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CONVERGENCE TO EQUILIBRIUM OF RELATIVELY COMPACT SOLUTIONS TO EVOLUTION EQUATIONS

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ABSTRACT. We prove convergence to equilibrium for relatively compact solutions to an abstract evolution equation satisfying energy estimates near the omega-limit set. These energy estimates generalize Łojasiewicz and Kurdyka-Łojasiewicz-Simon gradient inequalities. We apply the abstract results to several ODEs and PDEs of first and second order.

1. INTRODUCTION

Convergence results of the type "if φ is in the omega-limit set of $u : \mathbb{R}_+ \to X$ and a condition (C) holds, then $\lim_{t\to+\infty} u(t) = \varphi$" have been extensively studied (see, e.g., Haraux and Jendoubi [6], Albis et al. [1], Chill et al. [5], Lageman [7], Chergui [3, 4], Bárta et al. [2]). Each of the proofs of these results can be split into two parts: the first part shows the key estimate

$$-\frac{d}{dt}\mathbb{E}(u(t)) \geq c\|\dot{u}(t)\| \tag{1.1}$$

for some function $\mathbb{E} : X \to \mathbb{R}$ and the second part proves convergence with help of this estimate.

The second part of the proofs is always the same (see proof of Theorem 2.6 below or corresponding parts of proofs in the articles mentioned above). The first part follows from condition (C). Examples of condition (C) are the Łojasiewicz inequality

$$|\mathbb{E}(u) - \mathbb{E}(\varphi)|^{1-\theta} \leq c\|\mathbb{E}'(u)\| \quad \text{for all } u \text{ near } \varphi \tag{1.2}$$

or the more general Kurdyka-Łojasiewicz-Simon inequality

$$\Theta(|\mathbb{E}(u) - \mathbb{E}(\varphi)|) \leq c\|\mathbb{E}'(u)\| \quad \text{for all } u \text{ near } \varphi. \tag{1.3}$$

If u is a solution to the ordinary differential equation

$$\dot{u} + F(u) = 0, \tag{1.4}$$

one can write

$$-\frac{d}{dt}\mathbb{E}(u(t)) = -\langle \mathbb{E}'(u(t)), \dot{u}(t)\rangle = \langle \mathbb{E}'(u(t)), F(u(t))\rangle. \tag{1.5}$$

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In many important examples (e.g. if (1.4) is a gradient system with $F = \nabla\mathbb{E}$) one can continue with

$$\langle \mathbb{E}'(u(t)), F(u(t))\rangle \geq c\|\mathbb{E}'(u(t))\| \cdot \|F(u(t))\|. \tag{1.6}$$

This inequality is known as angle condition and it plays an important role in proving (1.1).

For partial differential equations, the situation is more complicated since we usually have $\mathbb{E}' : V \to V'$ and $\dot{u}$ has values in V'. So, already the first equality in (1.5) is often unclear, since the expression on the right-hand side has no meaning.

Therefore, it seems to be a good idea to formulate a general convergence result assuming that (1.1) holds and then study, under which conditions (1.1) holds. Another reason for this splitting is that (1.1) is equivalent to the fact that u has finite length (and all the mentioned convergence results are based on proving that u has finite length).

Let us mention that another approach to convergence of (weak) solutions of first and second order evolution equations with maximal monotone operators can be found in the works by Djafari Rouhani and his co-workers, see [8] and references therein.

In Section 2 we formulate and prove general convergence results assuming that (1.1) holds. In Sections 3 and 4 we give several applications to first and second order equations, respectively. Although the results in Sections 3 and 4 are known, we present some proofs to illustrate the applicability of the results in Section 2.

2. General convergence results

Before we formulate and prove the main results, we introduce some notations. Let V, H, be Hilbert spaces such that $V \hookrightarrow H \hookrightarrow V'$. Then $\|\cdot\|$, $\|\cdot\|_V$, $\|\cdot\|_*$ will be the norms in H, V, and V', respectively. Corresponding scalar products will be denoted by the same subscripts. The open ball in V of radius r centered at $\phi \in V$ is denoted by $B_V(\phi, r)$.

