Electronic Journal of Differential Equations, Vol. 2014 (2014), No. 86, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE OF SOLUTIONS TO NONLOCAL KIRCHHOFF EQUATIONS OF ELLIPTIC TYPE VIA GENUS THEORY

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Abstract. In this article, we study the existence and multiplicity of solutions to the nonlocal Kirchhoff fractional equation

$$\Big(a+b\int_{\mathbb{R}^{2N}}|u(x)-u(y)|^2K(x-y)\,dx\,dy\Big)(-\Delta)^s u-\lambda u=f(x,u(x))\quad\text{in }\Omega,$$
$$u=0\quad\text{in }\mathbb{R}^N\setminus\Omega,$$

where $a,b>0$ are constants, $(-\Delta)^s$ is the fractional Laplace operator, $s\in(0,1)$ is a fixed real number, λ is a real parameter and Ω is an open bounded subset of $\mathbb{R}^N$, $N>2s$, with Lipschitz boundary, $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a continuous function. The proofs rely essentially on the genus properties in critical point theory.

1. Introduction

Recently, a great attention has been focused on the study of fractional and non-local operators of elliptic type, both for the pure mathematical research and in view of concrete real-world applications. This type of operators arises in a quite natural way in many different contexts, such as, among the others, the thin obstacle problem, optimization, finance, phase transitions, stratified materials, conservation laws. The literature on non-local operators and on their applications is, therefore, very interesting and, up to now, quite large, we refer the interested readers to [7, 8, 9, 11, 15, 16, 17, 21, 22, 25].

In this article, we are concerned with a class of nonlocal Kirchhoff fractional equations of the type

$$\begin{gathered}-\Big(a+b\int_{\mathbb{R}^{2N}}|u(x)-u(y)|^2K(x-y)\,dx\,dy\Big)\mathcal{L}_K u-\lambda u=f(x,u(x))\quad\text{in }\Omega,\\ u=0\quad\text{in }\mathbb{R}^N\setminus\Omega,\end{gathered}\tag{1.1}$$

e1.1

where Ω is an open bounded subset of $\mathbb{R}^N$ with Lipschitz boundary, $N>2s$ with $s\in(0,1)$, $a,b>0$ are constants, $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a continuous function, λ is a

2000 *Mathematics Subject Classification.* 34B27, 35J60, 35B05.
Key words and phrases. Kirchhoff nonlocal operators; fractional differential equations; genus properties; critical point theory.

Submitted December 15, 2013. Published April 2, 2014.

parameter and

$$\mathcal{L}_K u(x) := \int_{\mathbb{R}^N} \Big(u(x+y) + u(x-y) - 2u(x) \Big) K(y)\, dy, \quad x \in \mathbb{R}^N, \tag{1.2}$$ e1.2

where $K : \mathbb{R}^N \setminus \{0\} \to (0, +\infty)$ is a kernel function satisfying the following properties:

- (K1) $mK \in L^1(\mathbb{R}^N)$, where $m(x) = \min\{|x|^2, 1\}$;
- (K2) there exists $\theta > 0$ such that $K(x) \geq \theta |x|^{-(N+2s)}$ for any $x \in \mathbb{R}^N \setminus \{0\}$;
- (K3) $K(x) = K(-x)$ for any $x \in \mathbb{R}^N \setminus \{0\}$.

The homogeneous Dirichlet datum in (1.1) is given in $\mathbb{R}^N \setminus \Omega$ and not simply on the boundary $\partial\Omega$, consistent with the nonlocal character of the kernel operator $\mathcal{L}_K$.

A typical model for K is given by the singular kernel $K(x) = |x|^{-(N+2s)}$ which gives rise to the fractional Laplace operator $-(-\Delta)^s$ where $s \in (0,1)$ $(N > 2s)$ is fixed, which, up to normalization factors, may be defined as

$$-(-\Delta)^s u(x) := \int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2s}}\, dy, \quad x \in \mathbb{R}^N. \tag{1.3}$$ e1.3

The problem (1.1) in the model case $\mathcal{L}_K = -(-\Delta)^s$ becomes

$$\begin{gathered} \Big(a + b \int_{\mathbb{R}^N \times \mathbb{R}^N} |u(x) - u(y)|^2 |x-y|^{-(N+2s)}\, dx\, dy \Big) (-\Delta)^s u - \lambda u = f(x, u(x)), \\ u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{gathered} \tag{1.4}$$ e1.4

which is related to Kirchhoff type problems. These problems model several physical and biological systems, where u describes a process which depends on the average of itself, such as the population density, see [3, 10]. Problem (1.4) with the p-Laplacian operator $-\Delta_p u$ has been studied in many papers, see [1, 2, 4, 5, 6, 13, 19, 24]. Motivated by [2, 17, 21, 22, 23], in this paper, we study the existence and multiplicity of solutions for Kirchhoff type problem (1.1) driven by the nonlocal operator $\mathcal{L}_K$.

