions, the study of the asymptotic behavior becomes difficult. The main difficulty is that the limit problem, after a change of variable, which is

$$\Delta^2 u = |u|^{p-1}u \quad \text{in } \mathbb{R}^n, \tag{1.2}$$

has many sign-changing solutions which are unknown. However, an interesting information about the energy shows that [14, Lemma 2]

$$\int_{\mathbb{R}^n} |\Delta w|^2 > 2S^{n/4}, \tag{1.3}$$

for each sign-changing solution w of (1.2), where S denotes the best minimizers of the Sobolev inequality on the whole space, that is

$$S = \inf\{|\Delta u|^2_{L^2(\mathbb{R}^n)} |u|^{-2}_{L^{2n/(n-4)}(\mathbb{R}^n)} : \Delta u \in L^2, u \in L^{2n/(n-4)}, u \not\equiv 0\}.$$

When we add the positivity assumption, the solutions of (1.2) are the family

$$\delta_{(a,\lambda)}(x) = c_0 \frac{\lambda^{(n-4)/2}}{(1 + \lambda^2|x-a|^2)^{(n-4)/2}}, \quad c_0 = \big(n(n-4)(n^2-4)\big)^{(n-4)/8}, \tag{1.4}$$

with $\lambda > 0$ and $a \in \mathbb{R}^n$.

The space $H^2(\Omega)\cap H_0^1(\Omega)$ is equipped with the norm $\|\cdot\|$ and its corresponding inner product $\langle\cdot,\cdot\rangle$ defined by

$$\|u\| = \Big(\int_\Omega |\Delta u|^2\Big)^{1/2}, \quad \langle u,v\rangle = \int_\Omega \Delta u \Delta v, \quad u,v\in H^2(\Omega)\cap H_0^1(\Omega). \tag{1.5}$$

When we study problem (1.2) in a bounded smooth domain Ω, we need to introduce the function $P\delta_{(a,\lambda)}$ which is the projection of $\delta_{(a,\lambda)}$ on $H_0^1(\Omega)$. It satisfies

$$\Delta^2 P\delta_{(a,\lambda)} = \Delta^2\delta_{(a,\lambda)} \quad \text{in } \Omega; \quad \Delta P\delta_{(a,\lambda)} = P\delta_{(a,\lambda)} = 0 \quad \text{on } \partial\Omega.$$

These functions are almost positive solutions of (1.1). Our first result deals with the low energy sign-changing solution of (1.1) with $\varepsilon>0$. We prove that there is no solution which blows up at exactly two points. More precisely, we have the following result.

Theorem 1.1. *Let Ω be any smooth bounded domain in $\mathbb{R}^n$ with $n\ge 5$. There exists $\varepsilon_0>0$, such that for each $\varepsilon\in(0,\varepsilon_0)$, problem* (1.1) *has no sign-changing solution u_ε which satisfies*

$$u_\varepsilon = P\delta_{(a_{\varepsilon,1},\lambda_{\varepsilon,1})} - P\delta_{(a_{\varepsilon,2},\lambda_{\varepsilon,2})} + v_\varepsilon, \tag{1.6}$$

with the L^∞-norm of u_ε at the power ε ($|u_\varepsilon|_\infty^\varepsilon$) begin bounded and

$$a_{\varepsilon,i}\in\Omega, \quad \lambda_{\varepsilon,i} d(a_{\varepsilon,i},\partial\Omega)\to\infty \quad \text{for } i=1,2$$
$$\langle P\delta_{(a_{\varepsilon,1},\lambda_{\varepsilon,1})}, P\delta_{(a_{\varepsilon,2},\lambda_{\varepsilon,2})}\rangle \to 0 \quad \text{and} \quad \|v_\varepsilon\|\to 0 \quad \text{as } \varepsilon\to 0.$$

We point out that there are other important phenomena in sign-changing solutions. Indeed, it is possible to find solutions having bubble over bubble (bubble-tower solutions). In the case of the Laplacian operator, Pistoia and Weth [21] constructed a family of sign-changing solutions of (1.1) ($\varepsilon<0$) with k bubbles, $k\ge 2$, concentrated at the same point. This result gives a new phenomenon compared with the positive case. In their paper, they conjectured that this phenomenon cannot appear when $\varepsilon>0$. In [8], we gave an affirmative answer for the conjecture of Pistoia and Weth. The following result deals with phenomenon of bubble-tower solutions for the biharmonic problem (1.1) with supercritical exponent.

Theorem 1.2. *Let Ω be any smooth bounded domain in $\mathbb{R}^n$ with $n\ge 5$. There exists $\varepsilon_0>0$, such that for each $\varepsilon\in(0,\varepsilon_0)$, problem* (1.1) *has no solution u_ε of the form*

$$u_\varepsilon = \sum_{i=1}^k (-1)^{i+1} P\delta_{(a_{\varepsilon,i},\lambda_{\varepsilon,i})} + v_\varepsilon, \tag{1.7}$$

with $\lambda_{\varepsilon,1}\le\lambda_{\varepsilon,2}\le\dots\le\lambda_{\varepsilon,k}$ and $|u_\varepsilon|_\infty^\varepsilon$ bounded, where $k\ge 2$, $a_{\varepsilon,i}\in\Omega$, $\min(\lambda_{\varepsilon,i},\lambda_{\varepsilon,j})|a_{\varepsilon,i}-a_{\varepsilon,j}|$ is bounded, and $v_\varepsilon\to 0$ in $H_0^1(\Omega)$, $\lambda_{\varepsilon,i}d(a_{\varepsilon,i},\partial\Omega)\to+\infty$, $\langle P\delta_{(a_{\varepsilon,i},\lambda_{\varepsilon,i})}, P\delta_{(a_{\varepsilon,j},\lambda_{\varepsilon,j})}\rangle\to 0$, for $i\ne j$, as $\varepsilon\to 0$.