If $u : \mathbb{R}_+ \to V$ then the omega-limit set of u in V is

$$\omega_V(u) := \{\phi \in V : \exists t_n \nearrow +\infty \text{ such that } \|u(t_n) - \phi\|_V \to 0\}.$$

We say that $u \in C^1(\mathbb{R}_+, H)$ *has finite length* in H if $\int_0^{+\infty} \|\dot{u}(s)\| \,\mathrm{d}s < +\infty$.

We say that a function $\mathbb{E}$ *satisfies Łojasiewicz (or Simon-Łojasiewicz) inequality on a neighborhood of* φ, if there exists $\theta \in (0, 1/2]$ and $c > 0$ such that (1.2) holds ('u near ϕ' means $u \in B_V(\phi, \varepsilon)$ for some $\varepsilon > 0$). We say that $\mathbb{E}$ *satisfies Kurdyka-Łojasiewicz-Simon inequality on a neighborhood of* φ, if there exists $c > 0$ and a function $\Theta \in C([0, +\infty))$ satisfying $\Theta(s) > 0$ for all $s > 0$, $1/\Theta \in L^1_{loc}([0, +\infty))$ and condition (1.3). We will call functions Θ with the above properties *Kurdyka functions*. Taking $\Theta(s) = s^{1-\theta}$ yields that Łojasiewicz inequality is a special case of Kurdyka-Łojasiewicz-Simon inequality. If Θ is a Kurdyka function, we define $\Phi_\Theta(t) := \int_0^t 1/\Theta(s) \,\mathrm{d}s$.

The following are well known results.

Lemma 2.1. *If u has finite length in H, then it has a limit in H.*

Lemma 2.2. *Let $u : \mathbb{R}_+ \to V$. If $\lim_{t\to+\infty} u(t) = \psi$ in H and u has precompact range in V, then $\lim_{t\to+\infty} u(t) = \psi$ in V.*

Lemma 2.3. *Let $u : \mathbb{R}_+ \to V$. If u has finite length in H and precompact range in V, then it converges in V (as $t \to +\infty$).*

We formulate the general convergence result proposed in the introduction. Its proof follows immediately from Theorem 2.6. Let us emphasize that H can be an arbitrarily large space. So, in the applications, it is sufficient to verify (1.1) with a very weak norm on the right-hand side.

Theorem 2.4. *Let $u \in C(\mathbb{R}_+, V) \cap C^1(\mathbb{R}_+, H)$ with V-precompact range and $\varphi \in \omega_V(u)$. Let $\rho > 0$ and $\mathbb{E} \in C(V, \mathbb{R})$ be such that $t \mapsto \mathbb{E}(u(t))$ is nonincreasing on $\mathbb{R}_+$ and (1.1) holds for almost every $t \in \{s \in \mathbb{R}_+ : \ u(s) \in B := B_V(\varphi, \rho)\}$. Then $\lim_{t\to+\infty} \|u(t) - \varphi\|_V = 0$.*

Remark 2.5. By the previous Lemmas, it is sufficient to show that u has finite length in H. One can see from the proof of Theorem 2.6 below, that the theorem remains valid if $\mathbb{E}$ is only defined on the closure of the range of u and continuous in V-norm on this set. Moreover, if u is injective, then this weaker condition is not only sufficient but also necessary for u to have finite length in H. In fact, one can define $\mathbb{E}(u(t)) := \int_t^{+\infty} \|\dot{u}(s)\| \,\mathrm{d}s$, then (1.1) holds on $\mathbb{R}_+$, so $t \mapsto \mathbb{E}(u(t))$ is nonincreasing on $\mathbb{R}_+$ and continuity of $\mathbb{E}$ also holds.

Theorem 2.4 does not speak about differential equations but it can be applied immediately to a solution of a first order equation

$$\dot{u}(t) + F(u) = 0$$

if $\mathbb{E}$ is nonicreasing along the solution (e.g. a Lyapunov function) and (1.1) holds. Here F may be an unbounded nonlinear operator. Second order equations

$$\ddot{u}(t) + F(u(t), \dot{u}(t)) + M(u(t)) = 0$$

can be reformulated as a first order equation on a product space denoting $v := \dot{u}$. But then the energy or Lyapunov function typically depends on u and v but we are interested in convergence of the first coordinate u only (the second coordinate converges to zero "automatically" — see Theorem 2.8). So, we will formulate Theorem 2.6 suitable for this situation. It is easy to see that Theorem 2.4 follows immediately from Theorem 2.6 (take $V_2 = \{0\} = H_2$ and $V := V_1 \times V_2$, $H := H_1 \times H_2$), so we will not prove it.