Before proving the main results, some preliminary material on function spaces and norms is needed. In the following, we briefly recall the definition of the functional space X_0, firstly introduce in [21], and we give some notations. We denote $\mathrm{Q} = \mathbb{R}^{2N} \setminus \mathcal{O}$, where $\mathcal{O} = \mathbb{R}^N \setminus \Omega \times \mathbb{R}^N \setminus \Omega$. We denote the set X by

$$X = \big\{ u : \mathbb{R}^N \to \mathbb{R} : u|_\Omega \in L^2(\Omega),\ (u(x) - u(y))\sqrt{K(x-y)} \in L^2(\mathbb{R}^{2N} \setminus \mathcal{O}) \big\},$$

where $u|_\Omega$ represents the restriction to Ω of function $u(x)$. Also, we denote by X_0 the following linear subspace of X

$$X_0 = \{ g \in X :\ g = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \}.$$

We know that X and X_0 are nonempty, since $C_0^2(\Omega) \subseteq X_0$ by Lemma 11 of [21]. Moreover, the linear space X is endowed with the norm defined as

$$\|u\|_X := \|u\|_{L^2(\Omega)} + \Big(\int_{\mathrm{Q}} |u(x) - u(y)|^2 K(x-y)\, dx\, dy \Big)^{1/2}.$$

It is easy seen that $\| \cdot \|_X$ is a norm on X (see, for instance, [22] for a proof). By Lemmas 6 and 7 of [22], in the sequel we can take the function

$$X_0 \ni v \mapsto \|v\|_{X_0} = \Big(\int_{\mathrm{Q}} |v(x) - v(y)|^2 K(x-y)\, dx\, dy \Big)^{1/2} \tag{1.5}$$ e1.5

as norm on X_0. Also $(X_0, \|\cdot\|_{X_0})$ is a Hilbert space, with scalar product

$$\langle u, v\rangle_{X_0} := \int_Q (u(x)-u(y))(v(x)-v(y))K(x-y)\,dxdy. \tag{1.6}$$ e1.6

Note that in (1.5) the integral can be extended to all $\mathbb{R}^N \times \mathbb{R}^N$, since $v \in X_0$ and so $v = 0$ a.e. in $\mathbb{R}^N \setminus \Omega$.

In what follows, we denote by λ_1 the first eigenvalue of the operator $\mathcal{L}_K$ with homogeneous Dirichlet boundary data, namely the first eigenvalue of the problem

$$\begin{gathered}\mathcal{L}_K u = \lambda u, \quad \text{in } \Omega,\\ u = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega.\end{gathered}$$

We refer to [23, Proposition 9 and Appendix A], for the existence and the basic properties of this eigenvalue, where a spectral theory for general integro-differential nonlocal operators was developed.

When $\lambda < \lambda_1$ we can take as a norm on X_0 the function

$$X_0 \ni v \mapsto \|v\|_{X_0,\lambda} = \Big(\int_Q |v(x)-v(y)|^2 K(x-y)\,dx\,dy - \lambda\int_\Omega |v(x)|^2\,dx\Big)^{1/2}, \tag{1.7}$$ e1.7

since for any $v \in X_0$ it holds true (for this see [23, Lemma 10])

$$m_\lambda\|v\|_{X_0} \le \|v\|_{X_0,\lambda} \le M_\lambda\|v\|_{X_0}, \tag{1.8}$$ e1.8

where

$$m_\lambda := \min\Big\{\sqrt{\frac{\lambda_1-\lambda}{\lambda_1}}, 1\Big\}, \quad M_\lambda := \max\Big\{\sqrt{\frac{\lambda_1-\lambda}{\lambda_1}}, 1\Big\}.$$

Let $H^s(\mathbb{R}^N)$ be the usual fractional Sobolev space endowed with the norm (the so-called Gagliardo norm)

$$\|u\|_{H^s(\mathbb{R}^N)} = \|u\|_{L^2(\mathbb{R}^N)} + \Big(\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\Big)^{1/2}. \tag{1.9}$$ e1.9

Also, we recall the embedding properties of X_0 into the usual Lebesgue spaces (see [22, Lemma 8]). The embedding $j : X_0 \hookrightarrow L^v(\mathbb{R}^N)$ is continuous for any $v \in [1, 2^*]$ ($2^* = \frac{2N}{N-2s}$), while it is compact whenever $v \in [1, 2^*)$. Hence, for any $v \in [1, 2^*]$ there exists a positive constant c_v such that

$$\|v\|_{L^v(\mathbb{R}^N)} \le c_v\|v\|_{X_0} \le c_v m_\lambda^{-1}\|v\|_{X_0,\lambda}, \tag{1.10}$$ e1.10

for any $v \in X_0$.

