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EXISTENCE AND UPPER SEMICONTINUITY OF RANDOM ATTRACTORS FOR STOCHASTIC p-LAPLACIAN EQUATIONS ON UNBOUNDED DOMAINS

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Abstract. The existence of a pullback attractor is established for a stochastic p-Laplacian equation on $\mathbb{R}^n$. Furthermore, the limiting behavior of random attractors of the random dynamical systems as stochastic perturbations approach zero is studied and the upper semicontinuity is proved.

1. Introduction

It is known that p-Laplacian equation is always used to model a variety of physical phenomena. In this paper, we investigate the asymptotic behavior of solutions to the following stochastic p-Laplacian equation with multiplicative noise defined on the entire space $\mathbb{R}^n$:

$$du + (-\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda u)dt = (f(x,u) + g(x))dt + \varepsilon u \circ dW(t), \qquad (1.1)$$

where $\varepsilon > 0$ is a small positive parameter, $\lambda > 0$, $p \geq 2$ are fixed constants. g is a given function defined on $\mathbb{R}^n$, f is a smooth nonlinear function satisfying certain conditions, and W is a two-sided real-valued Wiener processes on a probability space which will be specified later.

The long-term behavior of random systems is captured by a pullback random attractor, which was introduced by [8, 9] as an extension of the attractors theory of deterministic systems in [2, 11, 20, 22, 23]. In the case of bounded domains, the existence of random attractors for stochastic PDEs has been studied extensively by many authors (see [1, 8, 9, 17, 18, 19, 26, 29, 32]) and the reference therein. Since sobolev embeddings are not compact on unbounded domains, it is more difficult to discuss the existence of random attractors for PDEs defined on unbounded domains. Nevertheless, the existence of such attractors for some stochastic PDEs on unbounded domains has been proved in [3, 10, 24, 25, 27, 28, 31].

The first aim of this paper is to investigate the existence of random attractors for the stochastic p-Laplacian equation (1.1) defined on $\mathbb{R}^n$. We mention that the existence of global attractors for the p-Laplacian equation in the deterministic case has been discussed by many authors, for examples, in [30, 6] for bounded domains and in [15, 16] for unbounded domains. Recently [29] investigate the

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existence of random attractors for the p-Laplacian equation with multiplicative noise. However, in the paper [29], the p-Laplace equation is defined in a bounded domain where compactness of Sobolev embeddings is available. To overcome the difficulty caused by the non-compactness of Sobolev embedding on $\mathbb{R}^n$, we use the tail-estimate method which is always used to deal with the problem caused by the unboundedness of domains (see [3, 24, 25, 27, 28]). So far as we know, there were no results on random attractors for stochastic p-Laplacian equation with multiplicative noise on unbounded domains.

The second aim of this paper is to examine the limiting behavior of random attractors when $\varepsilon \to 0$ and prove the upper semicontinuity of these perturbed random attractors. It is worth mentioning that such continuity of attractors has been investigated, for examples, in [12, 13, 14, 23] for deterministic equations, in [4, 5, 21, 29] for stochastic PDEs in bounded domains and in [10, 28] for stochastic PDEs on unbounded domains.

The paper is arranged as follows. In the next section, we review the pullback random attractors theory for random dynamical systems. In section 3, we define a continuous random dynamical system for the stochastic p-Laplacian equation on $\mathbb{R}^n$. Section 4 is devoted to obtaining uniform estimates of solutions as $t \to \infty$. These estimate are necessary for proving the existence of bounded absorbing sets and the asymptotic compactness of the solution operator. Finally, we prove the upper semicontinuity of random attractors for (1.1) in the last section.

We denote by $\|\cdot\|$ and $(\cdot,\cdot)$ the norm and the inner product in $L^2(\mathbb{R}^n)$ respectively and $\|\cdot\|_p$ to denote the norm in $L^p(\mathbb{R}^n)$. Otherwise, the norm of a general Banach space X is written as $\|\cdot\|_X$.

The letters c and $C(\omega)$ are generic positive constants and positive random variable respectively, which don't depend on ε and may change their values from line to line or even in the same line.

2. Preliminaries

In this section, we recall some basic concepts related to RDS (see [1, 7, 8, 9, 28] for details).

Let $(X, \|\cdot\|_X)$ be a Banach space with Borel σ-algebra.

Definition 2.1. Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space and $\{\theta_t : \Omega \to \Omega, t \in \mathbb{R}\}$ a family of measure preserving transformations such that $(t,\omega) \mapsto \theta_t\omega$ is measurable, $\theta_0 = id$ and $\theta_{t+s} = \theta_t\theta_s$ for all $s, t \in \mathbb{R}$. The flow θ_t together with the corresponding probability space $(\Omega, \mathcal{F}, \mathrm{P}, \theta_t)$ is called a measurable dynamical system.

Definition 2.2. A continuous random dynamical system(RDS) on X over θ on $(\Omega, \mathcal{F}, \mathrm{P})$ is a measurable map

$$\varphi : \mathbb{R}^+ \times \Omega \times X \mapsto X,\ (t, \omega, x) \mapsto \varphi(t,\omega)x$$

such that P-a.s.

(i) $\varphi(0,\omega) = id$ on X;
(ii) $\varphi(t+s,\omega) = \varphi(t,\theta_s\omega)\varphi(s,\omega)$ for all $s, t \in \mathbb{R}^+$ (cocycle property);
(iii) $\varphi(t,\omega) : X \mapsto X$ is continuous.

Definition 2.3. A random compact set $\{K(\omega)\}_{\omega\in\Omega}$ is a family of compact sets indexed by ω such that for every $x \in X$ the mapping $\omega \mapsto d(x, K(\omega))$ is measurable with respect to $\mathcal{F}$. A random set $\{K(\omega)\}_{\omega\in\Omega}$ is said to be bounded if there exist $u_0 \in X$ and a random variable $R(\omega) > 0$ such that

$$K(\omega) \subset \{u \in X, \|u - u_0\|_X \le R(\omega)\} \quad \text{for all } \omega \in \Omega.$$

Definition 2.4. A random bounded set $\{B(\omega)\}_{\omega\in\Omega}$ of X is called tempered with respect to $(\theta_t)_{t\in\mathbb{R}}$ if for P-a.e.$\omega \in \Omega$,

$$\lim_{t\to\infty} e^{-\beta t} d(B(\theta_{-t}\omega)) = 0 \quad \text{for all } \beta > 0,$$

where $d(B) = \sup_{x\in B} \|x\|_X$.

Definition 2.5. Let $\mathcal{D}$ be a collection of random subsets of X and $\{K(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$. Then $\{K(\omega)\}_{\omega\in\Omega}$ is called a random absorbing set for φ in $\mathcal{D}$ if for every $B \in \mathcal{D}$ and P-a.e. $\omega \in \Omega$, there exists $t_B(\omega) > 0$ such that

$$\varphi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)) \subseteq K(\omega) \quad \text{for all } t \ge t_B(\omega).$$

Definition 2.6. Let $\mathcal{D}$ be a collection of random subsets of X. Then φ is said to be $\mathcal{D}$-pullback asymptotically compact in X if for P-a.e. $\omega \in \Omega$, $\{\varphi(t_n, \theta_{-t_n}\omega, x_n)\}_{n=1}^{\infty}$ has a convergent subsequence in X whenever $t_n \to \infty$, and $x_n \in B(\theta_{-t}\omega)$ with $\{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$.

Definition 2.7. Let $\mathcal{D}$ be a collection of random subsets of X. Then a random set $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ of X is called a $\mathcal{D}$-random attractor(or $\mathcal{D}$-pullback attractor) for φ if the following conditions are satisfied, for P-a.e.$\omega \in \Omega$,

(i) $\mathcal{A}(\omega)$ is a random compact set;
(ii) $\mathcal{A}(\omega)$ is invariant, that is, $\varphi(t, \omega, \mathcal{A}(\omega))=\mathcal{A}(\theta_t\omega)$, for all $t \ge 0$;
(iii) $\mathcal{A}(\omega)$ attracts every set in $\mathcal{D}$, that is, for every $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$,

$$\lim_{t\to\infty} d(\varphi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)), \mathcal{A}(\omega)) = 0,$$

where d is the Hausdorff semi-metric.

Proposition 2.8. *Let $\mathcal{D}$ be an inclusion-closed collection of random subsets of X and φ a continuous RDS on X over $(\Omega, \mathcal{F}, \mathrm{P}, (\theta_t)_{t\in\mathbb{R}})$. Suppose that $\{K(\omega)\}_{\omega\in\Omega}$ is a closed random absorbing set for φ in $\mathcal{D}$ and φ is $\mathcal{D}$-pullback asymptotically compact in X. Then φ has a unique $\mathcal{D}$-random attractor $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ which is given by*

$$\mathcal{A}(\omega) = \cap_{T\ge 0}\overline{\cup_{t\ge T}\varphi(t, \theta_{-t}\omega, K(\theta_{-t}\omega))}.$$

Proposition 2.9. *Let $\mathcal{D}$ be an inclusion-closed collection of random subsets of X. Given $\sigma > 0$, suppose φ_σ is a random dynamical system over a metric system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ which has a $\mathcal{D}$-random attractor $\mathcal{A}_\sigma$ and φ_0 is a deterministic dynamical system defined on X which has a global attractor $\mathcal{A}_0$. Assume that the following conditions be satisfied: (i) For $\mathbb{P}$-a.e. $\omega \in \Omega$, $t \ge 0$, $\sigma_n \to 0$, and x_n, $x \in X$ with $x_n \to x$, there holds*

$$\lim_{n\to\infty} \varphi_{\sigma_n}(t, \omega, x_n) = \varphi_0(t)x. \tag{2.1}$$

(ii) Every φ_σ has a random absorbing set $E_\sigma = \{E_\sigma(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ such that for some deterministic positive constant c and for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\limsup_{\sigma\to 0} \|E_\sigma(\omega)\|_X \le c, \tag{2.2}$$

where $\|E_\sigma(\omega)\|_X = \sup_{x\in E_\sigma(\omega)} \|x\|_X$.

(iii) There exists $\sigma_0 > 0$ such that for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\cup_{0<\sigma\le\sigma_0} \mathcal{A}_\sigma(\omega) \quad \textit{is precompact in } X. \tag{2.3}$$

Then for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\operatorname{dist}(\mathcal{A}_\sigma(\omega), \mathcal{A}_0) \to 0, \quad \textit{as } \sigma \to 0. \tag{2.4}$$

3. Stochastic p-Laplacian equation with multiplicative noise

Here we show that there is a continuous random dynamical system generated by the stochastic p-Laplacian equation defined on $\mathbb{R}^n$ with multiplicative noise:

$$du + (-\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda u)dt = (f(x,u) + g(x))dt + \varepsilon u \circ dW(t), \tag{3.1}$$

for $x \in \mathbb{R}^n$, $t > 0$, with the initial condition

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^n. \tag{3.2}$$

where $\varepsilon > 0$, $\lambda > 0$, $p \ge 2$ are constants, $g \in L^2(\mathbb{R}^n)$, W is a two-sided real-valued Wiener processes on a probability space which will be specified below, and f is a smooth nonlinear function satisfying the following conditions: For all $x \in \mathbb{R}^n$ and $s \in \mathbb{R}^n$,

$$f(x,s)s \le -\alpha_1|s|^p + \psi_1(x), \tag{3.3}$$

$$|f(x,s)| \le \alpha_2|s|^{p-1} + \psi_2(x), \tag{3.4}$$

where α_1, α_2 are positive constants, $\psi_1 \in L^1(\mathbb{R}^n) \cap L^{\frac{p}{2}}(\mathbb{R}^n)$, $\psi_2 \in L^2(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ with $\frac{1}{p} + \frac{1}{q} = 1$.