Theorem 2.6. *Let $u = (u_1, u_2)$ satisfy $u_1 \in C(\mathbb{R}_+, V_1) \cap C^1(\mathbb{R}_+, H_1)$ and $u_2 \in C(\mathbb{R}_+, V_2) \cap C^1(\mathbb{R}_+, H_2)$ with $V_1 \hookrightarrow H_1$, and let $(u_1(\cdot), u_2(\cdot))$ have a precompact range in $V_1 \times V_2$. Let $\varphi \in \omega_{V_1}(u_1)$, $\rho > 0$ and $\mathbb{E} \in C(V_1 \times V_2, \mathbb{R})$ be such that $t \mapsto \mathbb{E}(u(t))$ is nonincreasing on $\mathbb{R}_+$ and*

$$-\frac{d}{dt}\mathbb{E}(u(t)) \geq \|\dot{u}_1(t)\|_{H_1} \tag{2.1}$$

for almost every $t \in \{s \in \mathbb{R}_+ : \ u_1(s) \in B := B_{V_1}(\varphi, \rho)\}$. Then $\lim_{t\to+\infty} \|u_1(t) - \varphi\|_{V_1} = 0$.

Remark 2.7. (i) It will be clear from the proof that Theorem 2.6 remains valid if (2.1) holds only for almost every $t \in \{s \in [T, +\infty) : \ u_1(s) \in B := B_{V_1}(\varphi, \rho)\}$ for some $T > 0$.

Proof of Theorem 2.6. Let $t_n \nearrow +\infty$ be an increasing sequence such that $\|u_1(t_n) - \varphi\|_{V_1} \to 0$. By precompactness of the range we may assume that $\|u_2(t_n) - \psi\|_{V_2} \to 0$ for some $\psi \in V_2$ (passing to a subsequence of t_n if necessary).

Since $t \mapsto \mathbb{E}(u(t))$ is nonincreasing it has a limit for $t \to +\infty$. Since it is continuous, we have $\lim_{t\to+\infty} \mathbb{E}(u(t)) = \mathbb{E}(\varphi,\psi)$ and we can assume without loss of generality $\mathbb{E}(\varphi,\psi) = 0$ and $\mathbb{E}(u(t)) \geq 0$ for all $t \in \mathbb{R}_+$ (redefining $\mathbb{E}(u) := \mathbb{E}(u) - \mathbb{E}(\varphi,\psi)$).

Since $\|u_1(t_n) - \varphi\|_{V_1} \to 0$, we have $u_1(t_n) \in B$ for all $n \geq n_0$. Let us denote $s_n := \inf_{s\geq t_n}\{u_1(s) \notin B\}$ and assume for contradiction that $s_n < +\infty$ for all n. From continuity of u we have $s_n > t_n$ and $\|u_1(s_n) - \varphi\|_{V_1} = \rho$.

For $t \in (t_n, s_n)$ inequality (2.1) holds, so

$$\mathbb{E}(u(t_n)) - \mathbb{E}(u(t)) \geq \int_{t_n}^t \|\dot{u}_1(s)\|_{H_1}\,\mathrm{d}s.$$

So, we can estimate

$$\begin{aligned}
\|u_1(t) - \varphi\|_{H_1} &\leq \|u_1(t) - u_1(t_n)\|_{H_1} + \|u_1(t_n) - \varphi\|_{H_1}\\
&\leq \int_{t_n}^t \|\dot{u}_1(s)\|_{H_1}\,\mathrm{d}s + \|u_1(t_n) - \varphi\|_{H_1}\\
&\leq \mathbb{E}(u(t_n)) - \mathbb{E}(u(t)) + \|u_1(t_n) - \varphi\|_{H_1}\\
&\leq \mathbb{E}(u(t_n)) + \|u_1(t_n) - \varphi\|_{H_1}
\end{aligned}$$

and by continuity of u this inequality holds for $t = s_n$. Hence, $\|u_1(s_n) - \varphi\|_{H_1} \leq \mathbb{E}(u(t_n)) + \|u_1(t_n) - \varphi\|_{H_1} \to 0$ as $n \to \infty$ (since $V_1 \hookrightarrow H_1$).