We are now in the position to state the notation of solution and to state the main results of this article.

def1.1 **Definition 1.1.** We say that $u \in X_0$ is a weak solution of problem (1.1), if it satisfies

$$\begin{aligned}&\Big(a + b\int_Q |u(x)-u(y)|^2 K(x-y)\,dx\,dy\Big)\\ &\int_Q (u(x)-u(y))(v(x)-v(y))K(x-y)\,dx\,dy - \lambda\int_\Omega u(x)v(x)\,dx\\ &-\int_\Omega f(x,u(x))v(x)\,dx = 0, \quad \forall v \in X_0.\end{aligned}$$

the1.2 **Theorem 1.2.** *Assume that f satisfies the following conditions:*

(F1) $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$ *and there exist constants* $1 < \gamma_1 < \gamma_2 < \cdots < \gamma_m < 2$ *and functions* $a_i \in L^{\frac{2}{2-\gamma_i}}(\Omega, [0, +\infty))$, $i = 1, 2, \ldots, m$ *such that*

$$|f(x,z)| \le \sum_{i=1}^{m} a_i(x)|z|^{\gamma_i - 1}, \quad \forall (x,z) \in \Omega \times \mathbb{R}.$$

(F2) *There exist and open set* $\Omega_0 \subset \Omega$ *and three constants* $\delta > 0$, $\gamma_0 \in (1,2)$ *and* $\eta > 0$ *such that*

$$F(x,z) \ge \eta |z|^{\gamma_0}, \quad \forall (x,z) \in \Omega_0 \times [-\delta, \delta],$$

where $F(x,z) := \int_0^z f(x,s)\,ds$, $x \in \Omega$, $z \in \mathbb{R}$.

Then for any $\lambda < \lambda_1 . \min\{a, 1\}$, *problem* (1.1) *has at least one nontrivial solutions.*

the1.3 **Theorem 1.3.** *Assume that* f *and* F *satisfy the conditions* (F1), (F2) *and*

(F3) $F(x, -z) = F(x, z)$ *for all* $(x,z) \in \Omega \times \mathbb{R}$.

Then for any $\lambda < \lambda_1 . \min\{a, 1\}$, *problem* (1.1) *has infinitely many nontrivial solutions.*

2. Proofs of main results

Our idea is to obtain the existence and multiplicity of solutions for problem (1.1) by using critical point theory. Consider the functional $J : X_0 \to \mathbb{R}$ defined by

$$\begin{aligned} J(u) = \frac{a}{2}\int_Q |u(x)-u(y)|^2 K(x-y)\,dx\,dy + \frac{b}{4}\Big(\int_Q |u(x)-u(y)|^2 K(x-y)\,dx\,dy\Big)^2 \\ - \frac{\lambda}{2}\int_\Omega |u(x)|^2\,dx - \int_\Omega F(x,u(x))\,dx \end{aligned} \tag{2.1}$$ e2.1

and set

$$\Psi(u) = \int_\Omega F(x,u(x))\,dx.$$

Let us recall the following definitions and results which are used to prove our main results, see for instance [14, 18].

def2.1 **Definition 2.1.** *We say that* J *satisfies the Palais-Smale (PS) condition if any sequence* $(u_n) \in X$ *for which* $J(u_n)$ *is bounded and* $J'(u_n) \to 0$ *as* $n \to \infty$ *possesses a convergent subsequence.*

lem2.2 **Lemma 2.2** ([14])**.** *Let* X *be a real Banach space and* $J \in C^1(X, \mathbb{R})$ *satisfy the* (PS) *condition. If* J *is bounded from below, then* $c = \inf_X J$ *is a critical value of* J.

Let $\mathcal{X}$ be a Banach space, $g \in C^1(\mathcal{X}, \mathbb{R})$ and $c \in \mathbb{R}$. We set

$$\Sigma = \{A \subset \mathcal{X} \setminus \{0\} : \ A \text{ is closed in } X \text{ and symmetric with respect to } 0)\},$$
$$K_c = \{x \in \mathcal{X} : g(x) = c,\ g'(x) = 0\},$$
$$g^c = \{x \in \mathcal{X} : g(x) \le c\}.$$

def2.3 **Definition 2.3** ([14])**.** *For* $A \in \Sigma$, *we say genus of* A *is* j *(denoted by* $\gamma(A) = j$*) if there is an odd map* $\psi \in C(A, \mathbb{R}^j \setminus \{0\})$, *and* j *is the smallest integer with this property.*

lem2.4 **Lemma 2.4** ([18]). *Let g be an even C^1 functional on $\mathcal{X}$ which satisfies the Palais-Smale condition. If $j \in \mathbb{N}$, $j > 0$, let*

$$\Sigma_j = \{A \in \Sigma : \ \gamma(A) \geq j\}, c_j = \inf_{A\in\Sigma_j} \sup_{u\in A} g(u).$$