In the sequel, we consider the probability space $(\Omega, \mathcal{F}, \mathrm{P})$ where

$$\Omega = \{\omega \in C(\mathbb{R},\mathbb{R}) : \omega(0) = 0\},$$

$\mathcal{F}$ is the Borel σ-algebra induced by the compact-open topology of Ω, and P the corresponding Wiener measure on $(\Omega, \mathcal{F})$. Define the time shift by

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \omega \in \Omega,\ t \in \mathbb{R}.$$

Then $(\Omega, \mathcal{F}, \mathrm{P}, (\theta_t)_{t\in\mathbb{R}})$ is a metric dynamical system.

We now associate a continuous random dynamical system with the equation over $(\Omega, \mathcal{F}, \mathrm{P}, (\theta_t)_{t\in\mathbb{R}})$. To this end, we need to convert the stochastic equation with a random multiplicative term into a deterministic equation with a random parameter.

Consider the stationary solutions of the one-dimensional Ornstein-Uhlenbeck equation:

$$dz + zdt = dW(t). \tag{3.5}$$

The solution to (3.5) is given by

$$z(\theta_t\omega) = -\int_{-\infty}^0 e^\tau(\theta_t\omega)(\tau)d\tau, \quad t \in \mathbb{R}^n. \tag{3.6}$$

From [1, 3, 24, 25], the random variable $|z(\omega)|$ is tempered, and there is a θ_t-invariant set $\widetilde{\Omega} \subset \Omega$ of full P measure such that $z(\theta_t\omega)$ is continuous in t for every $\omega \in \widetilde{\Omega}$ and

$$\lim_{t\to\pm\infty} \frac{|z(\theta_t\omega)|}{|t|} = 0; \tag{3.7}$$

$$\lim_{t\to\pm\infty} \frac{1}{t}\int_0^t z(\theta_s\omega)ds = 0. \tag{3.8}$$

Following the properties of the O-U process and (3.7),(3.8), it is easy to show for for all $0 < \varepsilon \le 1$,

$$\begin{aligned} 0 \le M_\varepsilon(\omega) &:= \int_{-\infty}^0 e^{-2\varepsilon z(\theta_s\omega)+2\varepsilon\int_s^0 z(\theta_\tau\omega)d\tau+\lambda s} ds \\ &\le \int_{-\infty}^0 e^{2|z(\theta_s\omega)|+2|\int_s^0 z(\theta_\tau\omega)d\tau|+\lambda s} ds < +\infty, \end{aligned} \tag{3.9}$$

and

$$0 \le K(\omega) := \max_{-2\le\tau\le 0} |z(\theta_\tau(\omega))| < +\infty, \tag{3.10}$$

which will be used frequently in the following paper. And it is easy to see that $M_\varepsilon(\omega)$ and $K(\omega)$ are both tempered.

To show that problem (3.1)-(3.2) generates a random dynamical system, let

$$v(t) = e^{-\varepsilon z(\theta_t\omega)} u(t), \tag{3.11}$$

where u is a solution of problem (3.1)-(3.2). Then v satisfies

$$\frac{dv}{dt} = e^{\varepsilon(p-2)z(\theta_t\omega)} \operatorname{div}(|\nabla v|^{p-2}\nabla v) - \lambda v + e^{-\varepsilon z(\theta_t\omega)}(f(x,u)+g(x)) + \varepsilon v z(\theta_t\omega), \tag{3.12}$$

and the initial condition

$$v(x,0) = v_0(x) = e^{-\varepsilon z(\omega)} u_0(x), \quad x \in \mathbb{R}^n. \tag{3.13}$$

Using the standard Galerkin method, one may show that for all $v_0 \in L^2(\mathbb{R}^n)$, problem (3.12)-(3.13) has a unique solution

$$v(\cdot,\omega,v_0) \in C([0,\infty), L^2(\mathbb{R}^n)) \cap L^2((0,T), W^{1,p}(\mathbb{R}^n)).$$

Furthermore, the solution is continuous with respect to v_0 in $L^2(\mathbb{R}^n)$ for all $t \ge 0$. Let

$$u(t,\omega,u_0) = e^{\varepsilon z(\theta_t\omega)} v(t,\omega,v_0), \tag{3.14}$$

where

$$v_0 = e^{-\varepsilon z(\omega)} u_0. \tag{3.15}$$

We can associate a random dynamical system Φ_ε with problem (3.1)-(3.2) via u for each $\varepsilon > 0$, where $\Phi_\varepsilon : \mathbb{R}^+ \times \Omega \times L^2(\mathbb{R}^n) \mapsto L^2(\mathbb{R}^n)$ is given by

$$\Phi_\varepsilon(t,\omega)u_0 = u(t,\omega,u_0), \quad \text{for every } (t,\omega,u_0) \in \mathbb{R}^+ \times \Omega \times L^2(\mathbb{R}^n). \tag{3.16}$$

Then Φ_ε is a continuous random dynamical system over $(\Omega, \mathcal{F}, \mathrm{P}, (\theta_t)_{t\in\mathbb{R}})$ in $L^2(\mathbb{R}^n)$. In the sequel, we always assume that $\mathcal{D}$ is the collection of all tempered random subsets of $L^2(\mathbb{R}^n)$.

In the following, we will first prove that Φ_ε has a unique $\mathcal{D}$-pullback random attractor $\{\mathcal{A}_\varepsilon(\omega)\}_{\omega\in\Omega}$. When $\varepsilon = 0$, problem (3.1)-(3.2) defines a continuous deterministic dynamical system Φ in $L^2(\mathbb{R}^n)$. We use $\mathcal{A}_0$ to denote the global attractor for the deterministic dynamical system. At last, we will establish the relationship of $\{\mathcal{A}_\varepsilon(\omega)\}_{\omega\in\Omega}$ and $\mathcal{A}_0$ when $\varepsilon \to 0$.

4. Uniform estimates of solutions

In this section, we derive uniform estimates on the solution of the stochastic p-Laplacian equation on $\mathbb{R}^n$ when $t \to \infty$ with the purpose of proving the existence of a bounded random absorbing set and the asymptotic compactness of the random dynamical system associated with the equation. In particular, we will show that the tails of the solutions, i.e.,solutions evaluated at large values of $|x|$, are uniformly small when time is sufficiently large.

Lemma 4.1. *Let $0 < \varepsilon \leq 1, g \in L^2(\mathbb{R}^n)$ and (3.3)-(3.4) hold. Then for every $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and P-a.e.$\omega \in \Omega$, there is $T(B,\omega) > 0$, independent of ε, such that for all $u_0(\theta_{-t}\omega) \in B(\theta_{-t}\omega)$ and $t \geq T(B,\omega)$,*

$$\|u(t, \theta_{-t}\omega, u_0(\theta_{-t}\omega))\|^2 \leq \rho_1^2(\omega) := 1 + ce^{2\varepsilon z(\omega)} M_\varepsilon(\omega). \tag{4.1}$$

Furthermore $\rho_1(\omega)$ is a tempered function.

Proof. Multiplying (3.12) by v and then integrating over $\mathbb{R}^n$, we find that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v\|^2 = &-e^{\varepsilon(p-2)z(\theta_t\omega)}\int_{\mathbb{R}^n}|\nabla v|^p dx - \lambda\|v\|^2 + e^{-\varepsilon z(\theta_t\omega)}\int_{\mathbb{R}^n} f(x,u)vdx \\ &+ e^{-\varepsilon z(\theta_t\omega)}\int_{\mathbb{R}^n} g(x)vdx + \varepsilon z(\theta_t\omega)\|v\|^2\end{aligned}. \tag{4.2}$$

For the nonlinear term, by (3.3), we have

$$\begin{aligned}e^{-\varepsilon z(\theta_t\omega)}\int_{\mathbb{R}^n} f(x,u)vdx &= e^{-2\varepsilon z(\theta_t\omega)}\int_{\mathbb{R}^n} f(x,u)udx \\ &\leq -\alpha_1 e^{-2\varepsilon z(\theta_t\omega)}\int_{\mathbb{R}^n}|u|^p dx + e^{-2\varepsilon z(\theta_t\omega)}\int_{\mathbb{R}^n}|\psi_1(x)|dx \\ &= -\alpha_1 e^{-2\varepsilon z(\theta_t\omega)}\|u\|_p^p + e^{-2\varepsilon z(\theta_t\omega)}\|\psi_1\|_{L_1},\end{aligned} \tag{4.3}$$

And

$$e^{-\varepsilon z(\theta_t\omega)}\int_{\mathbb{R}^n} g(x)vdx \leq e^{-\varepsilon z(\theta_t\omega)}\|g\|\cdot\|v\| \leq \frac{1}{2\lambda}e^{-2\varepsilon z(\theta_t\omega)}\|g\|^2 + \frac{\lambda}{2}\|v\|^2. \tag{4.4}$$

Then it from (4.2)-(4.4) it follows that

$$\begin{aligned}\frac{d}{dt}\|v\|^2 \leq &-2e^{\varepsilon(p-2)z(\theta_t\omega)}\int_{\mathbb{R}^n}|\nabla v|^p dx + (2\varepsilon z(\theta_t\omega) - \lambda)\|v\|^2 \\ &- 2\alpha_1 e^{-2\varepsilon z(\theta_t\omega)}\|u\|_p^p + (\frac{1}{\lambda}\|g\|^2 + 2\|\psi_1\|_{L_1})e^{-2\varepsilon z(\theta_t\omega)}.\end{aligned} \tag{4.5}$$

Thus,

$$\frac{d}{dt}\|v\|^2 \leq (2\varepsilon z(\theta_t\omega) - \lambda)\|v\|^2 + ce^{-2\varepsilon z(\theta_t\omega)}. \tag{4.6}$$

By the Gronwall Lemma,

$$\begin{aligned}&\|v(t,\omega,v_0(\omega)\|^2 \\ &\leq \|v_0(\omega)\|^2 e^{2\varepsilon\int_0^t z(\theta_s\omega)ds - \lambda t} + c\int_0^t e^{-2\varepsilon z(\theta_s\omega) + 2\varepsilon\int_s^t z(\theta_\tau\omega)d\tau - \lambda(t-s)}ds.\end{aligned} \tag{4.7}$$

Replace ω by $\theta_{-t}\omega$ in (4.7), we have

$$\begin{aligned}
\|v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega)\|^2 &\le \|v_0(\theta_{-t}\omega)\|^2 e^{2\varepsilon\int_0^t z(\theta_{s-t}\omega)ds-\lambda t}\\
&\quad + c\int_0^t e^{-2\varepsilon z(\theta_{s-t}\omega)+2\varepsilon\int_s^t z(\theta_{\tau-t}\omega)d\tau-\lambda(t-s)}ds\\
&= \|v_0(\theta_{-t}\omega)\|^2 e^{2\varepsilon\int_{-t}^0 z(\theta_s\omega)ds-\lambda t}\\
&\quad + c\int_{-t}^0 e^{-2\varepsilon z(\theta_s\omega)+2\varepsilon\int_s^0 z(\theta_\tau\omega)d\tau+\lambda s}ds\\
&\le \|v_0(\theta_{-t}\omega)\|^2 e^{2\varepsilon\int_{-t}^0 z(\theta_s\omega)ds-\lambda t} + cM_\varepsilon(\omega),
\end{aligned} \tag{4.8}$$

where $M_\varepsilon(\omega)$ is defined in (3.9).