Note that Theorem 1.2 deals with the bubble-tower solutions at one point. However Theorem 1.1 says that there are no solutions which blow up at two points. Combining the ideas of the proof of Theorems 1.1 and 1.2, we are able to prove the following result.

Theorem 1.3. *Let Ω be any smooth bounded domain in $\mathbb{R}^n$ with $n\ge 5$. There exists $\varepsilon_0>0$, such that for each $\varepsilon\in(0,\varepsilon_0)$, problem* (1.1) *has no solution u_ε of the* *form*

$$u_\varepsilon=\sum_{i=1}^{m}(-1)^{i+1}P\delta_{(a_{\varepsilon,i},\lambda_{\varepsilon,i})}+\sum_{i=m+1}^{k}(-1)^{i-m}P\delta_{(a_{\varepsilon,i},\lambda_{\varepsilon,i})}+v_\varepsilon:=u_\varepsilon^1+u_\varepsilon^2+v_\varepsilon, \tag{1.8}$$

with $|u_\varepsilon|_\infty^\varepsilon$ *bounded,* $\|v_\varepsilon\|\to 0$*,* $a_{\varepsilon,i}\to a$ *for each* $i\le m$*,* $a_{\varepsilon,i}\to b$ *for each* $i\ge m+1$ *with* $a\neq b$*, and, for* $j=1,2$*, if* u_ε^j *contains more than one bubble then it satisfies the assumptions of Theorem 1.2.*

The proof of our results will be by contradiction. Thus, throughout this paper we will assume that there exist solutions (u_ε) of (1.1) which satisfy (1.6) or (1.7). In Section 2, we will obtain some information about such (u_ε) which allow us to develop Sections 3 which deal with some useful estimates to the proof of our Theorems. Finally, in Section 4, we combine these estimates to obtain a contradiction. Hence the proof of our results.

2. Preliminary results

In this Section, we assume that there exist solutions (u_ε) of (1.1) which satisfy

$$u_\varepsilon=\sum_{i=1}^{k}(-1)^{i+1}P\delta_{(a_{\varepsilon,i},\lambda_{\varepsilon,i})}+v_\varepsilon, \tag{2.1}$$

with $|u_\varepsilon|_\infty^\varepsilon$ bounded, $k\ge 2$, $a_{\varepsilon,i}\in\Omega$, and as $\varepsilon\to 0$, $\|v_\varepsilon\|\to 0$, $\lambda_{\varepsilon,i}d(a_{\varepsilon,i},\partial\Omega)\to +\infty$, $\langle P\delta_{(a_{\varepsilon,i},\lambda_{\varepsilon,i})},P\delta_{(a_{\varepsilon,j},\lambda_{\varepsilon,j})}\rangle\to 0$ for $i\neq j$. We will collect some useful information used in the next sections. First, from (2.1), it is easy to see that the following remark holds.

Remark 2.1. Let (u_ε) be a family of sign-changing solutions of (1.1) satisfying (2.1). Then

(i) $u_\varepsilon\rightharpoonup 0$ as $\varepsilon\to 0$,
(ii) $\int_\Omega|u_\varepsilon|^{p+1+\varepsilon}=\int_\Omega|\Delta u_\varepsilon|^2=kS^{n/4}+o(1)$,
(iii) $M_{\varepsilon,+}:=\max_\Omega u_\varepsilon\to+\infty$, $M_{\varepsilon,-}:=-\min_\Omega u_\varepsilon\to+\infty$ as $\varepsilon\to 0$.

Secondly, arguing as in [2] and [22], we see that for u_ε satisfying (2.1), there is a unique way to choose α_i, a_i, λ_i and v such that

$$u_\varepsilon=\sum_{i=1}^{k}(-1)^{i+1}\alpha_iP\delta_{(a_i,\lambda_i)}+v, \tag{2.2}$$

with

$$\begin{gathered}\alpha_i\in\mathbb{R},\quad \alpha_i\to 1,\\ a_i\in\Omega,\quad \lambda_i\in\mathbb{R}_+^*,\quad \lambda_id(a_i,\partial\Omega)\to+\infty,\\ v\to 0\quad\text{in } H^2(\Omega)\cap H_0^1(\Omega),\quad v\in E,\end{gathered} \tag{2.3}$$

where E denotes the subspace of $H_0^1(\Omega)$ defined by

$$E:=\big\{w:\langle w,\varphi\rangle=0,\ \forall\varphi\in\operatorname{span}\{P\delta_i,\partial P\delta_i/\partial\lambda_i,\partial P\delta_i/\partial a_i^j,\ i\le k;j\le n\}\big\}. \tag{2.4}$$

Here, a_i^j denotes the j-th component of a_i and in the sequel, in order to simplify the notations, we set

$$\delta_{(a_i,\lambda_i)}=\delta_i,\quad P\delta_{(a_i,\lambda_i)}=P\delta_i. \tag{2.5}$$

In the following, we always assume that u_ε (which satisfies (2.1)) is written as in (2.2) and (2.3) holds.