(i) *If $\Sigma_j \neq \emptyset$ and $c_j \in \mathbb{R}$, then c_j is a critical value of g.*
(ii) *If there exists $r \in \mathbb{N}$ such that $c_j = c_{j+1} = \cdots = c_{j+r} = c \in \mathbb{R}$ and $c \neq g(0)$, then $\gamma(K_c) \geq r + 1$.*

Remark 2.5. From [18, Remark 7.3], we know that if $K_c \subset \Sigma$ and $\gamma(K_c) > 1$, then K_c contains infinitely many distinct points, i.e., J has infinitely many distinct critical points in $\mathcal{X}$.

lem2.5 **Lemma 2.6.** *Assume that* (F1) *and* (F2) *hold. Then the functional $J : X_0 \to \mathbb{R}$ is well-defined and is of class $C^1(X_0, \mathbb{R})$ and*

$$\begin{aligned} J'(u)(v) = \Big(a + b \int_Q |u(x) - u(y)|^2 K(x-y)\,dx\,dy\Big) \\ \times \int_Q (u(x) - u(y))(v(x) - v(y))K(x-y)\,dx\,dy \\ - \lambda \int_\Omega u(x)v(x)\,dx - \Psi'(u)(v), \quad \text{for all } v \in X_0, \end{aligned} \tag{2.2}$$

e2.2

where $\Psi'(u)(v) = \int_\Omega f(x, u(x))v(x)\,dx$. Moreover, the critical points of J are the solutions of problem (1.1).

Proof. For any $u \in X_0$, by (F1) and the Hölder inequality, one have

$$\begin{aligned} \int_\Omega |F(x,u)|\,dx &\leq \sum_{i=1}^m \frac{1}{\gamma_i} \int_\Omega a_i(x)|u|^{\gamma_i}\,dx \\ &\leq \sum_{i=1}^m \frac{1}{\gamma_i} \Big(\int_\Omega |a_i(x)|^{\frac{2}{2-\gamma_i}}\,dx\Big)^{\frac{2-\gamma_i}{2}} \Big(\int_\Omega |u|^2\,dx\Big)^{\frac{\gamma_i}{2}} \\ &\leq C_1 \sum_{i=1}^m \frac{1}{\gamma_i} \|a_i\|_{\frac{2-\gamma_i}{2}} \|u\|_{X_0}^{\gamma_i}, \end{aligned} \tag{2.3}$$

e2.3

and so J is defined by (2.1) is well-defined on X_0 by (F1).

Next, we prove that (2.2) holds. For any $u, v \in X_0$, any function $\theta : \Omega \to (0,1)$ and any number $h \in (0,1)$, by (F1) and the Hölder inequality, we have

$$\begin{aligned} &\int_\Omega \max_{h\in(0,1)} |f(x, u(x) + \theta(x)hv(x))v(x)|\,dx \\ &\leq \int_\Omega \max_{h\in(0,1)} |f(x, u(x) + \theta(x)hv(x))||v(x)|\,dx \\ &\leq \sum_{i=1}^m \int_\Omega a_i(x)|u(x) + \theta(x)v(x)|^{\gamma_i - 1}|v(x)|\,dx \\ &\leq \sum_{i=1}^m \int_\Omega a_i(x)(|u(x)|^{\gamma_i-1} + |v(x)|^{\gamma_i-1})|v(x)|\,dx \\ &\leq C_2 \sum_{i=1}^m \|a_i\|_{\frac{2-\gamma_i}{2}} (\|u\|_{X_0}^{\gamma_i-1} + \|v\|_{X_0}^{\gamma_i-1})\|v\|_{X_0} < +\infty. \end{aligned} \tag{2.4}$$

e2.4

Then by (2.4) and the Lebesgue dominated convergence theorem, we have

$$
\begin{aligned}
\Psi'(u)(v) &= \lim_{h\to 0^+} \frac{\Psi(u+hv)-\Psi(u)}{h} \\
&= \lim_{h\to 0^+} \frac{1}{h}\int_\Omega [F(x,u(x)+hv(x))-F(x,u(x))]\,dx \\
&= \lim_{h\to 0^+} \int_\Omega f(x,u(x)+\theta(x)v(x))v(x)\,dx \\
&= \int_\Omega f(x,u(x))v(x)\,dx.
\end{aligned} \tag{2.5}
$$

e2.5

By (2.5), relation (2.2) holds. Furthermore, by a standard argument, it is easy to show that the critical points of the functional J in X_0 are the solutions of problem (1.1).

Let us prove now that J' is continuous. It is sufficient to verify that Ψ' is continuous. Let $u_n \to u$ in X_0, then $u_n \to u$ in $L^2(\Omega)$ and

$$
\begin{gathered}
u_n \to u, \quad \text{strongly in } L^2(\Omega), \\
u_n \to u, \quad \text{a.e. in } \Omega.
\end{gathered} \tag{2.6}
$$

e2.6

Then there exists $h \in L^2(\Omega)$ such that $|u_n(x)| \le h(x)$ a.e. $x \in \Omega$ and for any $n \in \mathbb{N}$.