It follows from (4.8) and (3.11) that

$$\begin{aligned}
&\|u(t,\theta_{-t}\omega,u_0(\theta_{-t}\omega)\|^2\\
&= \|e^{\varepsilon z(\omega)}v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega)\|^2\\
&= e^{2\varepsilon z(\omega)}\|v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega)\|^2\\
&\le e^{2\varepsilon z(\omega)}(\|v_0(\theta_{-t}\omega)\|^2 e^{2\varepsilon\int_{-t}^0 z(\theta_s\omega)ds-\lambda t} + cM_\varepsilon(\omega))\\
&= e^{2\varepsilon z(\omega)}(e^{-2\varepsilon z(\theta_{-t}\omega)}\|u_0(\theta_{-t}\omega)\|^2 e^{2\varepsilon\int_{-t}^0 z(\theta_s\omega)ds-\lambda t} + cM_\varepsilon(\omega)).
\end{aligned} \tag{4.9}$$

Since $B=\{B(\omega)\}_{\omega\in\Omega}\in\mathcal{D}$ and $u_0(\theta_{-t}\omega)\in B(\theta_{-t}\omega)$, due to (3.7)(3.8), there exists $T(B,\omega)>0$, independent of ε, such that for all $t\ge T(B,\omega)$

$$\begin{aligned}
&\|u_0(\theta_{-t}\omega)\|^2 e^{2\varepsilon z(\omega)-2\varepsilon z(\theta_{-t}\omega)+2\varepsilon\int_{-t}^0 z(\theta_s\omega)ds-\lambda t}\\
&\le \|u_0(\theta_{-t}\omega)\|^2 e^{2|z(\omega)|+2|z(\theta_{-t}\omega)|+2|\int_{-t}^0 z(\theta_s\omega)ds|-\lambda t}\\
&\le \|u_0(\theta_{-t}\omega)\|^2 e^{-\frac{\lambda}{2}t}\le 1,
\end{aligned} \tag{4.10}$$

which along with (4.9) implies that for all $t\ge T(B,\omega)$

$$\|u(t,\theta_{-t}\omega,u_0(\theta_{-t}\omega))\|^2 \le \rho_1^2(\omega) := 1+ce^{2\varepsilon z(\omega)}M_\varepsilon(\omega). \tag{4.11}$$

It is easy to prove that $\rho_1(\omega)$ is a tempered function. $\square$

Lemma 4.2. *Let $0<\varepsilon\le 1, g\in L^2(\mathbb{R}^n)$ and (3.3)-(3.4) hold. Then for every $B=\{B(\omega)\}_{\omega\in\Omega}\in\mathcal{D}$ and P-a.e. $\omega\in\Omega$, there is $T(B,\omega)>0$, independent of ε, such that for all $u_0(\theta_{-t}\omega)\in B(\theta_{-t}\omega)$ and $t\ge T(B,\omega)$*

$$\int_t^{t+1}\|\nabla u(s,\theta_{-t-1}\omega,u_0(\theta_{-t-1}\omega))\|_p^p \le 1+ce^{2p\varepsilon K(\omega)}M_\varepsilon(\omega), \tag{4.12}$$

$$\int_{t-1}^{t+1}\|u(s,\theta_{-t-1}\omega,u_0(\theta_{-t-1}\omega))\|_p^p \le 1+ce^{6\varepsilon K(\omega)}M_\varepsilon(\omega). \tag{4.13}$$

Proof. From (4.5), we have

$$\begin{aligned}
\frac{d}{dt}\|v\|^2 &\le -2e^{\varepsilon(p-2)z(\theta_t\omega)}\|\nabla v\|_p^p dx + (2\varepsilon z(\theta_t\omega)-\lambda)\|v\|^2\\
&\quad - 2\alpha_1 e^{-2\varepsilon z(\theta_t\omega)}\|u\|_p^p + ce^{-2\varepsilon z(\theta_t\omega)}.
\end{aligned} \tag{4.14}$$

Using Gronwall Lemma, for all $t \geq T \geq 0$, we have

$$
\begin{aligned}
&\|v(t,\omega,v_0(\omega))\|^2 \\
&\leq e^{2\varepsilon\int_T^t z(\theta_s\omega)ds-\lambda(t-T)}\|v(T,\omega,v_0(\omega))\|^2 \\
&\quad + c\int_T^t e^{-2\varepsilon z(\theta_s\omega)+2\varepsilon\int_s^t z(\theta_\tau\omega)d\tau+\lambda(s-t)} \\
&\quad - 2\alpha_1\int_T^t e^{-2\varepsilon z(\theta_s\omega)+2\varepsilon\int_s^t z(\theta_\tau\omega)d\tau+\lambda(s-t)}\|u(s,\omega,u_0(\omega))\|_p^p ds \\
&\quad - 2\int_T^t e^{\varepsilon(p-2)z(\theta_s\omega)+2\varepsilon\int_s^t z(\theta_\tau\omega)d\tau+\lambda(s-t)}\|\nabla v(s,\omega,v_0(\omega))\|_p^p ds.
\end{aligned}
\tag{4.15}
$$

Replace ω by $\theta_{-t}\omega$ and t by T in (4.7), we have

$$
\begin{aligned}
\|v(T,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2 &\leq \|v_0(\theta_{-t}\omega)\|^2 e^{2\varepsilon\int_0^T z(\theta_{s-t}\omega)ds-\lambda T} \\
&\quad + c\int_0^T e^{-2\varepsilon z(\theta_{s-t}\omega)+2\varepsilon\int_s^T z(\theta_{\tau-t}\omega)d\tau-\lambda(T-s)}ds.
\end{aligned}
\tag{4.16}
$$

Multiplying the two sides of (4.16) by $e^{2\varepsilon\int_T^t z(\theta_{s-t}\omega)ds-\lambda(t-T)}$, then for all $t \geq T$,

$$
\begin{aligned}
&e^{2\varepsilon\int_T^t z(\theta_{s-t}\omega)ds-\lambda(t-T)}\|v(T,\theta_{-t}\omega,v_0(\theta_{-t}\omega)\|^2 \\
&\leq \|v_0(\theta_{-t}\omega)\|^2 e^{2\varepsilon\int_0^t z(\theta_{s-t}\omega)ds-\lambda t} + c\int_0^T e^{-2\varepsilon z(\theta_{s-t}\omega)+2\varepsilon\int_s^t z(\theta_{\tau-t}\omega)d\tau-\lambda(t-s)}ds.
\end{aligned}
\tag{4.17}
$$

Thus, replace ω by $\theta_{-t}\omega$ in (4.15) and together with (4.17), it follows that

$$
\begin{aligned}
&2\alpha_1\int_T^t e^{-2\varepsilon z(\theta_{s-t}\omega)+2\varepsilon\int_s^t z(\theta_{\tau-t}\omega)d\tau+\lambda(s-t)}\|u(s,\theta_{-t}\omega,u_0(\theta_{-t}\omega))\|_p^p ds \\
&+ 2\int_T^t e^{\varepsilon(p-2)z(\theta_{s-t}\omega)+2\varepsilon\int_s^t z(\theta_{\tau-t}\omega)d\tau+\lambda(s-t)}\|\nabla v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|_p^p ds \\
&\leq \|v_0(\theta_{-t}\omega)\|^2 e^{2\varepsilon\int_0^t z(\theta_{s-t}\omega)ds-\lambda t} + c\int_0^t e^{-2\varepsilon z(\theta_{s-t}\omega)+2\varepsilon\int_s^t z(\theta_{\tau-t}\omega)d\tau-\lambda(t-s)}ds \\
&= \|v_0(\theta_{-t}\omega)\|^2 e^{2\varepsilon\int_{-t}^0 z(\theta_s\omega)ds-\lambda t} + c\int_{-t}^0 e^{-2\varepsilon z(\theta_s\omega)+2\varepsilon\int_s^0 z(\theta_\tau\omega)d\tau+\lambda s}ds.
\end{aligned}
\tag{4.18}
$$

Replace t by $t+1$ and T by t in (4.18), we have

$$
\begin{aligned}
&2\int_t^{t+1} e^{\varepsilon(p-2)z(\theta_{s-t-1}\omega)+2\varepsilon\int_s^{t+1} z(\theta_{\tau-t-1}\omega)d\tau+\lambda(s-t-1)} \\
&\times \|\nabla v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|_p^p ds \\
&\leq \|v_0(\theta_{-t-1}\omega)\|^2 e^{2\varepsilon\int_{-t-1}^0 z(\theta_s\omega)ds-\lambda(t+1)} + c\int_{-t-1}^0 e^{-2\varepsilon z(\theta_s\omega)+2\varepsilon\int_s^0 z(\theta_\tau\omega)d\tau+\lambda s}ds.
\end{aligned}
$$

Using (3.10), we have

$$
\begin{aligned}
&\int_t^{t+1} e^{\varepsilon(p-2)z(\theta_{s-t-1}\omega)+2\varepsilon\int_s^{t+1} z(\theta_{\tau-t-1}\omega)d\tau+\lambda(s-t-1)} \\
&\times \|\nabla v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|_p^p ds
\end{aligned}
$$

$$
\begin{aligned}
&\geq \int_t^{t+1} e^{-\varepsilon(p-2)K(\omega)-2\varepsilon K(\omega)-\lambda}\|\nabla v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|_p^p ds\\
&= e^{-p\varepsilon K(\omega)-\lambda}\int_t^{t+1}\|\nabla v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|_p^p ds.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\int_t^{t+1}\|\nabla v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|_p^p ds\\
&\leq \frac{1}{2}\|v_0(\theta_{-t-1}\omega)\|^2 e^{p\varepsilon K(\omega)+2\varepsilon\int_{-t-1}^0 z(\theta_s\omega)ds-\lambda t}\\
&\quad + ce^{p\varepsilon K(\omega)+\lambda}\int_{-t-1}^0 e^{-2\varepsilon z(\theta_s\omega)+2\varepsilon\int_s^0 z(\theta_\tau\omega)d\tau+\lambda s}ds\\
&\leq \frac{1}{2}\|v_0(\theta_{-t-1}\omega)\|^2 e^{p\varepsilon K(\omega)+2\varepsilon\int_{-t-1}^0 z(\theta_s\omega)ds-\lambda t}+ce^{p\varepsilon K(\omega)+\lambda}M_\varepsilon(\omega).
\end{aligned}
\tag{4.19}
$$