Lemma 2.2. *Let u_ε satisfying the assumption of above theorems. Then λ_i occurring in* (2.2) *satisfies*

$$\lambda_i^\varepsilon \to 1 \quad \text{as } \varepsilon \to 0, \text{ for each } i \le k. \tag{2.6}$$

Proof. By Remark 2.1, we know that

$$\int_\Omega |u_\varepsilon|^{p+1+\varepsilon} = kS^{n/4} + o(1) \quad \text{as } \varepsilon \to 0. \tag{2.7}$$

Furthermore,

$$\int_\Omega |u_\varepsilon|^{p+1+\varepsilon} = \int_\Omega -\Delta u_\varepsilon u_\varepsilon = \int_\Omega |u_\varepsilon|^{p-1+\varepsilon} u_\varepsilon \Big(\sum (-1)^{i+1}\alpha_i P\delta_i\Big) + O(\|v\|^2). \tag{2.8}$$

Observe that

$$\begin{aligned} &\int_\Omega |u_\varepsilon|^{p-1+\varepsilon} u_\varepsilon \Big(\sum (-1)^{i+1}\alpha_i P\delta_i\Big) \\ &= \sum \alpha_i^{p+\varepsilon+1} \int_\Omega P\delta_i^{p+\varepsilon+1} + O\Big(\sum_{j\neq i} \int_\Omega P\delta_i^{p+\varepsilon} P\delta_j\Big) \\ &\quad + O\Big(\sum \int_\Omega \alpha_i P\delta_i^{p+\varepsilon}|v| + \sum \int_\Omega \alpha_i P\delta_i |v|^{p+\varepsilon}\Big) := \sum A_i, \end{aligned} \tag{2.9}$$

where

$$A_i := \alpha_i^{p+\varepsilon+1} \int_\Omega P\delta_i^{p+\varepsilon+1} + O\Big(\sum_{j\neq i} \int_\Omega P\delta_i^{p+\varepsilon} P\delta_j + \int_\Omega P\delta_i^{p+\varepsilon}|v| + \int_\Omega P\delta_i |v|^{p+\varepsilon}\Big).$$

Easy computations show that

$$\begin{gathered} \int_\Omega P\delta_i^{p+1+\varepsilon} = \lambda_i^{\varepsilon(n-4)/2}\Big(S^{n/4} + o(1)\Big), \\ \int_\Omega P\delta_i^{p+\varepsilon}|v| = \lambda_i^{\varepsilon(n-4)/2} O(|v|_{L^{p+1}}), \\ \int_\Omega P\delta_i |v|^{p+\varepsilon} = \lambda_i^{\varepsilon(n-4)/2} O(|v|_{L^{p+1}}^{p+\varepsilon}). \end{gathered}$$

Recall that for $i \neq j$ (see [1])

$$\int_{\mathbb{R}^n} \delta_i^p \delta_j = \int_{\mathbb{R}^n} \delta_j^p \delta_i = c\varepsilon_{ij} + O\Big(\varepsilon_{ij}^{n/(n-4)} \log \varepsilon_{ij}^{-1}\Big),$$

where c is a positive constant and, for $i \neq j$, ε_{ij} is defined by

$$\varepsilon_{ij} = \Big(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j|a_i - a_j|^2\Big)^{(4-n)/2}. \tag{2.10}$$

Hence, we obtain

$$\int_\Omega P\delta_i^{p+\varepsilon} P\delta_j = O\Big(\lambda_i^{\varepsilon(n-4)/2} \int_\Omega P\delta_i^p P\delta_j\Big) = \lambda_i^{\varepsilon(n-4)/2} O(\varepsilon_{ij}), \quad \text{for } i \neq j.$$

Thus

$$A_i = \alpha_i^{p+\varepsilon+1} \lambda_i^{\varepsilon(n-4)/2}\Big(S^{n/4} + o(1)\Big). \tag{2.11}$$

Therefore (2.8), (2.9), and (2.11) provide us with

$$\int_\Omega |u_\varepsilon|^{p+1+\varepsilon} = \Big(\sum \alpha_i^{p+1+\varepsilon} \lambda_i^{\varepsilon(n-4)/2}\Big)\Big(S^{n/4} + o(1)\Big) + o(1). \tag{2.12}$$

Combining (2.7), (2.12), and the fact that α_i satisfies (2.3), the lemma follows. $\square$

Remark 2.3 ([6, 24]). We recall the estimate

$$\delta_i^\varepsilon(x) - c_0^\varepsilon \lambda_i^{\varepsilon(n-4)/2} = O\left(\varepsilon \log(1+\lambda_i^2|x-a_i|^2)\right) \quad \text{in } \Omega, \tag{2.13}$$

which will be very useful in the next section.

3. Some useful estimates

As usual in this type of problems, we first deal with the v-part of u_ε, in order to show that it is negligible with respect to the concentration phenomenon.

Lemma 3.1. *The function v defined in* (2.2)*, satisfies the estimate*

$$\|v\| \\ \leq c\varepsilon + c \begin{cases} \sum_i \frac{1}{(\lambda_i d_i)^{n-4}} + \sum_{i\neq j} \varepsilon_{ij} (\log \varepsilon_{ij}^{-1})^{(n-4)/n} & \text{if } n < 12, \\ \sum_i \frac{1}{(\lambda_i d_i)^{(n+4)/2-\varepsilon(n-4)}} + \sum_{i\neq j} \varepsilon_{ij}^{(n+4)/2(n-4)} (\log \varepsilon_{ij}^{-1})^{(n+4)/2n} & ifn \geq 12, \end{cases}$$

where ε_{ij} is defined in (2.10) *and $d_i := d(a_i, \partial\Omega)$ for $i \leq k$.*