By (F1), we have

$$
\begin{aligned}
&|f(x,u_n(x))-f(x,u(x))|^2 \\
&\le 2(|f(x,u_n(x))|^2+|f(x,u(x))|^2) \\
&\le C_2\sum_{i=1}^m |a_i(x)|^2\left(|u_n(x)|^{2(\gamma_i-1)}+|u(x)|^{2(\gamma_i-1)}\right) \\
&\le C_2\sum_{i=1}^m |a_i(x)|^2\left(|h(x)|^{2(\gamma_i-1)}+|u(x)|^{2(\gamma_i-1)}\right) \\
&:= g(x), \quad \forall n\in\mathbb{N}, \quad x\in\Omega
\end{aligned} \tag{2.7}
$$

e2.9

and

$$
\begin{aligned}
\int_\Omega g(x)\,dx &= C_2\sum_{i=1}^m \int_\Omega |a_i(x)|^2\left(|h(x)|^{2(\gamma_i-1)}+|u(x)|^{2(\gamma_i-1)}\right)\,dx \\
&\le C_2\sum_{i=1}^m \|a_i\|^2_{\frac{2-\gamma_i}{2}}\left(\|h\|_{L^2}^{2(\gamma_i-1)}+\|u\|_{L^2}^{2(\gamma_i-1}\right) < +\infty.
\end{aligned} \tag{2.8}
$$

e2.10

By (2.6), (2.7), (2.8), and the Lebesgue dominated convergence theorem, we have

$$
\lim_{n\to\infty}\int_\Omega |f(x,u_n(x))-f(x,u(x))|^2\,dx = 0. \tag{2.9}
$$

e2.7

From (1.10), (2.2), (F1) and the Hölder inequality, we have

$$
\begin{aligned}
|(\Psi'(u_n)-\Psi'(u),v)| &= \left|\int_\Omega [f(x,u_n(x))-f(x,u(x))]v(x)\,dx\right| \\
&\le \int_\Omega |f(x,u_n(x))-f(x,u(x))\|v(x)|\,dx \\
&\le \Big(\int_\Omega |f(x,u_n(x))-f(x,u(x))|^2\,dx\Big)^{1/2}\|v\|_{L^2}
\end{aligned}
$$

$$\le C_3\Big(\int_\Omega |f(x,u_n(x)) - f(x,u(x))|^2\,dx\Big)^{1/2}\|v\|_{X_0},$$

which converges to 0 as $n\to\infty$. This implies that Ψ' is continuous and the proof of Lemma 2.6 is complete. $\square$

Proof of Theorem 1.2. In view of Lemma 2.6, $J\in C^1(X_0,\mathbb{R})$. In what follows, we first show that J is bounded from below. Since $\lambda<\lambda_1.\min\{a,1\}$ we have $a-1+m_\lambda^2>0$, where m_λ is defined by (1.8). By (F1), (1.5), (1.7), (1.8) and the Hölder inequality, we have

$$\begin{aligned}
&J(u)\\
&= \frac{a}{2}\int_Q |u(x)-u(y)|^2K(x-y)\,dx\,dy + \frac{b}{4}\Big(\int_Q |u(x)-u(y)|^2K(x-y)\,dx\,dy\Big)^2\\
&\quad - \frac{\lambda}{2}\int_\Omega |u(x)|^2\,dx - \int_\Omega F(x,u(x))\,dx\\
&\ge \frac12(a-1+m_\lambda^2)\|u\|_{X_0}^2 - \sum_{i=1}^m \frac{1}{\gamma_i}\int_\Omega a_i(x)|u|^{\gamma_i}\,dx\\
&\ge \frac12(a-1+m_\lambda^2)\|u\|_{X_0}^2 - C_1\sum_{i=1}^m \frac{1}{\gamma_i}\|a_i\|_{\frac{2-\gamma_i}{2}}\|u\|_{X_0}^{\gamma_i}.
\end{aligned}\tag{2.10}$$

e2.11

As $\gamma_i\in(1,2)$, $i=1,2,\dots,m$, it follows from (2.10) that $J(u)\to+\infty$ as $\|u\|_{X_0}\to+\infty$ and J is bounded from below.