It follows from (4.19) that

$$
\begin{aligned}
&\int_t^{t+1}\|\nabla u(s,\theta_{-t-1}\omega,u_0(\theta_{-t-1}\omega))\|_p^p ds\\
&= \int_t^{t+1} e^{p\varepsilon z(\theta_{s-t-1}\omega)}\|\nabla v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|_p^p ds\\
&\leq e^{p\varepsilon K(\omega)}\int_t^{t+1}\|\nabla v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|_p^p ds\\
&\leq \frac{1}{2}\|v_0(\theta_{-t-1}\omega)\|^2 e^{2p\varepsilon K(\omega)+2\varepsilon\int_{-t-1}^0 z(\theta_s\omega)ds-\lambda t}+ce^{2p\varepsilon K(\omega)+\lambda}M_\varepsilon(\omega)\\
&\leq c\|u_0(\theta_{-t-1}\omega)\|^2 e^{-2\varepsilon z(\theta_{-t-1}\omega)+2p\varepsilon K(\omega)+2\varepsilon\int_{-t-1}^0 z(\theta_s\omega)ds-\lambda t}\\
&\quad + ce^{2p\varepsilon K(\omega)+\lambda}M_\varepsilon(\omega).
\end{aligned}
\tag{4.20}
$$

On the other hand, since

$$
\begin{aligned}
&\int_{t-1}^{t+1} e^{-2\varepsilon z(\theta_{s-t-1}\omega)+2\varepsilon\int_s^{t+1} z(\theta_{\tau-t-1}\omega)d\tau+\lambda(s-t-1)}\|u(s,\theta_{-t-1}\omega,u_0(\theta_{-t-1}\omega))\|_p^p ds\\
&\geq e^{-6\varepsilon K(\omega)-2\lambda}\int_{t-1}^{t+1}\|u(s,\theta_{-t-1}\omega,u_0(\theta_{-t-1}\omega))\|_p^p ds.
\end{aligned}
\tag{4.21}
$$

Replace t by $t+1$ and T by $t-1$ in (4.18) and using (4.21), we obtain

$$
\begin{aligned}
&\int_{t-1}^{t+1}\|u(s,\theta_{-t-1}\omega,u_0(\theta_{-t-1}\omega))\|_p^p ds\\
&\leq c\|v_0(\theta_{-t-1}\omega)\|^2 e^{6\varepsilon K(\omega)+2\varepsilon\int_{-t-1}^0 z(\theta_s\omega)ds-\lambda t+\lambda}\\
&\quad + ce^{6\varepsilon K(\omega)+2\lambda}\int_{-t-1}^0 e^{-2\varepsilon z(\theta_s\omega)+2\varepsilon\int_s^0 z(\theta_\tau\omega)d\tau+\lambda s}ds\\
&\leq c\|v_0(\theta_{-t-1}\omega)\|^2 e^{6\varepsilon K(\omega)+2\varepsilon\int_{-t-1}^0 z(\theta_s\omega)ds-\lambda t+\lambda}+ce^{6\varepsilon K(\omega)+2\lambda}M_\varepsilon(\omega)\\
&\leq c\|u_0(\theta_{-t-1}\omega)\|^2 e^{-2\varepsilon z(\theta_{-t-1}\omega)+6\varepsilon K(\omega)+2\varepsilon\int_{-t-1}^0 z(\theta_s\omega)ds-\lambda t+\lambda}\\
&\quad + ce^{6\varepsilon K(\omega)+2\lambda}M_\varepsilon(\omega).
\end{aligned}
\tag{4.22}
$$

Since $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and $u_0(\theta_{-t}(\omega)) \in B(\theta_{-t}(\omega))$, similar to (4.10), there exists $T(B,\omega) > 0$, independent of ε, such that for all $t \ge T(B,\omega)$

$$c\|u_0(\theta_{-t-1}\omega)\|^2 e^{-2\varepsilon z(\theta_{-t-1}\omega)+2p\varepsilon K(\omega)+2\varepsilon\int_{-t-1}^0 z(\theta_s\omega)ds-\lambda t} \le 1, \tag{4.23}$$

$$c\|u_0(\theta_{-t-1}\omega)\|^2 e^{-2\varepsilon z(\theta_{-t-1}\omega)+6\varepsilon K(\omega)+2\varepsilon\int_{-t-1}^0 z(\theta_s\omega)ds-\lambda t+\lambda} \le 1. \tag{4.24}$$

From (4.20), (4.22) and using (4.23), (4.24), we obtain that for all $t \ge T(B,\omega)$,

$$\int_t^{t+1} \|\nabla u(s,\theta_{-t-1}\omega, u_0(\theta_{-t-1}\omega))\|_p^p \le 1 + ce^{2p\varepsilon K(\omega)} M_\varepsilon(\omega), \tag{4.25}$$

$$\int_{t-1}^{t+1} \|u(s,\theta_{-t-1}\omega, u_0(\theta_{-t-1}\omega))\|_p^p \le 1 + ce^{6\varepsilon K(\omega)} M_\varepsilon(\omega). \tag{4.26}$$

□

Lemma 4.3. *Let $0 < \varepsilon \le 1$, $g \in L^2(\mathbb{R}^n)$ and (3.3)-(3.4) hold. Then for every $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and P-a.e. $\omega \in \Omega$, there is $T(B,\omega) > 0$, independent of ε, such that for all $u_0(\theta_{-t}\omega) \in B(\theta_{-t}\omega)$ and $t \ge T(B,\omega), \forall\tau \in [t,t+1]$*

$$\|v(\tau,\theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))\|_p^p \le ce^{p\varepsilon K(\omega)}(\varepsilon K(\omega)+1)(e^{6\varepsilon K(\omega)}M_\varepsilon(\omega)+1). \tag{4.27}$$

Proof. Multiplying (3.12) with $|v|^{p-2}v$ and then integrating over $\mathbb{R}^n$, it yields that

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\|v\|_p^p &= e^{\varepsilon(p-2)z(\theta_t\omega)}(\mathrm{div}(|\nabla v|^{p-2}\nabla v), |v|^{p-2}v) - \lambda\|v\|_p^p \\ &\quad + e^{-\varepsilon z(\theta_t\omega)}(f(x,u), |v|^{p-2}v) + e^{-\varepsilon z(\theta_t\omega)}(g(x), |v|^{p-2}v) \\ &\quad + \varepsilon z(\theta_t\omega)\|v\|_p^p.\end{aligned} \tag{4.28}$$

We now estimate every term of (4.28). First by our assumption $p \ge 2$, we have

$$\begin{aligned}&e^{\varepsilon(p-2)z(\theta_t\omega)}(\mathrm{div}(|\nabla v|^{p-2}\nabla v), |v|^{p-2}v) \\ &= e^{\varepsilon(p-2)z(\theta_t\omega)}\int_{\mathbb{R}^n}\sum_{i=1}^n \frac{\partial}{\partial x_i}(|\nabla v|^{p-2}\frac{\partial v}{\partial x_i})|v|^{p-2}v dx \\ &= -e^{\varepsilon(p-2)z(\theta_t\omega)}\sum_{i=1}^n\Big[\int_{\mathbb{R}^n}(|\nabla v|^{p-2}\frac{\partial v}{\partial x_i})(p-2)|v|^{p-2}\frac{\partial v}{\partial x_i}dx \\ &\quad + \int_{\mathbb{R}^n}(|\nabla v|^{p-2}\frac{\partial v}{\partial x_i})|v|^{p-2}\frac{\partial v}{\partial x_i}dx\Big] \\ &= -e^{\varepsilon(p-2)z(\theta_t\omega)}(p-1)\int_{\mathbb{R}^n}|\nabla v|^p|v|^{p-2}dx \le 0.\end{aligned} \tag{4.29}$$

To estimate the nonlinear term, from (3.3), we have

$$\begin{aligned}f(x,u)v = e^{-\varepsilon z(\theta_t\omega)}f(x,u)u &\le -\alpha_1 e^{-\varepsilon z(\theta_t\omega)}|u|^p + e^{-\varepsilon z(\theta_t\omega)}\psi_1(x) \\ &= -\alpha_1 e^{(p-1)\varepsilon z(\theta_t\omega)}|v|^p + e^{-\varepsilon z(\theta_t\omega)}\psi_1(x).\end{aligned} \tag{4.30}$$

From which it follows by Young's inequality that

$$\begin{aligned} &e^{-\varepsilon z(\theta_t\omega)}(f(x,u),|v|^{p-2}v)\\ &= e^{-\varepsilon z(\theta_t\omega)}\int_{\mathbb{R}^n} f(x,u)|v|^{p-2}v\\ &\le -\alpha_1 e^{(p-2)\varepsilon z(\theta_t\omega)}\|v\|_{2p-2}^{2p-2} + e^{-2\varepsilon z(\theta_t\omega)}\int_{\mathbb{R}^n}\psi_1(x)|v|^{p-2}dx\\ &\le -\alpha_1 e^{(p-2)\varepsilon z(\theta_t\omega)}\|v\|_{2p-2}^{2p-2} + ce^{-p\varepsilon z(\theta_t\omega)}\|\psi_1\|_{\frac{p}{2}}^{\frac{p}{2}} + \frac{(p-1)\lambda}{p}\|v\|_p^p. \end{aligned} \tag{4.31}$$

On the other hand, the fourth term on the right-hand side of (4.28) is bounded by

$$|e^{-\varepsilon z(\theta_t\omega)}(g(x),|v|^{p-2}v)| \le \frac{\alpha_1}{2}e^{(p-2)\varepsilon z(\theta_t\omega)}\|v\|_{2p-2}^{2p-2} + ce^{-p\varepsilon z(\theta_t\omega)}\|g\|^2. \tag{4.32}$$

Then it follows from (4.28)(4.29), (4.31),(4.32) that

$$\frac{d}{dt}\|v\|_p^p \le (p\varepsilon z(\theta_t\omega)-\lambda)\|v\|_p^p + ce^{-p\varepsilon z(\theta_t\omega)}. \tag{4.33}$$

Integrating (4.33) from $s(t-1\le s\le t)$ to $\tau(t\le\tau\le t+1)$, we obtain

$$\begin{aligned} &\|v(\tau,\omega,v_0(\omega))\|_p^p\\ &\le \|v(s,\omega,v_0(\omega))\|_p^p + \int_s^\tau |p\varepsilon z(\theta_{s'}\omega)-\lambda|\|v(s',\omega,v_0(\omega))\|_p^p ds'\\ &\quad + c\int_s^\tau e^{-p\varepsilon z(\theta_{s'}\omega)}ds'. \end{aligned} \tag{4.34}$$

Replace ω by $\theta_{-t-1}\omega$ in (4.34), we have

$$\begin{aligned} &\|v(\tau,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|_p^p\\ &\le \|v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|_p^p + c\int_{t-1}^{t+1} e^{-p\varepsilon z(\theta_{s'-t-1}\omega)}ds'\\ &\quad + \int_{t-1}^{t+1}|p\varepsilon z(\theta_{s'-t-1}\omega)-\lambda|\|v(s',\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|_p^p ds'. \end{aligned} \tag{4.35}$$