Proof. Since $u_\varepsilon = \sum(-1)^{i+1}\alpha_i P\delta_i + v$ is a solution of (1.1) and $v \in E$ (see (2.4)), we obtain

$$\begin{aligned} \int_\Omega -\Delta u_\varepsilon v = \|v\|^2 &= \int_\Omega |u_\varepsilon|^{p-1+\varepsilon} u_\varepsilon v \\ &= \int_\Omega |\sum(-1)^{i+1}\alpha_i P\delta_i|^{p-1+\varepsilon} (\sum(-1)^{i+1}\alpha_i P\delta_i) v \\ &\quad + p \int_\Omega |\sum(-1)^{i+1}\alpha_i P\delta_i|^{p-1+\varepsilon} v^2 + o(\|v\|^2). \end{aligned}$$

Hence, we have

$$Q(v,v) = f(v) + o(\|v\|^2), \tag{3.1}$$

where

$$\begin{aligned} Q(v,v) &= \|v\|^2 - p\int_\Omega |\sum(-1)^{i+1}\alpha_i P\delta_i|^{p-1+\varepsilon} v^2, \\ f(v) &= \int_\Omega |\sum(-1)^{i+1}\alpha_i P\delta_i|^{p-1+\varepsilon} (\sum(-1)^{i+1}\alpha_i P\delta_i) v. \end{aligned}$$

Using Remark 2.3 and according to [1], it is easy to see that

$$Q(v,v) = \|v\|^2 - p\sum_{i=1}^k \int_\Omega (P\delta_i)^{p-1+\varepsilon} v^2 + o(\|v\|^2)$$

is positive definite; that is, there exists $c > 0$ independent of ε, satisfying $Q(v,v) \geq c\|v\|^2$, for each $v \in E$. Then, from (3.1) we get

$$\|v\|^2 = O(\|f(v)\|).$$

Now, using Lemma 2.2, we obtain

$$\begin{aligned} f(v) &= \sum(-1)^{i+1} \int_\Omega (\alpha_i P\delta_i)^{p+\varepsilon} v \\ &\quad + O\Big(\sum_{i\neq j} \int_\Omega (\delta_i\delta_j)^{p/2}|v| + \sum_{i\neq j} \int_\Omega \delta_i^{p-1}\delta_j |v| (\text{if } n < 12)\Big). \end{aligned} \tag{3.2}$$

Using Remark 2.3 and the fact that $v \in E$, we obtain

$$
\begin{aligned}
&|\int_\Omega P\delta_i^{p+\varepsilon} v| \\
&= |\int \delta_i^{p+\varepsilon} v| + O\Big(\int \delta_i^{p-1+\varepsilon}\theta_i |v|\Big) \\
&\le c\varepsilon \int \log(1+\lambda_i^2|x-a_i|^2)\delta_i^p|v| + c|\theta_i|_{L^\infty}\int \delta_i^{p-1+\varepsilon}|v| \\
&\le c\|v\|\Big(\varepsilon + \frac{1}{(\lambda_i d_i)^{n-4}}(\text{if } n<12) + \frac{1}{(\lambda_i d_i)^{\frac{n+4}{2}+\varepsilon(n-4)}}(\text{if } n \ge 12)\Big),
\end{aligned}
$$

where $\theta_i := \theta_{a_i,\lambda_i} := \delta_i - P\delta_i$.

For the other integrals of (3.2), we use Holder's inequality and we obtain for $i \neq j$

$$
\begin{aligned}
\int_\Omega (\delta_i\delta_j)^{p/2}|v| &\le c\|v\|\Big(\int_\Omega (\delta_i\delta_j)^{n/(n-4)}\Big)^{(n+4)/2n} \\
&\le c\|v\|\varepsilon_{ij}^{(n+4)/2(n-4)}(\log\varepsilon_{ij}^{-1})^{(n+4)/2n}
\end{aligned}
$$

and if $n<12$, we have $p-1 = 8/(n-4) > 1$; therefore

$$
\int_\Omega \delta_i^{p-1}\delta_j|v| \le c\|v\|\Big(\int_\Omega (\delta_i\delta_j)^{n/(n-4)}\Big)^{(n-4)/n} \le c\|v\|\varepsilon_{ij}(\log\varepsilon_{ij}^{-1})^{(n-4)/n}. \tag{3.3}
$$

Combining (3.2)–(3.3), the proof follows. □

Now, we need to introduce some notations before to state the crucial point in the proof of our Theorems. We denote by G the Green's function defined by : $\forall x \in \Omega$

$$
\begin{gathered}
\Delta^2 G(x,.) = c_n\delta_x \quad \text{in } \Omega, \\
\Delta G(x,.) = G(x,.) = 0 \quad \text{on } \partial\Omega,
\end{gathered}
$$

where δ_x is the Dirac mass at x and $c_n = (n-4)(n-2)\omega_n$, with ω_n is the area of the unit sphere of $\mathbb{R}^n$. We denote by H the regular part of G, that is,

$$
H(x_1,x_2) = |x_1-x_2|^{4-n} - G(x_1,x_2) \quad \text{for } (x_1,x_2) \in \Omega^2 \setminus \Gamma
$$

with $\Gamma = \{(y,y) : y \in \Omega\}$.