Next, we prove that J satisfies the (PS)-condition. Assume that $\{u_n\}\subset X_0$ is a sequence such that $\{J(u_n)\}$ is bounded and $J'(u_n)\to 0$ as $n\to\infty$. Since $\{u_n\}$ is a (PS)-sequence and using the definition of J, there exists a constant $C_4>0$ such that

$$\|u_n\|_{X_0}\le C_4,\quad \forall n\in\mathbb{N}.\tag{2.11}$$

e2.12

So passing to a subsequence it necessary, it can be assumed that $\{u_n\}$ converges weakly to u_0 in X_0 and thus $\{u_n\}$ converges strongly to u_0 in $L^2(\Omega)$. By (2.11) and (F1), we have

$$\begin{aligned}
&\Big|\int_\Omega (f(x,u_n(x)) - f(x,u(x)))(u_n(x)-u_0(x))\,dx\Big|\\
&\le \int_\Omega |f(x,u_n(x)) - f(x,u(x))||u_n(x)-u_0(x)|\,dx\\
&\le \Big(\int_\Omega |f(x,u_n(x)) - f(x,u_0(x))|^2\,dx\Big)^{1/2}\Big(\int_\Omega |u_n(x)-u_0(x)|^2\,dx\Big)^{1/2}\\
&\le \Big(\int_\Omega 2(|f(x,u_n(x))|^2 + |f(x,u_0(x))|^2)\,dx\Big)^{1/2}\Big(\int_\Omega |u_n(x)-u_0(x)|^2\,dx\Big)^{1/2}\\
&\le C_5\Big(\sum_{i=1}^m \|a_i\|^2_{\frac{2}{2-\gamma_i}}(\|u_n\|_{X_0}^{2(\gamma_i-1)} + \|u_0\|_{X_0}^{2(\gamma_i-1)})\,dx\Big)^{1/2}\|u_n-u_0\|_{L^2(\Omega)},
\end{aligned}\tag{2.12}$$

e2.13

which approaches 0 as $n\to\infty$.

Since $\lambda<\lambda_1\min\{a,1\}$, by (1.7) and (1.8), we have

$$(J'(u_n)-J'(u_0))(u_n-u_0)$$

$$
\begin{aligned}
&= \Big(a + b\int_Q |u_n(x) - u_n(y)|^2 K(x-y)\,dx\,dy\Big)\\
&\quad \times \int_Q (u_n(x) - u_n(y))\Big((u_n(x) - u_0(x)) - (u_n(y) - u_0(y))\Big) K(x-y)\,dx\,dy\\
&\quad - \Big(a + b\int_Q |u_0(x) - u_0(y)|^2 K(x-y)\,dx\,dy\Big)\\
&\quad \times \int_Q (u_0(x) - u_0(y))\Big((u_n(x) - u_0(x)) - (u_n(y) - u_0(y))\Big) K(x-y)\,dx\,dy\\
&\quad - \lambda\int_\Omega |u_n(x) - u_0(x)|^2\,dx - \int_\Omega [f(x,u_n(x)) - f(x,u_0(x))](u_n - u_0)\,dx\\
&= \Big(a + b\int_Q |u_n(x) - u_n(y)|^2 K(x-y)\,dx\,dy\Big)\\
&\quad \times \int_Q |(u_n(x) - u_0(x)) - (u_n(y) - u_0(y)|^2 K(x-y)\,dx\,dy\\
&\quad - b\Big(\int_Q |u_0(x) - u_0(y)|^2 K(x-y)\,dx\,dy - \int_Q |u_n(x) - u_n(y)|^2 K(x-y)\,dx\,dy\Big)\\
&\quad \times \int_Q (u_0(x) - u_0(y))\Big((u_n(x) - u_0(x)) - (u_n(y) - u_0(y))\Big) K(x-y)\,dx\,dy\\
&\quad - \lambda\int_\Omega |u_n(x) - u_0(x)|^2\,dx - \int_\Omega [f(x,u_n(x)) - f(x,u_0(x))](u_n - u_0)\,dx\\
&\geq (a - 1 + m_\lambda^2)\|u_n - u_0\|_{X_0}^2 - \int_\Omega [f(x,u_n(x)) - f(x,u_0(x))](u_n - u_0)\,dx\\
&\quad - b\Big(\|u_0\|_{X_0}^2 - \|u_n\|_{X_0}^2\Big)\int_Q (u_0(x) - u_0(y))\\
&\quad \times \Big((u_n(x) - u_0(x)) - (u_n(y) - u_0(y))\Big) K(x-y)\,dx\,dy\,.
\end{aligned}
$$

Then

$$
\begin{aligned}
&(a - 1 + m_\lambda^2)\|u_n - u_0\|_{X_0}^2\\
&\leq (J'(u_n) - J'(u_0))(u_n - u_0) + \int_\Omega [f(x,u_n(x)) - f(x,u_0(x))](u_n - u_0)\,dx\\
&\quad + b\left(\|u_0\|_{X_0}^2 - \|u_n\|_{X_0}^2\right)\\
&\quad \times \int_Q (u_0(x) - u_0(y))\Big((u_n(x) - u_0(x)) - (u_n(y) - u_0(y))\Big) K(x-y)\,dx\,dy.
\end{aligned}
\tag{2.13}
$$

e2.14

As $\{u_n\}$ converges weakly u_0 in X_0, $\{\|u_n\|_{X_0}\}$ is bounded and we have

$$
\begin{aligned}
&\lim_{n\to\infty} b\left(\|u_0\|_{X_0}^2 - \|u_n\|_{X_0}^2\right)\int_Q (u_0(x) - u_0(y))\\
&\times \Big((u_n(x) - u_0(x)) - (u_n(y) - u_0(y))\Big) K(x-y)\,dx\,dy = 0.
\end{aligned}
\tag{2.14}
$$

e2.15

It follows from (2.12), (2.13) and (2.14) that $\{u_n\}$ converges strongly to u_0 in X_0 and the functional J satisfies the (PS) condition.