Integrating (4.35) with respect to s from $t-1$ to t, we obtain that for all $\tau\in[t,t+1]$,

$$\begin{aligned} &\|v(\tau,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|_p^p\\ &\le \int_{t-1}^{t}\|v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|_p^p ds + c\int_{-2}^{0} e^{-p\varepsilon z(\theta_s\omega)}ds\\ &\quad + c(\varepsilon K(\omega)+1)\int_{t-1}^{t+1}\|v(s',\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|_p^p ds'\\ &\le \int_{t-1}^{t+1}\|v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|_p^p ds + ce^{p\varepsilon K(\omega)}\\ &\quad + c(\varepsilon K(\omega)+1)\int_{t-1}^{t+1}\|v(s',\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|_p^p ds'\\ &\le \int_{t-1}^{t+1} e^{-p\varepsilon z(\theta_{s-t-1}\omega)}\|u(s,\theta_{-t-1}\omega,u_0(\theta_{-t-1}\omega))\|_p^p ds + ce^{p\varepsilon K(\omega)}\\ &\quad + c(\varepsilon K(\omega)+1)\int_{t-1}^{t+1} e^{-p\varepsilon z(\theta_{s'-t-1}\omega)}\|u(s',\theta_{-t-1}\omega,u_0(\theta_{-t-1}\omega))\|_p^p ds' \end{aligned} \tag{4.36}$$

$$\le c(\varepsilon K(\omega)+1)e^{p\varepsilon K(\omega)}\int_{t-1}^{t+1}\|u(s',\theta_{-t-1}\omega,u_0(\theta_{-t-1}\omega))\|_p^p ds' + ce^{p\varepsilon K(\omega)}. \quad (4.37)$$

Let $T(B,\omega)$ be the positive constant in Lemma 4.2 and $t \ge T(B,\omega)$, together with (4.13) and (4.37), we have for $t \le \tau \le t+1$

$$\|v(\tau,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|_p^p \le ce^{p\varepsilon K(\omega)}(\varepsilon K(\omega)+1)(e^{6\varepsilon K(\omega)}M_\varepsilon(\omega)+1). \quad (4.38)$$

□

Lemma 4.4. *Let $0 < \varepsilon \le 1, g \in L^2(\mathbb{R}^n)$ and (3.3)-(3.4) hold. Then for every $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and P-a.e.$\omega \in \Omega$, there is $T(B,\omega) > 0$, independent of ε, such that for all $u_0(\theta_{-t}\omega) \in B(\theta_{-t}\omega)$ and $t \ge T(B,\omega)$*

$$\int_t^{t+1}\|u(s,\theta_{-t-1}\omega,u_0(\theta_{-t-1}\omega))\|_{2p-2}^{2p-2}ds \le ce^{c\varepsilon K(\omega)}(\varepsilon K(\omega)+1)(e^{c\varepsilon K(\omega)}M_\varepsilon(\omega)+1). \quad (4.39)$$

Proof. Using (4.28) together with (4.29) (4.31)and (4.32), we have

$$\frac{d}{dt}\|v\|_p^p \le -ce^{\varepsilon(p-2)z(\theta_t\omega)}\|v\|_{2p-2}^{2p-2} + (p\varepsilon z(\theta_t\omega)-\lambda)\|v\|_p^p + ce^{-p\varepsilon z(\theta_t\omega)}. \quad (4.40)$$

Using Gronwall's Lemma, for all $t \ge T \ge 0$

$$\begin{aligned}
&\|v(t,\omega,v_0(\omega))\|_p^p \\
&\le \|v(T,\omega,v_0(\omega))\|_p^p e^{p\varepsilon\int_T^t z(\theta_s\omega)ds-\lambda(t-T)} \\
&\quad + c\int_T^t e^{-p\varepsilon z(\theta_s\omega)+p\varepsilon\int_s^t z(\theta_\tau\omega)d\tau-\lambda(t-s)}ds \\
&\quad - c\int_T^t e^{\varepsilon(p-2)z(\theta_s\omega)+p\varepsilon\int_s^t z(\theta_\tau\omega)d\tau-\lambda(t-s)}\|v(s,\omega,v_0(\omega))\|_{2p-2}^{2p-2}ds.
\end{aligned} \quad (4.41)$$

Replace ω by $\theta_{-t}\omega$ in (4.41). It follows that

$$\begin{aligned}
&c\int_T^t e^{\varepsilon(p-2)z(\theta_{s-t}\omega)+p\varepsilon\int_s^t z(\theta_{\tau-t}\omega)d\tau-\lambda(t-s)}\|v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|_{2p-2}^{2p-2}ds \\
&\le \|v(T,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|_p^p e^{p\varepsilon\int_T^t z(\theta_{s-t}\omega)ds-\lambda(t-T)} \\
&\quad + c\int_T^t e^{-p\varepsilon z(\theta_{s-t}\omega)+p\varepsilon\int_s^t z(\theta_{\tau-t}\omega)d\tau-\lambda(t-s)}ds.
\end{aligned} \quad (4.42)$$

Replacing t by $t+1$ and T by t, we have

$$\begin{aligned}
&c\int_t^{t+1} e^{\varepsilon(p-2)z(\theta_{s-t-1}\omega)+p\varepsilon\int_s^{t+1} z(\theta_{\tau-t-1}\omega)d\tau-\lambda(t+1-s)} \\
&\times \|v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|_{2p-2}^{2p-2}ds \\
&\le \|v(t,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|_p^p e^{p\varepsilon\int_t^{t+1} z(\theta_{s-t-1}\omega)ds-\lambda} \\
&\quad + c\int_t^{t+1} e^{-p\varepsilon z(\theta_{s-t-1}\omega)+p\varepsilon\int_s^{t+1} z(\theta_{\tau-t-1}\omega)d\tau-\lambda(t+1-s)}ds \\
&\le \|v(t,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|_p^p e^{p\varepsilon K(\omega)-\lambda} + ce^{2p\varepsilon K(\omega)}.
\end{aligned} \quad (4.43)$$

Note that

$$\begin{aligned}
&\int_t^{t+1} e^{\varepsilon(p-2)z(\theta_{s-t-1}\omega)+p\varepsilon\int_s^{t+1} z(\theta_{\tau-t-1}\omega)d\tau-\lambda(t+1-s)} \\
&\times \|v(s,\theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))\|_{2p-2}^{2p-2}ds \\
&\geq e^{-2p\varepsilon K(\omega)+2\varepsilon K(\omega)-\lambda}\int_t^{t+1}\|v(s,\theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))\|_{2p-2}^{2p-2}ds.
\end{aligned} \tag{4.44}$$

Thus together with (4.43)-(4.44), we obtain

$$\begin{aligned}
&\int_t^{t+1}\|u(s,\theta_{-t-1}\omega, u_0(\theta_{-t-1}\omega))\|_{2p-2}^{2p-2}ds \\
&= \int_t^{t+1} e^{\varepsilon z(\theta_{s-t-1}\omega)(2p-2)}\|v(s,\theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))\|_{2p-2}^{2p-2}ds \\
&\leq e^{(2p-2)\varepsilon K(\omega)}\int_t^{t+1}\|v(s,\theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))\|_{2p-2}^{2p-2}ds \\
&\leq ce^{5p\varepsilon K(\omega)-4\varepsilon K(\omega)}\|v(t,\theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))\|_p^p + ce^{6p\varepsilon K(\omega)-4\varepsilon K(\omega)+\lambda}.
\end{aligned} \tag{4.45}$$

Let $T(B,\omega)$ be the positive constant in Lemma 4.3. Then for $t \geq T(B,\omega)$, from (4.27), we have

$$\begin{aligned}
&\int_t^{t+1}\|u(s,\theta_{-t-1}\omega, u_0(\theta_{-t-1}\omega))\|_{2p-2}^{2p-2}ds \\
&\leq ce^{6p\varepsilon K(\omega)-4\varepsilon K(\omega)}(\varepsilon K(\omega)+1)(e^{6\varepsilon K(\omega)}M_\varepsilon(\omega)+1) + ce^{6p\varepsilon K(\omega)-4\varepsilon K(\omega)+\lambda} \\
&= ce^{c\varepsilon K(\omega)}(\varepsilon K(\omega)+1)(e^{c\varepsilon K(\omega)}M_\varepsilon(\omega)+1).
\end{aligned} \tag{4.46}$$

□

Lemma 4.5. *Let $0 < \varepsilon \leq 1, g \in L^2(\mathbb{R}^n)$ and (3.3)-(3.4) hold. Then for every $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and P-a.e. $\omega \in \Omega$, there is $T(B,\omega) > 0$, independent of ε, such that for all $u_0(\theta_{-t}\omega) \in B(\theta_{-t}\omega)$ and $t \geq T(B,\omega)$*

$$\|\nabla u(t,\theta_{-t}\omega, u_0(\theta_{-t}\omega))\|_p^p \leq ce^{c\varepsilon K(\omega)}(\varepsilon K(\omega)+1)(M_\varepsilon(\omega)+1). \tag{4.47}$$

Proof. Take the inner product of (3.12) with v_t in $L^2(\mathbb{R}^n)$, we obtain

$$\begin{aligned}
\|v_t\|^2 &= -e^{\varepsilon(p-2)z(\theta_t\omega)}\frac{1}{p}\frac{d}{dt}\|\nabla v\|_p^p - \lambda(v,v_t) + e^{-\varepsilon z(\theta_t\omega)}(f(x,u),v_t) \\
&\quad + e^{-\varepsilon z(\theta_t\omega)}(g(x),v_t) + \varepsilon z(\theta_t\omega)(v,v_t).
\end{aligned} \tag{4.48}$$

By (3.4), the Cauchy-Schwartz inequality and the Young inequality, we find that

$$\begin{aligned}
&|e^{-\varepsilon z(\theta_t\omega)}(f(x,u),v_t)| \\
&\leq e^{-2\varepsilon z(\theta_t\omega)}\|f(x,u)\|^2 + \frac{1}{4}\|v_t\|^2 \\
&\leq 2\alpha_2^2 e^{-2\varepsilon z(\theta_t\omega)}\|u\|_{2p-2}^{2p-2} + 2e^{-2\varepsilon z(\theta_t\omega)}\|\psi_2\|^2 + \frac{1}{4}\|v_t\|^2,
\end{aligned} \tag{4.49}$$

$$|e^{-\varepsilon z(\theta_t\omega)}(g(x),v_t)| \leq e^{-2\varepsilon z(\theta_t\omega)}\|g\|^2 + \frac{1}{4}\|v_t\|^2, \tag{4.50}$$

$$|(\varepsilon z(\theta_t\omega)-\lambda)(v,v_t)| \leq |\varepsilon z(\theta_t\omega)-\lambda|^2\|v\|^2 + \frac{1}{4}\|v_t\|^2. \tag{4.51}$$

It follows from (4.48)-(4.51) that

$$
\begin{aligned}
&\frac{d}{dt}\|\nabla v\|_p^p \\
&\leq 2p\alpha_2^2 e^{-p\varepsilon z(\theta_t\omega)}\|u\|_{2p-2}^{2p-2} + 2pe^{-p\varepsilon z(\theta_t\omega)}\|\psi_2\|^2 \\
&\quad + pe^{-p\varepsilon z(\theta_t\omega)}\|g\|^2 + pe^{-\varepsilon(p-2)z(\theta_t\omega)}|\varepsilon z(\theta_t\omega) - \lambda|^2\|v\|^2 \\
&= ce^{-p\varepsilon z(\theta_t\omega)}\|u\|_{2p-2}^{2p-2} + ce^{-p\varepsilon z(\theta_t\omega)} + ce^{\varepsilon(2-p)z(\theta_t\omega)}|\varepsilon z(\theta_t\omega) - \lambda|^2\|v\|^2.
\end{aligned}
\tag{4.52}
$$