On the other hand, by continuity of u we have $\|u_1(s_n) - \varphi\|_{V_1} = \rho$ for all $n \in \mathbb{N}$. So, there is a subsequence of $u_1(s_n)$ converging to some $\tilde{\varphi} \in V_1$ (by precompactness of the range), $\tilde{\varphi} \neq \varphi$, which is a contradiction with $\|u_1(s_n) - \varphi\|_{H_1} \to 0$.

Hence, $s_n = +\infty$ for some n. Hence, $\dot{u}_1 \in L^1(\mathbb{R}_+, H_1)$, it has finite length in H_1 and converges to ϕ in the norm of V_1 by Lemma 2.2. □

In case of second order equations, if a solution has a limit then its derivative usually tends to zero. However, convergence of the derivative often needs much weaker assumptions (or different assumptions) and it is helpful to know the convergence of the derivative a-priori, before one shows convergence of the function itself. Therefore, we formulate the following theorem.

Theorem 2.8. *Let $V \hookrightarrow H \hookrightarrow V'$ be Hilbert spaces, $F \in C(V \times H, V')$, $E \in C^1(V, R)$ and $M = E' : V \to V'$. Assume that there exists a nondecreasing function $g : (0,+\infty) \to (0,+\infty)$ such that*

$$\langle F(u,v), v\rangle_{V',V} \geq g(\|v\|_*)$$

for all u, $v \in V$. If $u \in C^1(\mathbb{R}_+, V) \cap C^2(\mathbb{R}_+, H)$ is a classical solution of

$$\begin{gathered}
\ddot{u}(t) + F(u(t), \dot{u}(t)) + M(u(t)) = 0,\\
u(0) = u_0 \in V,\ \dot{u}(0) = u_1 \in H
\end{gathered} \tag{2.2}$$

such that $(u, \dot{u})$ is precompact in $V \times H$, then $\lim_{t\to+\infty} \|\dot{u}(t)\| = 0$.

Proof. Since range of $(u, \dot{u})$ is precompact in $V \times H$, range of $F(u, \dot{u}) + M(u)$ is bounded in V'. Hence, range of $\ddot{u}$ is bounded in V' and $\dot{u}$ is Lipschitz continuous in V'. Moreover, we have

$$\begin{aligned}
-\frac{d}{dt}\frac{1}{2}\|\dot{u}(t)\|^2 &= -\langle \ddot{u}(t), \dot{u}\rangle_{V',V}\\
&= \langle F(u(t), \dot{u}(t)), \dot{u}\rangle_{V',V} + \frac{d}{dt}E(u(t))
\end{aligned}$$

$$\geq g(\|\dot{u}(t)\|_*) + \frac{d}{dt}E(u(t)).$$

Since $|E(u(s))| \leq K$ for some $K > 0$ and all $s \geq 0$, integrating on $[t_0, t]$,

$$\begin{aligned}\int_{t_0}^{t} g(\|\dot{u}(s)\|_*)\,\mathrm{d}s &\leq \frac{1}{2}(-\|\dot{u}(t)\| + \|\dot{u}(t_0)\|) - E(u(t)) + E(u(t_0)) \\ &\leq \frac{1}{2}\|\dot{u}(t_0)\| + 2K.\end{aligned} \tag{2.3}$$

Hence, $s \mapsto g(\|\dot{u}(s)\|_*) \in L^1((0, +\infty))$ and due to Lipschitz continuity we have $\lim_{t\to+\infty} \|\dot{u}(t)\|_* = 0$. Since range of $\dot{u}$ is precompact in H, $\lim_{t\to+\infty} \|\dot{u}(t)\| = 0$. □

Corollary 2.9. *Let the assumptions of Theorem 2.8 be satisfied and let there exist* $\rho > 0$ *and* $\mathbb{E} \in C(V \times H, \mathbb{R})$ *such that* $t \mapsto \mathbb{E}(u(t), \dot{u}(t))$ *is nonincreasing on* $(0, +\infty)$ *and*

$$-\frac{d}{dt}\mathbb{E}(u(t), \dot{u}(t)) \geq c\|\dot{u}(t)\|_* \tag{2.4}$$

for almost every $t \in \{s \in \mathbb{R}_+ : u(s) \in B_V(\varphi, \rho) \times B_H(0, \varepsilon)\}$ *where* $\varepsilon > 0$ *is arbitrary. Then* $\lim_{t\to+\infty} \|u(t) - \varphi\|_V + \|\dot{u}(t)\| = 0$.