Then $d = \inf_{X_0} J(u)$ is a critical value of J, that is, there exists a critical point $u^* \in X_0$ such that $J(u^*) = d$.

Finally, we show that $u^* \neq 0$. Let $u_0 \in X_0 \cap C_0^\infty(\Omega_0)$ and $\|u_0\|_\infty \le 1$, where Ω_0 is given by (F2). By (F2), for $t \in (0,\delta)$, we have

$$
\begin{aligned}
J(tu_0) = &\frac{at^2}{2}\int_Q |u_0(x)-u_0(y)|^2 K(x-y)\,dx\,dy \\
&+ \frac{bt^4}{4}\Big(\int_Q |u_0(x)-u_0(y)|^2 K(x-y)\,dx\,dy\Big)^2 \\
&- \frac{\lambda t^2}{2}\int_\Omega |u_0(x)|^2\,dx - \int_\Omega F(x,tu_0(x))\,dx \qquad (2.15) \\
\le &\frac{t^2}{2}(a-1+M_\lambda^2)\|u_0\|_{X_0}^2 + \frac{bt^4}{4}\|u_0\|_{X_0}^4 - \int_{\Omega_0} F(x,tu_0(x))\,dx \\
\le &\frac{t^2}{2}(a-1+M_\lambda^2)\|u_0\|_{X_0}^2 + \frac{bt^4}{4}\|u_0\|_{X_0}^4 - \eta t^{\gamma_0}\int_{\Omega_0} |u_0(x)|^{\gamma_0}\,dx\,.
\end{aligned}
$$
$\boxed{\texttt{e2.16}}$

As $\gamma_0 \in (1,2)$, it follows from (2.15) that $J(tu_0) < 0$ for $t>0$ small enough. Hence, $J(u^*) = d < 0$ and therefore, u^* is a nontrivial critical point of J, and so u^* is a nontrivial solution of problem (1.1). □

Proof of Theorem 1.3. In view of Lemma 2.6, $J \in C^1(X_0,\mathbb{R})$ is bounded from below and satisfies the (PS) condition. It follows from (F3) that J is even and $J(0)=0$. In order to apply Lemma 2.4, we prove now that

$$
\text{for any } n \in \mathbb{N},\ \text{there exists } \epsilon > 0 \text{ such that } \gamma(J^{-\epsilon}) \ge n. \tag{2.16}
$$
$\boxed{\texttt{e2.17}}$

For any $n \in \mathbb{N}$, we take n disjoint open sets K_i such that

$$
\cup_{i=1}^n K_i \subset \Omega_0.
$$

For $i = 1,2,\dots,n$, let $u_i \in \big(X_0 \cap C_0^\infty(K_i)\big)\backslash\{0\}$ and $\|u_i\|_{X_0} = 1$, and

$$
E_n = \operatorname{span}\{u_1,u_2,\dots,u_n\}, \quad S_n = \{u \in E_n : \|u\|_{X_0} = 1\}.
$$

For each $u \in E_n$, there exist $\mu_i \in \mathbb{R}$, $i = 1,2,\dots,n$ such that

$$
u(x) = \sum_{i=1}^n \mu_i u_i(x) \quad \text{for } x \in \Omega. \tag{2.17}
$$
$\boxed{\texttt{e2.18}}$

Then

$$
\|u\|_{\gamma_0} = \Big(\int_\Omega |u(x)|^{\gamma_0}\,dx\Big)^{1/\gamma_0} = \Big(\sum_{i=1}^n |\mu_i|^{\gamma_0}\int_{K_i} |u_i(x)|^{\gamma_0}\,dx\Big)^{1/\gamma_0} \tag{2.18}
$$
$\boxed{\texttt{e2.19}}$

and

$$
\begin{aligned}
\|u\|_{X_0}^2 &= \int_Q |u(x)-u(y)|^2 K(x-y)\,dx\,dy \\
&= \sum_{i=1}^n \mu_i^2 \int_Q |u_i(x)-u_i(y)|^2 K(x-y)\,dx\,dy \qquad (2.19) \\
&= \sum_{i=1}^n \mu_i^2 \|u_i\|_{X_0}^2 = \sum_{i=1}^n \mu_i^2.
\end{aligned}
$$
$\boxed{\texttt{san}}$

As all norms of a finite dimensional normed space are equivalent, there is a constant $C_6 > 0$ such that

$$
C_6\|u\|_{X_0} \le \|u\|_{\gamma_0} \quad \text{for all } u \in E_n. \tag{2.20}
$$
$\boxed{\texttt{e2.20}}$