Proposition 3.2. *Assume that $n \ge 5$ and let α_i, a_i and λ_i be the variables defined in (2.2) with $k=2$. We have*

$$
\begin{aligned}
&\Big|\alpha_i c_1 \frac{n-4}{2}\frac{H(a_i,a_i)}{\lambda_i^{n-4}} - \alpha_j c_1\Big(\lambda_i\frac{\partial\varepsilon_{12}}{\partial\lambda_i} + \frac{n-4}{2}\frac{H(a_1,a_2)}{(\lambda_1\lambda_2)^{(n-4)/2}}\Big) + \alpha_i\frac{n-4}{2}c_2\varepsilon\Big| \\
&\le c\varepsilon^2 + c\begin{cases} \sum_{k=1,2}\frac{1}{(\lambda_k d_k)^{n-2}} + \varepsilon_{12}^{\frac{n}{n-4}}\log\varepsilon_{12}^{-1} + \varepsilon_{12}^2(\log\varepsilon_{12}^{-1})^{\frac{2(n-4)}{n}} & \text{if } n \ge 6), \\ \sum_{k=1,2}\frac{1}{(\lambda_k d_k)^2} + \varepsilon_{12}^2(\log\varepsilon_{12}^{-1})^{2/5} & \text{if } n=5, \end{cases}
\end{aligned} \tag{3.4}
$$

where $i,j \in \{1,2\}$ with $i \neq j$ and c_1, c_2 are positive constants.

Proof. Let

$$
c_1 = c_0^{\frac{2n}{n-4}}\int_{\mathbb{R}^n}\frac{dx}{(1+|x|^2)^{(n+4)/2}}, \quad c_2 = \frac{n-4}{2}c_0^{\frac{2n}{n-4}}\int_{\mathbb{R}^n}\log(1+|x|^2)\frac{|x|^2-1}{(1+|x|^2)^{n+1}}dx.
$$

It suffices to prove the proposition for $i=1$. Multiplying (1.1) by $\lambda_1 \partial P\delta_1/\partial\lambda_1$ and integrating on Ω, we obtain

$$\alpha_1 \int_\Omega \delta_1^p \lambda_1 \frac{\partial P\delta_1}{\partial \lambda_1} - \alpha_2 \int_\Omega \delta_2^p \lambda_1 \frac{\partial P\delta_2}{\partial \lambda_2} = \int_\Omega |u_\varepsilon|^{p-1+\varepsilon} u_\varepsilon \lambda_1 \frac{\partial P\delta_1}{\partial \lambda_1}. \tag{3.5}$$

Using [1], we derive

$$\begin{gathered}
\int_\Omega \delta_1^p \lambda_1 \frac{\partial P\delta_1}{\partial \lambda_1} = \frac{n-4}{2} c_1 \frac{H(a_1,a_1)}{\lambda_1^{n-4}} + O\Big(\frac{\log(\lambda_1 d_1)}{(\lambda_1 d_1)^{n-1}}\Big), \\
\int_\Omega \delta_2^p \lambda_1 \frac{\partial P\delta_1}{\partial \lambda_1} = c_1 \Big(\lambda_1 \frac{\partial \varepsilon_{12}}{\partial \lambda_1} + \frac{n-4}{2} \frac{H(a_1,a_2)}{(\lambda_1\lambda_2)^{(n-4)/2}}\Big) + R,
\end{gathered}$$

where R satisfies

$$R = O\Big(\sum_{k=1,2} \frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^{n-1}} + \varepsilon_{12}^{\frac{n}{n-4}} \log \varepsilon_{12}^{-1}\Big). \tag{3.6}$$

For the other term of (3.5), we have

$$\begin{aligned}
&\int_\Omega |u_\varepsilon|^{p-1+\varepsilon} u_\varepsilon \lambda_1 \frac{\partial P\delta_1}{\partial \lambda_1} \\
&= \int_\Omega |\alpha_1 P\delta_1 - \alpha_2 P\delta_2|^{p-1+\varepsilon} (\alpha_1 P\delta_1 - \alpha_2 P\delta_2) \lambda_1 \frac{\partial P\delta_1}{\partial \lambda_1} \\
&\quad + (p+\varepsilon) \int_\Omega |\alpha_1 P\delta_1 - \alpha_2 P\delta_2|^{p-1+\varepsilon} v \lambda_1 \frac{\partial P\delta_1}{\partial \lambda_1} + O\Big(\|v\|^2 + \varepsilon_{12}^{\frac{n}{n-4}} \log \varepsilon_{12}^{-1}\Big).
\end{aligned} \tag{3.7}$$

The above integral can be written as

$$\begin{aligned}
&\int_\Omega |\alpha_1 P\delta_1 - \alpha_2 P\delta_2|^{p-1+\varepsilon} v \lambda_1 \frac{\partial P\delta_1}{\partial \lambda_1} \\
&= \int_\Omega (\alpha_1 P\delta_1)^{p-1+\varepsilon} v \lambda_1 \frac{\partial P\delta_1}{\partial \lambda_1} + O\Big(\int_{\Omega\setminus A} P\delta_2^{p-1} P\delta_1 |v| + \int_A P\delta_1^{p-1} P\delta_2 |v|\Big),
\end{aligned} \tag{3.8}$$

where $A = \{x : 2\alpha_2 P\delta_2 \le \alpha_1 P\delta_1\}$. Observe that, for $n \ge 12$, we have $p-1 = 8/(n-4) \le 1$, thus

$$\begin{aligned}
\int_{\Omega\setminus A} P\delta_2^{p-1} P\delta_1 |v| + \int_A P\delta_1^{p-1} P\delta_2 |v| &\le c \int_\Omega |v| (\delta_1\delta_2)^{\frac{n+4}{2(n-4)}} \\
&\le c\|v\| \varepsilon_{12}^{(n+4)/2(n-4)} (\log \varepsilon_{12}^{-1})^{(n+4)/2n}.
\end{aligned}$$