Proof. The derivative converges to 0 by Theorem 2.8. Then $\dot{u}(t) \in B_H(0, \varepsilon)$ for all $t \geq T$. Then (2.1) is satisfied for $t \in [T, +\infty)$ and applying Theorem 2.6 with $H_1 = V'$ (see Remark 2.7) we obtain convergence of $u(t)$. □

Remark 2.10. We can see that the $*$-norm on the right-hand side of (2.4) can be replaced by any other norm weaker than H-norm.

3. Applications to first order equations

In this section, we show several known results that are covered by Theorem 2.4.

3.1. Łojasiewicz convergence result.

We start with the classical convergence result by Łojasiewicz. Let us remark that the following Proposition speaks about ordinary differential equations (then u has values in a finite-dimensional space $H = V$ and $E \in C^1(H)$) and also about partial differential equations (then $V \hookrightarrow H$ are Hilbert spaces, $u \in C(\mathbb{R}_+, V) \cap C^1(\mathbb{R}_+, H)$ and $E \in C^1(V)$).

Proposition 3.1. *Let* u *be a solution to the gradient system* $\dot{u} + \nabla E(u) = 0$, $\varphi \in \omega_V(u)$ *and let* E *satisfy the Łojasiewicz or Kurdyka-Łojasiewicz-Simon inequality on a neighborhood of* φ. *Then there exists a function* $\mathbb{E}$ *such that* $t \mapsto \mathbb{E}(u(t))$ *is nonincreasing and* (1.1) *holds on a neighborhood of* φ.

Proof. It is sufficient to define $\mathbb{E}(u) := E(u)^\theta$ in case of Łojasiewicz inequality and $\mathbb{E}(u) := \Phi_\Theta(E(u))$ in case of Kurdyka-Łojasiewicz-Simon inequality. □

3.2. Convergence result by Chill, Haraux, Jendoubi and its corollaries.

Theorem 1 in [5] is another corollary of Theorem 2.4. If we replace Łojasiewicz inequality by the more general Kurdyka-Łojasiewicz-Simon inequality, then the theorem in [5] reads as follows.

Theorem 3.2 ([5, Theorem 1]). *Let $u \in C(\mathbb{R}_+, V) \cap C^1(\mathbb{R}_+, H)$ with V-precompact range and $\varphi \in \omega_V(u)$. Let $\rho > 0$, $c > 0$ and $E \in C^2(V, \mathbb{R})$ be such that $t \mapsto \mathbb{E}(u(t))$ is differentiable almost everywhere and*

$$-\frac{d}{dt}E(u(t)) \geq c\|E'(u(t))\|_*\|\dot{u}(t)\|_*$$

for almost every $t \in \mathbb{R}_+$ with $u(t) \in B_V(\varphi, \rho)$. Assume in addition that

$$\text{if } E(u(\cdot)) \text{ is constant for } t \geq t_0, \text{ then } u \text{ is constant for } t \geq t_0$$

and that E satisfies the Kurdyka-Lojasiewicz-Simon gradient inequality with a Kurdyka function Θ. Then $\lim_{t\to+\infty} \|u(t) - \varphi\|_V = 0$.

Proof. We can assume that $E(\varphi) = 0$. If $E(u(t)) = 0$ for some t_0, then u is constant for all $t > t_0$ and the assertion holds. Otherwise, $E(u(t)) > 0$ for all $t \in \mathbb{R}_+$. In this case, let us define $\mathbb{E}(u) := \Phi_\Theta(E(u))$. Then

$$-\frac{d}{dt}\mathbb{E}(u(t)) \geq \frac{1}{\Theta(E(u(t)))} \cdot c\|E'(u(t))\|_*\|\dot{u}(t)\|_* \geq c\|\dot{u}(t)\|_*.$$

So, assumptions of Theorem 2.4 hold and $\|u(t) - \varphi\|_V \to 0$. □

For many applications and corollaries of Theorem 3.2 see [5].