By (2.17), (2.18), (2.20), we have

$$
\begin{aligned}
J(tu) &= \frac{at^2}{2}\int_Q |u(x)-u(y)|^2 K(x-y)\,dx\,dy \\
&\quad + \frac{bt^4}{4}\Big(\int_Q |u(x)-u(y)|^2 K(x-y)\,dx\,dy\Big)^2 \\
&\quad - \frac{\lambda t^2}{2}\int_\Omega |u(x)|^2\,dx - \int_\Omega F(x,tu(x))\,dx \\
&\le \frac{t^2}{2}(a-1+M_\lambda^2)\|u\|_{X_0}^2 + \frac{bt^4}{4}\|u\|_{X_0}^4 - \sum_{i=1}^n \int_{K_i} F(x,t\mu_i u_i(x))\,dx \\
&\le \frac{t^2}{2}(a-1+M_\lambda^2)\|u\|_{X_0}^2 + \frac{bt^4}{4}\|u\|_{X_0}^4 - \eta t^{\gamma_0}\sum_{i=1}^n |\mu_i|^{\gamma_0}\int_{K_i} |u_i(x)|^{\gamma_0}\,dx \\
&\le \frac{t^2}{2}(a-1+M_\lambda^2)\|u\|_{X_0}^2 + \frac{bt^4}{4}\|u\|_{X_0}^4 - \eta t^{\gamma_0}\|u\|_{\gamma_0}^{\gamma_0} \\
&\le \frac{t^2}{2}(a-1+M_\lambda^2)\|u\|_{X_0}^2 + \frac{bt^4}{4}\|u\|_{X_0}^4 - \eta (C_6 t)^{\gamma_0}\|u\|_{X_0}^{\gamma_0} \\
&\le \frac{t^2}{2}(a-1+M_\lambda^2)\|u\|_{X_0}^2 + \frac{bt^4}{4}\|u\|_{X_0}^4 - \eta (C_6 t)^{\gamma_0} \\
&= \frac{t^2}{2}(a-1+M_\lambda^2) + \frac{bt^4}{4} - \eta (C_6 t)^{\gamma_0}
\end{aligned}
\tag{2.21}
$$

e2.21

for all $u \in S_n$ and and sufficient small $t > 0$. In this case (F2) is applicable, since u is continuous on $\overline{\Omega}_0$ and so $|t\mu_i u_i(x)| \le \delta$, $\forall\, x \in \Omega_0$, $i = 1, 2, \dots, n$ can be true for sufficiently small t. Then, there exist $\epsilon > 0$ and $\sigma > 0$ such that

$$
J(\sigma u) < -\epsilon \quad \text{for } u \in S_n. \tag{2.22}
$$

e2.22

Let

$$
S_n^\sigma = \{\sigma u :\ u \in S_n\}, \quad \Lambda = \big\{(\mu_1, \mu_2, \dots, \mu_n) \in \mathbb{R}^n :\ \sum_{i=1}^n \mu_i^2 < \sigma^2\big\}.
$$

Then it follows from (2.22) that

$$
J(u) < -\epsilon \quad \text{for all } u \in S_n^\sigma,
$$

which, together with the fact that $J \in C^1(X_0, \mathbb{R})$ and is even, implies that

$$
S_n^\sigma \subset J^{-\epsilon} \in \Sigma. \tag{2.23}
$$

e2.23

On the other hand, it follows from (2.17) and (2.19), that

$$
S_n^\sigma = \big\{\sum_{i=1}^n \mu_i u_i :\ \sum_{i=1}^n \mu_i^2 = \sigma^2\big\}.
$$

So, we define a map $\psi : S_n^\sigma \to \partial\Lambda$ as follows:

$$
\psi(u) = (\mu_1, \mu_2, \dots, \mu_n), \qquad \forall\, u \in S_n^\sigma.
$$

It is easy to verify that $\psi : S_n^\sigma \to \partial\Lambda$ is an odd homeomorphic map. By Proposition 7.7 in [18], we get $\gamma(S_n^\sigma) = n$ and so by some properties of the genus (see 3° of [18, Proposition 7.5]), we have

$$
\gamma(J^{-\epsilon}) \ge \gamma(S_n^\sigma) = n, \tag{2.24}
$$

e2.24

so the proof of (2.16) follows. Set

$$c_n = \inf_{A\in\Sigma_n} \sup_{u\in A} J(u).$$

It follows from (2.24) and the fact that J is bounded from below on X_0 that $-\infty < c_n \le -\epsilon < 0$, that is, for any $n \in \mathbb{N}$, c_n is a real negative number. By Lemma 2.4, the functional J has infinitely many nontrivial critical points, and so problem (1.1) possesses infinitely many nontrivial solutions. □

Acknowledgments. The authors would like to thank the anonymous referees for their suggestions and helpful comments which improved the presentation of the original manuscript.

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