For $n < 12$, we have

$$\int_{\Omega\setminus A} P\delta_2^{p-1} P\delta_1 |v| + \int_A P\delta_1^{p-1} P\delta_2 |v| \le c\varepsilon_{12} (\log \varepsilon_{12}^{-1})^{(n-4)/n} \|v\|. \tag{3.9}$$

For the other integral in (3.8), using [1], [24] and Remark 2.3, we obtain

$$\begin{aligned}
&\int_\Omega P\delta_1^{p-1+\varepsilon} v \lambda_1 \frac{\partial P\delta_1}{\partial \lambda_1} \\
&= O\Big(\|v\| \Big[\varepsilon + \Big(\frac{1}{(\lambda_1 d_1)^{\inf(n-4,(n+4)/2)}} (\text{if } n \ne 12) + \frac{\log(\lambda_1 d_1)}{(\lambda_1 d_1)^4} (\text{if } n = 12)\Big)\Big]\Big).
\end{aligned}$$

It remains to estimate the second integral of (3.7). We have

$$\int_\Omega |\alpha_1 P\delta_1 - \alpha_2 P\delta_2|^{p-1+\varepsilon} (\alpha_1 P\delta_1 - \alpha_2 P\delta_2) \lambda_1 \frac{\partial P\delta_1}{\partial \lambda_1}$$

$$\begin{aligned}&= \int_\Omega (\alpha_1 P\delta_1)^{p+\varepsilon}\lambda_1 \frac{\partial P\delta_1}{\partial\lambda_1} - \int_\Omega (\alpha_2 P\delta_2)^{p+\varepsilon}\lambda_1 \frac{\partial P\delta_1}{\partial\lambda_1}\\ &\quad - (p+\varepsilon)\int_\Omega \alpha_2 P\delta_2(\alpha_1 P\delta_1)^{p-1+\varepsilon}\lambda_1 \frac{\partial P\delta_1}{\partial\lambda_1} + O\Big(\varepsilon_{12}^{\frac{n}{n-4}}\log\varepsilon_{12}^{-1}\Big).\end{aligned}$$

Now, using Remark 2.3 and [1], we have

$$\begin{aligned}\int_\Omega P\delta_1^{p+\varepsilon}\lambda_1\frac{\partial P\delta_1}{\partial\lambda_1} &= \frac{n-4}{2}\Big(c_2\varepsilon + 2c_1\frac{H(a_1,a_1)}{\lambda_1^{n-4}}\Big)\\ &\quad + O\Big(\varepsilon^2 + \frac{\log(\lambda_1 d_1)}{(\lambda_1 d_1)^{n-1}} + \frac{1}{(\lambda_1 d_1)^2}(\text{if } n=5)\Big),\\ \int_\Omega P\delta_2^{p+\varepsilon}\lambda_1\frac{\partial P\delta_1}{\partial\lambda_1} &= c_1\Big(\lambda_1\frac{\partial\varepsilon_{12}}{\partial\lambda_1} + \frac{n-4}{2}\frac{H(a_1,a_2)}{(\lambda_1\lambda_2)^{(n-4)/2}}\Big) + R_2,\\ p\int_\Omega P\delta_2 P\delta_1^{p-1+\varepsilon}\lambda_1\frac{\partial P\delta_1}{\partial\lambda_1} &= c_1\Big(\lambda_1\frac{\partial\varepsilon_{12}}{\partial\lambda_1} + \frac{n-4}{2}\frac{H(a_1,a_2)}{(\lambda_1\lambda_2)^{(n-4)/2}}\Big) + R_1,\end{aligned} \tag{3.10}$$

where for $i=1,2$,

$$\begin{aligned}R_i &= O\Big(\varepsilon\varepsilon_{12}(\log\varepsilon_{12}^{-1})^{\frac{n-4}{n}}\Big) + \Big(\varepsilon_{12}^{\frac{n}{n-4}}(\log\varepsilon_{12}^{-1}) + \frac{\log(\lambda_i d_i)}{(\lambda_i d_i)^n}\text{if } n\geq 8\Big)\\ &\quad + \Big(\frac{\varepsilon_{12}(\log\varepsilon_{12}^{-1})^{\frac{n-4}{n}}}{(\lambda_i d_i)^{n-4}}\text{if } n<8\Big)\end{aligned}$$

Therefore, combining (3.5)–(3.10), and Lemma 3.1, the proof of Proposition 3.2 follows. ☐

4. Proof of main theorems

Proof of Theorem 1.1. Arguing by contradiction, let us suppose that the problem (1.1) has a solution u_ε as stated in Theorem 1.1. This solution has to satisfy (2.2) and from Proposition 3.2, we have

$$\begin{aligned}&c_1\frac{n-4}{2}\frac{H(a_i,a_i)}{\lambda_i^{n-4}} - c_1\Big(\lambda_i\frac{\partial\varepsilon_{12}}{\partial\lambda_i} + \frac{n-4}{2}\frac{H(a_1,a_2)}{(\lambda_1\lambda_2)^{(n-4)/2}}\Big) + \frac{n-4}{2}c_2\varepsilon\\ &= o\Big(\varepsilon + \sum_{k=1,2}\frac{1}{(\lambda_k d_k)^{n-4}} + \varepsilon_{12}\Big), \quad \text{for } i=1,2.\end{aligned} \tag{4.1}$$

Furthermore, an easy computation shows that

$$\lambda_i\frac{\partial\varepsilon_{12}}{\partial\lambda_i} = -\frac{n-4}{2}\varepsilon_{12}\big(1 - 2\frac{\lambda_j}{\lambda_i}\varepsilon_{12}^{2/n-4}\big), \quad \text{for } i,j=1,2;\ j\neq i. \tag{4.2}$$

Without loss of generality, we can assume that $\lambda_2\geq\lambda_1$. We distinguish two cases and in each one, we will find a contradiction which implies our theorem.