Let $T_0(B,\omega)$ be the positive constant in Lemma 4.2, take $t \geq T_0(B,\omega)$ and $s \in (t, t+1)$. Integrate (4.52) over $(s, t+1)$ to get

$$
\begin{aligned}
&\|\nabla v(t+1,\omega,v_0(\omega))\|_p^p \\
&\leq \|\nabla v(s,\omega,v_0(\omega))\|_p^p + c\int_s^{t+1} e^{-p\varepsilon z(\theta_\tau\omega)}\|u(\tau,\omega,u_0(\omega))\|_{2p-2}^{2p-2}d\tau \\
&\quad + c\int_s^{t+1} e^{-p\varepsilon z(\theta_\tau\omega)}d\tau + c\int_s^{t+1} e^{\varepsilon(2-p)z(\theta_\tau\omega)}|\varepsilon z(\theta_\tau\omega) - \lambda|^2\|v(\tau,\omega,v_0(\omega))\|^2 d\tau.
\end{aligned}
\tag{4.53}
$$

Integrating with respect to s over $(t, t+1)$, it follows that

$$
\begin{aligned}
&\|\nabla v(t+1,\omega,v_0(\omega))\|_p^p \\
&\leq \int_t^{t+1} \|\nabla v(s,\omega,v_0(\omega))\|_p^p ds + c\int_t^{t+1} e^{-p\varepsilon z(\theta_\tau\omega)}\|u(\tau,\omega,u_0(\omega))\|_{2p-2}^{2p-2}d\tau \\
&\quad + c\int_t^{t+1} e^{-p\varepsilon z(\theta_\tau\omega)}d\tau + c\int_t^{t+1} e^{\varepsilon(2-p)z(\theta_\tau\omega)}|\varepsilon z(\theta_\tau\omega) - \lambda|^2\|v(\tau,\omega,v_0(\omega))\|^2 d\tau.
\end{aligned}
\tag{4.54}
$$

Replace ω by $\theta_{-t-1}\omega$ in the above inequality, we have

$$
\begin{aligned}
&\|\nabla v(t+1,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|_p^p \\
&\leq \int_t^{t+1} \|\nabla v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|_p^p ds \\
&\quad + c\int_t^{t+1} e^{-p\varepsilon z(\theta_{\tau-t-1}\omega)}\|u(\tau,\theta_{-t-1}\omega,u_0(\theta_{-t-1}\omega))\|_{2p-2}^{2p-2}d\tau \\
&\quad + c\int_t^{t+1} e^{-p\varepsilon z(\theta_{\tau-t-1}\omega)}d\tau \\
&\quad + c\int_t^{t+1} e^{\varepsilon(2-p)z(\theta_{\tau-t-1}\omega)}|\varepsilon z(\theta_{\tau-t-1}\omega) - \lambda|^2\|v(\tau,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2 d\tau \\
&\leq \int_t^{t+1} \|\nabla v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|_p^p ds \\
&\quad + ce^{p\varepsilon K(\omega)} + ce^{p\varepsilon K(\omega)}\int_t^{t+1} \|u(\tau,\theta_{-t-1}\omega,u_0(\theta_{-t-1}\omega))\|_{2p-2}^{2p-2}d\tau \\
&\quad + c(K^2(\omega)+1)\int_t^{t+1} e^{\varepsilon(2-p)z(\theta_{\tau-t-1}\omega)}\|v(\tau,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2 d\tau,
\end{aligned}
\tag{4.55}
$$

where we used $0 < \varepsilon \leq 1$.

We now estimate each part on the right. Firstly, by the Lemma 4.2, for $t \geq T_0(B,\omega)$,

$$
\begin{aligned}
&\int_t^{t+1} \|\nabla v(s,\theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))\|_p^p ds \\
&= \int_t^{t+1} e^{-p\varepsilon z(\theta_{s-t-1}\omega)} \|\nabla u(s,\theta_{-t-1}\omega, u_0(\theta_{-t-1}\omega))\|_p^p ds \\
&\leq e^{p\varepsilon K(\omega)} \int_t^{t+1} \|\nabla u(s,\theta_{-t-1}\omega, u_0(\theta_{-t-1}\omega))\|_p^p ds \\
&\leq e^{p\varepsilon K(\omega)} (1 + c e^{2p\varepsilon K(\omega)} M_\varepsilon(\omega)).
\end{aligned}
\tag{4.56}
$$

To estimate the last part, firstly, replace t by $\tau + t + 1$ and ω by $\theta_{-t-1}\omega$ in (4.7), we obtain

$$
\begin{aligned}
&\|v(\tau + t + 1, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)\|^2 \\
&\leq \|v_0(\theta_{-t-1}\omega)\|^2 e^{2\varepsilon \int_0^{\tau+t+1} z(\theta_{s-t-1}\omega) ds - \lambda(\tau+t+1)} \\
&\quad + c \int_0^{\tau+t+1} e^{-2\varepsilon z(\theta_{s-t-1}\omega) + 2\varepsilon \int_s^{\tau+t+1} z(\theta_{s'-t-1}\omega) ds' - \lambda(\tau+t+1-s)} ds.
\end{aligned}
\tag{4.57}
$$

Thus

$$
\begin{aligned}
&\int_t^{t+1} e^{\varepsilon(2-p) z(\theta_{\tau-t-1}\omega)} \|v(\tau, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))\|^2 d\tau \\
&\leq \int_{-1}^0 e^{\varepsilon(2-p) z(\theta_\tau\omega)} \|v(\tau + t + 1, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))\|^2 d\tau \\
&\leq \int_{-1}^0 \|v_0(\theta_{-t-1}\omega)\|^2 e^{2\varepsilon \int_0^{\tau+t+1} z(\theta_{s-t-1}\omega) ds - \lambda(\tau+t+1) + \varepsilon(2-p) z(\theta_\tau\omega)} d\tau \\
&+ c \int_{-1}^0 \int_0^{\tau+t+1} e^{\varepsilon(2-p) z(\theta_\tau\omega) - 2\varepsilon z(\theta_{s-t-1}\omega) + 2\varepsilon \int_s^{\tau+t+1} z(\theta_{s'-t-1}\omega) ds' - \lambda(\tau+t+1-s)} ds d\tau.
\end{aligned}
\tag{4.58}
$$

Since

$$
\begin{aligned}
&\int_{-1}^0 \|v_0(\theta_{-t-1}\omega)\|^2 e^{2\varepsilon \int_0^{\tau+t+1} z(\theta_{s-t-1}\omega) ds - \lambda(\tau+t+1) + \varepsilon(2-p) z(\theta_\tau\omega)} d\tau \\
&\leq \|u_0(\theta_{-t-1}\omega)\|^2 e^{-2\varepsilon z(\theta_{-t-1}\omega) - \lambda(t+1) + 2\varepsilon \int_{-t-1}^0 z(\theta_s\omega) ds} \\
&\quad \times \int_{-1}^0 e^{2\varepsilon \int_0^\tau z(\theta_s\omega) ds - \lambda\tau + \varepsilon(2-p) z(\theta_\tau\omega)} d\tau.
\end{aligned}
\tag{4.59}
$$

For $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and $u_0(\theta_{-t}(\omega)) \in B(\theta_{-t}(\omega))$, similar to (4.10), there exists $T_1(B,\omega) > 0$, independent of ε, such that for all $t \geq T_1(B,\omega)$,

$$
\begin{aligned}
&\|u_0(\theta_{-t-1}\omega)\|^2 e^{-2\varepsilon z(\theta_{-t-1}\omega) - \lambda(t+1) + 2\varepsilon \int_{-t-1}^0 z(\theta_s\omega) ds} \\
&\times \int_{-1}^0 e^{2\varepsilon \int_0^\tau z(\theta_s\omega) ds - \lambda\tau + \varepsilon(2-p) z(\theta_\tau\omega)} d\tau \\
&\leq \|u_0(\theta_{-t-1}\omega)\|^2 e^{-\frac{\lambda}{2}(t+1)} \leq 1.
\end{aligned}
\tag{4.60}
$$

For the second part of (4.58),

$$
\begin{aligned}
&c\int_{-1}^{0}\int_{0}^{\tau+t+1} e^{\varepsilon(2-p)z(\theta_\tau\omega)-2\varepsilon z(\theta_{s-t-1}\omega)+2\varepsilon\int_s^{\tau+t+1} z(\theta_{s'-t-1}\omega)ds'-\lambda(\tau+t+1-s)}dsd\tau\\
&= c\int_{-1}^{0}\int_{-t-1}^{\tau} e^{\varepsilon(2-p)z(\theta_\tau\omega)-2\varepsilon z(\theta_s\omega)+2\varepsilon\int_s^{\tau} z(\theta_{s'}\omega)ds'-\lambda(\tau-s)}dsd\tau\\
&\le c\int_{-1}^{0} e^{2\varepsilon\int_0^{\tau} z(\theta_{s'}\omega)ds'-\lambda\tau+\varepsilon(2-p)z(\theta_\tau\omega)}M_\varepsilon(\omega)d\tau\\
&\le ce^{p\varepsilon K(\omega)}M_\varepsilon(\omega).
\end{aligned}
\tag{4.61}
$$

Thus (4.58)-(4.61) imply that for all $t\ge T_1(B,\omega)$

$$
\begin{aligned}
&\int_t^{t+1} e^{\varepsilon(2-p)z(\theta_{\tau-t-1}\omega)}\|v(\tau,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2d\tau\\
&\le 1+ce^{p\varepsilon K(\omega)}M_\varepsilon(\omega).
\end{aligned}
\tag{4.62}
$$

Let $T(B,\omega)=max\{T_0(B,\omega),T_1(B,\omega)\}$, it follows from (3.10), (3.11), (4.39), (4.55)-(4.56), (4.62) that for all $t\ge T(B,\omega)$,

$$
\begin{aligned}
&\|\nabla u(t+1,\theta_{-t-1},u_0(\theta_{-t-1}\omega))\|_p^p\\
&= e^{p\varepsilon z(\omega)}\|\nabla v(t+1,\theta_{-t-1},v_0(\theta_{-t-1}\omega))\|_p^p\\
&\le e^{p\varepsilon z(\omega)}[ce^{p\varepsilon K(\omega)}(1+e^{2p\varepsilon K(\omega)}M_\varepsilon(\omega))+ce^{p\varepsilon K(\omega)}\\
&\quad + ce^{c\varepsilon K(\omega)}(\varepsilon K(\omega)+1)(1+e^{c\varepsilon K(\omega)}M_\varepsilon(\omega))+c(K^2(\omega)+1)(1+e^{p\varepsilon K(\omega)}M_\varepsilon(\omega))]\\
&\le ce^{c\varepsilon K(\omega)}(1+\varepsilon K(\omega))(1+M_\varepsilon(\omega)).
\end{aligned}
\tag{4.63}
$$