3.3. **Convergence result by Bárta, Chill, Fašangová.** In [2], Bárta, Chill and Fašangová proved a convergence theorem formulated on manifolds. If we reformulate it for $\mathbb{R}^N$, it becomes a corollary of Theorem 2.4.

Theorem 3.3 ([2, Theorem 3]). *Let $F \in C(\mathbb{R}^N, \mathbb{R}^N)$, $u : \mathbb{R}_+ \to \mathbb{R}^N$ be a global solution of the ordinary differential equation*

$$\dot{u}(t) + F(u(t)) = 0 \tag{3.1}$$

and let $E : \mathbb{R}^N \to \mathbb{R}$ be a continuously differentiable, strict Lyapunov function for (3.1). Assume that there exist a Kurdyka function Θ, $\varphi \in \omega(u)$ and a neighbourhood U of φ such that for every $v \in U$ we have $F(v) \neq 0$ and

$$\Theta(|E(v) - E(\varphi)|) \leq \langle E'(v), \frac{F(v)}{\|F(v)\|}\rangle. \tag{3.2}$$

Then u has finite length and, in particular, $\lim_{t\to+\infty} u(t) = \varphi$.

Proof. Let us recall that E is *a strict Lyapunov function* for (3.1), if $\langle E'(u), F(u)\rangle > 0$, whenever $u \in \mathbb{R}^N$, $F(u) \neq 0$. Since $E(u(\cdot))$ is nonincreasing and continuous, it has a limit which is equal to $E(\varphi)$. We can assume that $\mathbb{E}(\varphi) = 0$, so that $E(u(t)) \geq 0$ for all $t \in \mathbb{R}_+$. If $E(u(t_0)) = 0$ for some $t_0 \geq 0$, then $E(u(t)) = 0$ for every $t \geq t_0$, and therefore, since E is a strict Lyapunov function, the function u is constant for $t \geq t_0$. In this case, there remains nothing to prove.

Hence, we may assume that $E(u(t)) > 0$ for every $t \geq 0$ and define $\mathbb{E}(u) := \Phi_\Theta(E(u))$. Then

$$\begin{aligned}
-\frac{d}{dt}\mathbb{E}(u(t)) &= \frac{1}{\Theta(E(u(t))}\big(-\frac{d}{dt}E(u(t))\big) \\
&= \frac{1}{\Theta(E(u(t))}\langle E'(u(t)), F(u(t))\rangle \\
&\geq \|F(u(t))\| = \|\dot{u}(t))\|
\end{aligned}$$

in a neighborhood of φ. Hence the assumptions of Theorem 2.4 are satisfied and $\lim_{t\to\infty} u(t) = \varphi$. $\square$

4. Applications to second order equations

4.1. Second order ODE with weak nonlinear damping. The equation

$$\ddot{u}(t) + |\dot{u}(t)|^\alpha \dot{u}(t) + \nabla E((u(t))) = 0$$

with $\alpha > 0$ was studied by Chergui in [3] and the convergence result was then extended to more general dampings

$$\ddot{u}(t) + G(u(t), \dot{u}(t)) + \nabla E((u(t))) = 0 \tag{4.1}$$

by Bárta, Chill and Fašangová [2], where $G \in C^2(\mathbb{R}^N \times \mathbb{R}^N)$ and for every u, $v \in \mathbb{R}^N$ it holds that

$$\begin{gathered} \langle G(u,v), v\rangle \geq g(\|v\|)\,\|v\|^2, \\ \|G(u,v)\| \leq cg(\|v\|)\,\|v\|, \\ \|\nabla G(u,v)\| \leq c\,g(\|v\|), \end{gathered} \tag{4.2}$$

where $c \geq 0$ is a constant and $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a nonnegative, concave, nondecreasing function, $g(s) > 0$ for $s > 0$.

Under these assumptions we have

$$\langle G(u,v), v\rangle \geq g(\|v\|)\,\|v\|^2 = g(\|v\|_*)\,\|v\|_*^2 =: \tilde{g}(\|v\|_*),$$

so assumptions of Theorem 2.8 hold with $\tilde{g}$. By Corollary 2.9, it is sufficient to prove that

$$\mathbb{E}((u,v)) := \Phi_\Theta\Big(\frac{1}{2}\|v\|^2 + E(u) + \varepsilon\langle G(u, \nabla E(u)), v\rangle\Big)$$

satisfies the key estimate (2.4), which needs some work (see [2] for details).