Case 1. $\frac{\lambda_1\lambda_2|a_1-a_2|^2}{\lambda_2/\lambda_1}\to+\infty$. In this case, it is easy to obtain

$$\varepsilon_{12} = \frac{1}{(\lambda_1\lambda_2|a_1-a_2|^2)^{(n-4)/2}} + o(\varepsilon_{12}), \tag{4.3}$$

which implies that

$$\lambda_i\frac{\partial\varepsilon_{12}}{\partial\lambda_i} = -\frac{n-4}{2}\frac{1}{(\lambda_1\lambda_2|a_1-a_2|^2)^{(n-4)/2}} + o(\varepsilon_{12}) \quad \text{for } i=1,2. \tag{4.4}$$

Then from (4.1) and (4.4), we obtain

$$\begin{aligned}&\frac{c_1}{2}\Big(\frac{H(a_1,a_1)}{\lambda_1^{n-4}}+\frac{H(a_2,a_2)}{\lambda_2^{n-4}}\Big)+\frac{c_1}{(\lambda_1\lambda_2)^{(n-4)/2}}\Big(\frac{1}{|a_1-a_2|^{n-4}}-H(a_1,a_2)\Big)+c_2\varepsilon\\&=o\Big(\varepsilon+\sum_{k=1,2}\frac{1}{(\lambda_k d_k)^{n-4}}+\varepsilon_{12}\Big).\end{aligned}$$

Using the fact that

$$\begin{gathered}G(a_1,a_2):=\frac{1}{|a_1-a_2|^{n-4}}-H(a_1,a_2)>0,\\ \varepsilon_{12}=O\Big(\frac{H(a_1,a_2)}{(\lambda_1\lambda_2)^{(n-4)/2}}+\frac{G(a_1,a_2)}{(\lambda_1\lambda_2)^{(n-4)/2}}\Big),\end{gathered}$$

we derive a contradiction in this case.

Case 2. $\frac{\lambda_1\lambda_2|a_1-a_2|^2}{\lambda_2/\lambda_1}\to c\geq 0$. In this case, we remark that $\lambda_2/\lambda_1\to+\infty$ (since $\varepsilon_{12}\to 0$). Multiplying (4.1) by 2 for $i=2$ and adding to (4.1) for $i=1$, we obtain:

$$\begin{aligned}&c_1\Big(\frac{H(a_1,a_1)}{\lambda_1^{n-4}}+2\frac{H(a_2,a_2)}{\lambda_2^{n-4}}\Big)-\frac{2c_1}{n-4}\Big(\lambda_1\frac{\partial\varepsilon_{12}}{\partial\lambda_1}+2\lambda_2\frac{\partial\varepsilon_{12}}{\partial\lambda_2}\Big)\\&-\frac{3H(a_1,a_2)}{(\lambda_1\lambda_2)^{(n-4)/2}}+3c_2\varepsilon\\&=o\Big(\varepsilon+\sum_{k=1,2}\frac{1}{(\lambda_k d_k)^{n-4}}+\varepsilon_{12}\Big).\end{aligned}\tag{4.5}$$

Now, using (4.2) and the fact that $\lambda_2\geq\lambda_1$, an easy computation shows that

$$-\lambda_1\frac{\partial\varepsilon_{12}}{\partial\lambda_1}-2\lambda_2\frac{\partial\varepsilon_{12}}{\partial\lambda_2}\geq\frac{n-4}{4}\varepsilon_{12}.\tag{4.6}$$

Furthermore, since $H(a_1,a_2)\leq cd_1^{4-n}$ and $\lambda_2/\lambda_1\to\infty$, we obtain

$$\frac{H(a_1,a_2)}{(\lambda_1\lambda_2)^{(n-4)/2}}=o\Big(\frac{1}{(\lambda_1 d_1)^{n-4}}\Big).\tag{4.7}$$

Then we derive a contradiction from (4.5), (4.6) and (4.7). Our proof is thereby complete. □

Proof of Theorem 1.2. Arguing by contradiction, let us assume that problem (1.1) has solutions (u_ε) as stated in Theorem 1.2. From Section 2, these solutions have to satisfy (2.2) and (2.3). As in the proof of Proposition 3.2, we have for each $i=1,\dots,k$

$$\begin{aligned}&c_1\frac{n-4}{2}\frac{H(a_i,a_i)}{\lambda_i^{n-4}}+c_1\sum_{j\neq i}(-1)^{j+1}\Big(\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i}+\frac{n-4}{2}\frac{H(a_i,a_j)}{(\lambda_i\lambda_j)^{(n-4)/2}}\Big)+\frac{n-4}{2}c_2\varepsilon\\&=o\Big(\varepsilon+\sum_{j=1}^{k}\frac{1}{(\lambda_j d_j)^{n-4}}+\sum_{r\neq j}\varepsilon_{rj}\Big).\end{aligned}\tag{4.8}$$