□

Lemma 4.6. *Let $0<\varepsilon\le 1, g\in L^2(\mathbb{R}^n)$ and (3.3)-(3.4) hold. Then for every $\eta>0, B=\{B(\omega)\}_{\omega\in\Omega}\in\mathcal{D}$ and P-a.e. $\omega\in\Omega$, there is $T(B,\omega,\eta)>0$ and $R(\omega,\eta)>0$, independent of ε, such that for all $u_0(\theta_{-t}\omega)\in B(\theta_{-t}\omega)$ and $t\ge T(B,\omega,\eta)$*

$$
\int_{|x|\ge R}|u(t,\theta_{-t}\omega,u_0(\theta_{-t}\omega))|^2dx\le\eta. \tag{4.64}
$$

Proof. Choose a smooth function ξ, such that $0\le\xi(s)\le 1$ for $s\in\mathbb{R}^+$, and

$$
\xi(s)=\begin{cases}0, & 0\le s\le 1,\\ 1, & s\ge 2.\end{cases}
$$

Then, there exists a constant M, such that $|\xi'(s)|\le M$ for $s\in\mathbb{R}^+$. Taking the inner product of (3.12) with $\xi(\frac{|x|^2}{r^2})v$ in $L^2(\mathbb{R}^n)$, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n}\xi(\frac{|x|^2}{r^2})|v|^2dx\\
&= 2e^{\varepsilon(p-2)z(\theta_t\omega)}\int_{\mathbb{R}^n}(\mathrm{div}\,|\nabla v|^{p-2}\nabla v)\xi(\frac{|x|^2}{r^2})vdx+2(\varepsilon z(\theta_t\omega)-\lambda)\int_{\mathbb{R}^n}\xi(\frac{|x|^2}{r^2})|v|^2dx\\
&\quad + 2e^{-\varepsilon z(\theta_t\omega)}\int_{\mathbb{R}^n}f(x,u)\xi(\frac{|x|^2}{r^2})vdx+2e^{-\varepsilon z(\theta_t\omega)}\int_{\mathbb{R}^n}g(x)\xi(\frac{|x|^2}{r^2})vdx.
\end{aligned}
\tag{4.65}
$$

We now estimate the terms in (4.65) as follows. First we have

$$
\begin{aligned}
&2e^{\varepsilon(p-2)z(\theta_t\omega)}\int_{\mathbb{R}^n}(\operatorname{div}|\nabla v|^{p-2}\nabla v)\xi(\frac{|x|^2}{r^2})vdx\\
&=-2e^{\varepsilon(p-2)z(\theta_t\omega)}\int_{\mathbb{R}^n}\sum_{i=1}^n|\nabla v|^{p-2}\frac{\partial v}{\partial x_i}\xi(\frac{|x|^2}{r^2})\frac{\partial v}{\partial x_i}dx\\
&\quad-2e^{\varepsilon(p-2)z(\theta_t\omega)}\int_{\mathbb{R}^n}\sum_{i=1}^n|\nabla v|^{p-2}\frac{\partial v}{\partial x_i}\xi'(\frac{|x|^2}{r^2})\frac{2x_i}{r^2}vdx\\
&=-2e^{\varepsilon(p-2)z(\theta_t\omega)}\int_{\mathbb{R}^n}|\nabla v|^p\xi(\frac{|x|^2}{r^2})dx\\
&\quad-2e^{\varepsilon(p-2)z(\theta_t\omega)}\int_{r\le|x|\le\sqrt{2}r}v|\nabla v|^{p-2}\xi'(\frac{|x|^2}{r^2})\frac{2}{r^2}x\cdot\nabla vdx\\
&\le-2e^{\varepsilon(p-2)z(\theta_t\omega)}\int_{\mathbb{R}^n}|\nabla v|^p\xi(\frac{|x|^2}{r^2})dx\\
&\quad+\frac{4\sqrt{2}M}{r}e^{\varepsilon(p-2)z(\theta_t\omega)}\int_{r\le|x|\le\sqrt{2}r}|v||\nabla v|^{p-1}dx\\
&\le-2e^{\varepsilon(p-2)z(\theta_t\omega)}\int_{\mathbb{R}^n}|\nabla v|^p\xi(\frac{|x|^2}{r^2})dx+\frac{4\sqrt{2}M}{r}e^{\varepsilon(p-2)z(\theta_t\omega)}(\|v\|_p^p+\|\nabla v\|_p^p).
\end{aligned}
\tag{4.66}
$$

For the third term of (4.65), using (3.3), we have

$$
\begin{aligned}
&2e^{-\varepsilon z(\theta_t\omega)}\int_{\mathbb{R}^n}f(x,u)\xi(\frac{|x|^2}{r^2})vdx\\
&=2e^{-2\varepsilon z(\theta_t\omega)}\int_{\mathbb{R}^n}f(x,u)u\xi(\frac{|x|^2}{r^2})dx\\
&\le-2e^{-2\varepsilon z(\theta_t\omega)}\alpha_1\int_{\mathbb{R}^n}|u|^p\xi(\frac{|x|^2}{r^2})dx+2e^{-2\varepsilon z(\theta_t\omega)}\int_{\mathbb{R}^n}|\psi_1(x)|\xi(\frac{|x|^2}{r^2})dx.
\end{aligned}
\tag{4.67}
$$

For the last term of (4.65), we have

$$
\begin{aligned}
&2e^{-\varepsilon z(\theta_t\omega)}\int_{\mathbb{R}^n}g(x)\xi(\frac{|x|^2}{r^2})vdx\\
&\le\lambda\int_{\mathbb{R}^n}\xi(\frac{|x|^2}{r^2})|v|^2dx+\frac{1}{\lambda}e^{-2\varepsilon z(\theta_t\omega)}\int_{\mathbb{R}^n}g^2(x)\xi(\frac{|x|^2}{r^2})dx.
\end{aligned}
\tag{4.68}
$$

From (4.65)-(4.68), it follows that

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n}\xi(\frac{|x|^2}{r^2})|v|^2dx\\
&\le(2\varepsilon z(\theta_t\omega)-\lambda)\int_{\mathbb{R}^n}\xi(\frac{|x|^2}{r^2})|v|^2dx+\frac{c}{r}e^{\varepsilon(p-2)z(\theta_t\omega)}(\|\nabla v\|_p^p+\|v\|_p^p)\\
&\quad+2e^{-2\varepsilon z(\theta_t\omega)}\int_{\mathbb{R}^n}|\psi_1(x)|\xi(\frac{|x|^2}{r^2})dx+\frac{1}{\lambda}e^{-2\varepsilon z(\theta_t\omega)}\int_{\mathbb{R}^n}g^2(x)\xi(\frac{|x|^2}{r^2})dx.
\end{aligned}
\tag{4.69}
$$

By the Gronwall Lemma, for any $t \geq 0$,

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \xi(\frac{|x|^2}{r^2})|v(t,\omega,v_0(\omega))|^2dx \\
&\leq e^{2\varepsilon\int_0^t z(\theta_\tau\omega)d\tau-\lambda t}\int_{\mathbb{R}^n} \xi(\frac{|x|^2}{r^2})|v_0(\omega)|^2dx \\
&\quad + \frac{c}{r}\int_0^t e^{\varepsilon(p-2)z(\theta_s\omega)+2\varepsilon\int_s^t z(\theta_\tau\omega)d\tau-\lambda(t-s)}(\|\nabla v\|_p^p+\|v\|_p^p)ds \\
&\quad + 2\int_0^t e^{-2\varepsilon z(\theta_s\omega)+2\varepsilon\int_s^t z(\theta_\tau\omega)d\tau-\lambda(t-s)}\int_{\mathbb{R}^n} \xi(\frac{|x|^2}{r^2})|\psi_1(x)|\,dx\,ds \\
&\quad + \frac{1}{\lambda}\int_0^t e^{-2\varepsilon z(\theta_s\omega)+2\varepsilon\int_s^t z(\theta_\tau\omega)d\tau-\lambda(t-s)}\int_{\mathbb{R}^n} \xi(\frac{|x|^2}{r^2})g^2(x)\,dx\,ds.
\end{aligned}
\tag{4.70}
$$

By replacing ω by $\theta_{-t}\omega$, it follows that

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \xi(\frac{|x|^2}{r^2})|v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|^2dx \\
&\leq e^{2\varepsilon\int_0^t z(\theta_{\tau-t}\omega)d\tau-\lambda t}\int_{\mathbb{R}^n} \xi(\frac{|x|^2}{r^2})|v_0(\theta_{-t}\omega)|^2dx \\
&\quad + \frac{c}{r}\int_0^t e^{\varepsilon(p-2)z(\theta_{s-t}\omega)+2\varepsilon\int_s^t z(\theta_{\tau-t}\omega)d\tau-\lambda(t-s)}(\|\nabla v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|_p^p \\
&\quad + \|v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|_p^p)ds \\
&\quad + \int_0^t e^{-2\varepsilon z(\theta_{s-t}\omega)+2\varepsilon\int_s^t z(\theta_{\tau-t}\omega)d\tau-\lambda(t-s)}\int_{\mathbb{R}^n} \xi(\frac{|x|^2}{r^2})(2|\psi_1(x)|+\frac{1}{\lambda}g^2(x))\,dx\,ds.
\end{aligned}
\tag{4.71}
$$

We then estimate each term on the right-hand side of (4.71). Firstly, Since $B \in \mathcal{D}$ and $u_0(\theta_{-t}\omega) \in B(\theta_{-t}\omega)$, it follows from (3.7), (3.8) that there exists $T_1(B,\omega,\eta) > 0$, independent of ε, such that for all $t \geq T_1(B,\omega,\eta)$

$$
\begin{aligned}
&e^{2\varepsilon\int_0^t z(\theta_{\tau-t}\omega)d\tau-\lambda t}\int_{\mathbb{R}^n} \xi(\frac{|x|^2}{r^2})|v_0(\theta_{-t}\omega)|^2dx \\
&\leq e^{2\varepsilon\int_0^t z(\theta_{\tau-t}\omega)d\tau-\lambda t}\|v_0(\theta_{-t}\omega)\|^2 \\
&= e^{2\varepsilon\int_{-t}^0 z(\theta_\tau\omega)d\tau-\lambda t-2\varepsilon z(\theta_{-t}\omega)}\|u_0(\theta_{-t}\omega)\|^2 \\
&\leq e^{-\frac{\lambda}{2}t}\|u_0(\theta_{-t}\omega)\|^2 < \frac{\eta}{3}.
\end{aligned}
\tag{4.72}
$$

To estimate the second term on the right-hand side of (4.71), from (4.18) with T replaced by 0, we have

$$
\begin{aligned}
&\int_0^t e^{-2\varepsilon z(\theta_{s-t}\omega)+2\varepsilon\int_s^t z(\theta_{\tau-t}\omega)d\tau+\lambda(s-t)}\|u(s,\theta_{-t}\omega,u_0(\theta_{-t}\omega))\|_p^p \\
&+ \int_0^t e^{\varepsilon(p-2)z(\theta_{s-t}\omega)+2\varepsilon\int_s^t z(\theta_{\tau-t}\omega)d\tau+\lambda(s-t)}\|\nabla v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|_p^pds \\
&= \int_0^t e^{\varepsilon(p-2)z(\theta_{s-t}\omega)+2\varepsilon\int_s^t z(\theta_{\tau-t}\omega)d\tau+\lambda(s-t)}(\|v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|_p^p \\
&\quad + \|\nabla v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|_p^p)ds
\end{aligned}
$$

$$\le c\|v_0(\theta_{-t}\omega)\|^2 e^{2\varepsilon\int_{-t}^0 z(\theta_s\omega)ds-\lambda t} + c\int_{-t}^0 e^{-2\varepsilon z(\theta_s\omega)+2\varepsilon\int_s^0 z(\theta_\tau\omega)d\tau+\lambda s}ds.$$