4.2. A semilinear wave equation with nonlinear damping. The following problem was studied by Chergui in [4]. Consider the equation

$$u_{tt} + |u_t|^\alpha u_t = \Delta u + f(x,u) \tag{4.3}$$

in $\mathbb{R}_+ \times \Omega$ with Dirichlet boundary conditions and initial values

$$u(0,\cdot) = u_0 \in H_0^1(\Omega), \quad u_t(0,\cdot) = u_1 \in L^2(\Omega).$$

Function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies

- If $N = 1$: f, $\partial_2 f$ are bounded in $\Omega \times [-r, r]$ for all $r > 0$,
- If $N \geq 2$: $f(\cdot, 0) \in L^\infty(\Omega)$ and $|\partial_2 f(x,s)| \leq c(1 + |s|^\gamma)$ on $\Omega \times \mathbb{R}$,

where $c \geq 0$, $\gamma \geq 0$ and $(N-2)\gamma < 2$.

Then the main part of the proof of [4, Theorem 1.4] can be interpreted as proving that (for appropriate α and θ and small $\varepsilon > 0$)

$$\begin{aligned} &\mathbb{E}((u(t), \dot{u}(t))) \\ &:= \Big(\frac{1}{2}\|\dot{u}(t)\|_2^2 + E(u(t)) - \varepsilon\|\dot{u}(t)\|_*^\alpha\langle \Delta u(t) + f(x, u(t)), \dot{u}(t)\rangle_*\Big)^{\theta-(1-\theta)\alpha} \end{aligned}$$

satisfies estimate (2.4), where

$$E(u) := \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega F(x,u)\,\mathrm{d}x, \quad F(x,u) := \int_0^u f(x,s)\,\mathrm{d}s. \tag{4.4}$$

Let us mention that Corollary 2.9 can be applied in this case, if we consider classical solutions (the result in [4] refers to weak solutions).

4.3. **Abstract wave equation with linear damping.** The following abstract second-order equation is studied in [5]. We have $V \hookrightarrow H \hookrightarrow V'$, $\gamma \neq 0$, $E \in C^2(V)$, $M = E'$ and consider the equation

$$u_{tt} + \gamma u_t + M(u) = 0. \tag{4.5}$$

Let us introduce the duality mapping $K : V' \to V$ given by $\langle u, v\rangle_* = \langle u, Kv\rangle_{V',V} = \langle u, Kv\rangle$ for $u \in H$, $v \in V'$.

Theorem 4.1 ([5, Corollary 16])**.** *Assume that $\gamma > 0$ and*

(1) *for every $v \in V$, the operator $KM'(v)$ extends to a bounded operator on H and $\sup_v \|KM'(v)\|_{L(H)}$ is finite when v ranges over a compact subset of V, and*
(2) *$u \in C^1(\mathbb{R}_+, V) \cap C^2(\mathbb{R}_+, H)$ is a global solution to (4.5), $(u, \dot u)$ has precompact range in $V \times H$ and there exist $\varphi \in \omega(u)$, $C > 0$, $\rho > 0$ and a sublinear Kurdyka function Θ, such that E satisfies Kurdyka-Łojasiewicz-Simon gradient inequality in $B_V(\varphi, \rho)$.*

Then $\lim_{t\to+\infty} \|u(t) - \varphi\|_V = 0$.

Since

$$\langle \gamma \dot u, \dot u\rangle \geq \gamma c \|\dot u\|_*^2 =: g(\|\dot u\|_*),$$

the assumptions of Theorem 2.8 are satisfied and $\|\dot u\| \to 0$. It is not difficult to show that function $\mathbb{E}(u, \dot u) := \Phi_\Theta(\Psi(u, \dot u))$ satisfies the key estimate (2.4), where

$$\Psi(u, \dot u) := \frac{1}{2}\|\dot u\|^2 + E(u) + \varepsilon\langle M(u), \dot u\rangle_*$$

and $\varepsilon > 0$ is small enough. Then Corollary 2.9 proves the assertion.

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