Observe that, if $j<i$, we have $\lambda_j|a_i-a_j|$ is bounded (by the assumption) which implies that

$$|a_i-a_j|=o(d_j),\quad d_i/d_j=1+o(1)\,\forall i,j,\quad \varepsilon_{ij}\geq c(\lambda_j/\lambda_i)^{(n-4)/2}\,\forall j<i,\tag{4.9}$$

where c is a positive constant. Using (4.9), easy computations show that

$$\begin{gathered}\varepsilon_{(i-1)j}+\varepsilon_{i(j+1)}=o(\varepsilon_{ij})\quad\forall i<j,\\ \frac{H(a_i,a_j)}{(\lambda_i\lambda_j)^{(n-4)/2}}=o(\frac{1}{(\lambda_1 d_1)^{n-4}})\text{ if }(i,j)\neq(1,1).\end{gathered}\tag{4.10}$$

Thus, using (4.10), (4.8) can be written as

$$c_1\frac{n-4}{2}\frac{H(a_1,a_1)}{\lambda_1^{n-4}}-c_1\lambda_1\frac{\partial\varepsilon_{12}}{\partial\lambda_1}+\frac{n-4}{2}c_2\varepsilon=o\Big(\varepsilon+\frac{1}{(\lambda_1 d_1)^{n-4}}+\sum_{r\neq j}\varepsilon_{rj}\Big),\tag{4.11}$$

$$-c_1\frac{\partial\varepsilon_{(k-1)k}}{\partial\lambda_k}+\frac{n-4}{2}c_2\varepsilon=o\Big(\varepsilon+\frac{1}{(\lambda_1 d_1)^{n-4}}+\sum_{r\neq j}\varepsilon_{rj}\Big),\tag{4.12}$$

and for $1<i<k$,

$$-c_1\lambda_i\frac{\partial\varepsilon_{(i-1)i}}{\partial\lambda_i}-c_1\lambda_i\frac{\partial\varepsilon_{i(i+1)}}{\partial\lambda_i}+\frac{n-4}{2}c_2\varepsilon=o\Big(\varepsilon+\frac{1}{(\lambda_1 d_1)^{n-4}}+\sum_{r\neq j}\varepsilon_{rj}\Big).\tag{4.13}$$

Using (4.2) and (4.12), we derive that

$$\varepsilon=o\Big(\frac{1}{(\lambda_1 d_1)^{n-4}}+\sum_{r\neq j}\varepsilon_{ij}\Big),\quad\varepsilon_{(k-1)k}=o\Big(\frac{1}{(\lambda_1 d_1)^{n-4}}+\sum_{r\neq j}\varepsilon_{rj}\Big).\tag{4.14}$$

Now, using (4.14) and (4.12) with $k-1$ instead of k, we derive the estimate of $\varepsilon_{(k-2)(k-1)}$ and by induction we get

$$\varepsilon_{(i-1)i}=o\Big(\frac{1}{(\lambda_1 d_1)^{n-4}}+\sum_{r\neq j}\varepsilon_{rj}\Big)\quad\text{for }i=2,\dots,k.\tag{4.15}$$

Finally, using (4.10), (4.14), (4.15) and (4.11), we obtain

$$\frac{H(a_1,a_1)}{\lambda_1^{n-4}}=o(\frac{1}{(\lambda_1 d_1)^{n-4}}),$$

which gives a contradiction. Hence, our theorem is proved. $\square$

Proof of Theorem 1.3. Arguing by contradiction, let us assume that problem (1.1) has solutions (u_ε) as stated in Theorem 1.3. From Section 2, these solutions have to satisfy (2.2) and (2.3). Without loss of generality, in the sequel, we will assume that $\lambda_1 d_1\leq\lambda_{m+1}d_{m+1}$. As in the proof of Theorem 1.2, (4.8) is satisfied for each $i=1,\dots,k$. Furthermore, (4.10) holds if $i,j\leq m$ or $i,j>m$ (in the last case, we require that $(i,j)\neq(m+1,m+1)$).

Observe that since $a\neq b$, it is easy to obtain that $|a_i-a_j|\geq c>0$ for each $i\leq m$ and $j\geq m+1$. Hence for $i\leq m$ and $j\geq m+1$ we have

$$\lambda_r\frac{\partial\varepsilon_{ij}}{\partial\lambda_r}=-\frac{n-4}{2}\frac{1}{(\lambda_i\lambda_j|a_i-a_j|^2)^{(n-4)/2}}+o(\varepsilon_{ij}),\quad\text{for }r=i,j,\tag{4.16}$$

$$\varepsilon_{ij}+\frac{H(a_i,a_j)}{(\lambda_i\lambda_j)^{(n-4)/2}}=o\Big(\varepsilon_{1(m+1)}+\frac{1}{(\lambda_1 d_1)^{n-4}}\Big)\quad\text{for }(i,j)\neq(1,m+1).\tag{4.17}$$

Now using (4.10), (4.16) and (4.17), we derive that (4.13) holds for each $i\notin\{1,m+1\}$. However, since the first bubble in the second bubble tower u_ε^2 has negative sign, for $i=1,m+1$, we have

$$c_1\frac{H(a_i,a_i)}{\lambda_i^{n-4}}-\frac{2c_1}{n-4}\lambda_i\frac{\partial\varepsilon_{i(i+1)}}{\partial\lambda_i}+c_1\varepsilon_{1(m+1)}+c_2\varepsilon=o\Big(\varepsilon+\frac{1}{(\lambda_1 d_1)^{n-4}}+\sum_{r\neq j}\varepsilon_{rj}\Big).$$

Finally, arguing as in Theorem 1.2, we derive a contradiction. Hence our result is proved. □

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