Thus

$$\begin{aligned} &\frac{c}{r}\int_0^t e^{\varepsilon(p-2)z(\theta_{s-t}\omega)+2\varepsilon\int_s^t z(\theta_{\tau-t}\omega)d\tau-\lambda(t-s)}(\|\nabla v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|_p^p \\ &+\|v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|_p^p)ds \\ &\le \frac{c}{r}[\|u_0(\theta_{-t}\omega)\|^2 e^{-2\varepsilon z(\theta_{-t}\omega)+2\varepsilon\int_{-t}^0 z(\theta_\tau\omega)d\tau-\lambda t}+M_\varepsilon(\omega)]. \end{aligned} \tag{4.73}$$

Since $B\in\mathcal{D}, u_0(\theta_{-t}\omega)\in B(\theta_{-t}\omega)$, together with (3.7)-(3.9), it follows that there exists $T_2(B,\omega,\eta)>0$ and $R_1(\eta,\omega)>0$, both independent of ε, such that for $t\ge T_2(B,\omega,\eta)$ and $r\ge R_1(\eta,\omega)$,

$$\begin{aligned} &\frac{c}{r}\int_0^t e^{\varepsilon(p-2)z(\theta_{s-t}\omega)+2\varepsilon\int_s^t z(\theta_{\tau-t}\omega)d\tau-\lambda(t-s)} \\ &\times(\|\nabla v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|_p^p+\|v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|_p^p)ds<\frac{\eta}{3}. \end{aligned} \tag{4.74}$$

For the last term of (4.71), since $g\in L^2(\mathbb{R}^n)$ and $\psi_1\in L^1(\mathbb{R}^n)$, there exists $R_2(\omega,\eta)>0$, independent of ε, such that

$$\begin{aligned} &\int_0^t e^{-2\varepsilon z(\theta_{s-t}\omega)+2\varepsilon\int_s^t z(\theta_{\tau-t}\omega)d\tau-\lambda(t-s)}\int_{\mathbb{R}^n}\xi(\frac{|x|^2}{r^2})(2|\psi_1(x)|+\frac{1}{\lambda}g^2(x))\,dx\,ds \\ &\le\int_0^t e^{-2\varepsilon z(\theta_{s-t}\omega)+2\varepsilon\int_s^t z(\theta_{\tau-t}\omega)d\tau-\lambda(t-s)}ds(2\int_{|x|\ge r}|\psi_1(x)|dx+\frac{1}{\lambda}\int_{|x|\ge r}g^2(x)dx) \\ &\le M_\varepsilon(\omega)(2\int_{|x|\ge r}|\psi_1(x)|dx+\frac{1}{\lambda}\int_{|x|\ge r}g^2(x))dx<\frac{\eta}{3}. \end{aligned} \tag{4.75}$$

Let

$$T(B,\omega,\eta)=\max\{T_1,T_2\},\quad R(\omega,\eta)=\max\{R_1,R_2\}.$$

Note that $T(B,\omega,\eta),R(\omega,\eta)$ are both independent of ε. And for $t\ge T(B,\omega,\eta)$ and $r\ge R(\omega,\eta)$, it follows from (4.71), (4.72), (4.74) and (4.75) that

$$\int_{|x|>r}|v(t,\theta_{-t\omega},v_0(\theta_{-t}\omega))|^2dx=\int_{\mathbb{R}^n}\xi(\frac{|x|^2}{r^2})|v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|^2dx<\eta, \tag{4.76}$$

which along with (3.11) implies the Lemma. □

Lemma 4.7. *Let $0<\varepsilon\le 1$, $g\in L^2(\mathbb{R}^n)$ and (3.3)-(3.4) hold. Then the random dynamical system generated by (3.1) is $\mathcal{D}$-pullback asymptotically compact in $L^2(\mathbb{R}^n)$, that is, for P-a.e. $\omega\in\Omega$, the sequence $\{\Phi(t_n,\theta_{-t_n}\omega,u_{0,n}(\theta_{-t_n}\omega))\}$ has a convergent subsequence in $L^2(\mathbb{R}^n)$ provided $t_n\to\infty,\{B(\omega)\}\in\mathcal{D}$ and $u_{0,n}(\theta_{-t_n}\omega)\in B(\theta_{-t_n}\omega)$.*

Proof. Let $t_n\to\infty,\{B(\omega)\}\in\mathcal{D}$ and $u_{0,n}(\theta_{-t_n}\omega)\in B(\theta_{-t_n}\omega)$. Then by Lemma 4.1, for P-a.e. $\omega\in\Omega$, we have that

$$\{\Phi(t_n,\theta_{-t_n}\omega,u_{0,n}(\theta_{-t_n}\omega))\}_{n=1}^\infty \text{ is bounded in } L^2(\mathbb{R}^n). \tag{4.77}$$

Hence, there is $\zeta\in L^2(\mathbb{R}^n)$ such that, up to a subsequence which is still denoted by $\{\Phi(t_n,\theta_{-t_n}\omega,u_{0,n}(\theta_{-t_n}\omega))\}$, such that

$$\Phi(t_n,\theta_{-t_n}\omega,u_{0,n}(\theta_{-t_n}\omega))\to\zeta\quad\text{weakly in }\ L^2(\mathbb{R}^n). \tag{4.78}$$

Next, we prove the weak convergence of (4.78) is actually strong convergence.

Given $\eta > 0$, by Lemma 4.6, there is $T_1(B,\omega,\eta) > 0$ and $R_1(\eta,\omega) > 0$, independent of ε, such that for all $t \geq T_1$

$$\int_{|x|\geq R_1} |\Phi(t,\theta_{-t}\omega,u_0(\theta_{-t}\omega))|^2 \leq \eta. \tag{4.79}$$

Since $t_n \to \infty$, there is N_1 such that $t_n \geq T_1$ for every $n \geq N_1$. Hence, it follows from (4.79) that for all $n \geq N_1$

$$\int_{|x|\geq R_1} |\Phi(t_n,\theta_{-t_n}\omega,u_{0,n}(\theta_{-t_n}\omega))|^2 \leq \eta. \tag{4.80}$$

Note that $\zeta \in L^2(\mathbb{R}^n)$, therefore there exists $R_2(\eta) > 0$ such that

$$\int_{|x|\geq R_2} |\zeta(x)|^2 \leq \eta. \tag{4.81}$$

Let $R_3 = \max\{R_1,R_2\}$, denote $Q_{R_3} = \{x \in \mathbb{R}^n, |x| \leq R_3\}$, from Lemma 4.5, there is $T_2 = T_2(B,\omega) > 0$, such that for all $t \geq T_2$,

$$\|\Phi(t,\theta_{-t}\omega,u_0(\theta_{-t}\omega))\|^p_{W^{1,p}(Q_{R_3})} \leq ce^{c\varepsilon K(\omega)}(1+\varepsilon K(\omega))(1+M_\varepsilon(\omega)). \tag{4.82}$$

Then, by the compactness of the embedding $W^{1,p}(Q_{R_3}) \hookrightarrow L^2(Q_{R_3})$, from (4.82) it follows that, up to a subsequence,

$$\Phi(t_n,\theta_{-t_n}\omega,u_{0,n}(\theta_{-t_n}\omega)) \to \zeta \quad \text{strongly in } L^2(Q_{R_3}), \tag{4.83}$$

which implies that for the given $\eta > 0$, there exists $N_2 > 0$ such that for all $n \geq N_2$,

$$\|\Phi(t_n,\theta_{-t_n}\omega,u_{0,n}(\theta_{-t_n}\omega)) - \zeta\|^2_{L^2(Q_{R_3})} \leq \eta. \tag{4.84}$$

Let $N = \max\{N_1,N_2\}$. By (4.80) (4.81) (4.84), we find that for all $n \geq N$,

$$\begin{aligned}
&\|\Phi(t_n,\theta_{-t_n}\omega,u_{0,n}(\theta_{-t_n}\omega)) - \zeta\|^2_{L^2(\mathbb{R}^n)} \\
&= \int_{|x|\leq R_3} |\Phi(t_n,\theta_{-t_n}\omega,u_{0,n}(\theta_{-t_n}\omega)) - \zeta|^2 dx \\
&\quad + \int_{|x|\geq R_3} |\Phi(t_n,\theta_{-t_n}\omega,u_{0,n}(\theta_{-t_n}\omega)) - \zeta|^2 dx \\
&\leq \eta + 2\int_{|x|\geq R_3} |\Phi(t_n,\theta_{-t_n}\omega,u_{0,n}(\theta_{-t_n}\omega))|^2 dx + 2\int_{|x|\geq R_3} |\zeta(x)|^2 dx \\
&\leq 5\eta,
\end{aligned} \tag{4.85}$$

which shows that

$$\Phi(t_n,\theta_{-t_n}\omega,u_{0,n}(\theta_{-t_n}\omega)) \to \zeta \quad \text{strongly in } L^2(\mathbb{R}^n) \tag{4.86}$$

as desired. □

It is now sufficient to show the existence of a $\mathcal{D}$-random attractor for Φ in $L^2(\mathbb{R}^n)$.

Theorem 4.8. *Let $0 < \varepsilon \leq 1, g \in L^2(\mathbb{R}^n)$ and (3.3)-(3.4) hold. Then the random dynamical system Φ generated by (3.1) has a unique $\mathcal{D}$-random attractor in $L^2(\mathbb{R}^n)$.*

Proof. Notice that Φ has a closed random absorbing set $\{K(\omega)\}_{\omega\in\Omega}$ in $\mathcal{D}$ by Lemma 4.1, and is $\mathcal{D}$-pullback asymptotically compact in $L^2(\mathbb{R}^n)$ by Lemma 4.7. Hence the existence of a unique $\mathcal{D}$-random attractor for Φ follows from Proposition 2.8 immediately. □

5. Upper semicontinuity of random attractors for p-Laplacian equation on $\mathbb{R}^n$

In this section, we prove the upper semicontinuity of random attractors for the p-Laplacian equation defined on $\mathbb{R}^n$ when the stochastic perturbations approach zero. We may further assume that the nonlinear function f satisfies, for all $x \in \mathbb{R}^n$ and $s \in \mathbb{R}$

$$\frac{\partial f}{\partial s}(x,s) \le \beta, \tag{5.1}$$

$$|\frac{\partial f}{\partial s}(x,s)| \le \alpha_3 |s|^{p-2} + \psi_3(x), \tag{5.2}$$

where α_3, β are positive constants, $\psi_3 \in L^\infty(\mathbb{R}^n)$ if $p=2$ and $\psi_3 \in L^{\frac{p}{p-2}}(\mathbb{R}^n)$ if $p>2$.

To indicate dependence of solutions on ε, in this section, we write the solution of problem (3.1)-(3.2) as u^ε, and the corresponding cocycle as Φ_ε. And we denote the solution and the semigroup of the deterministic equation

$$\frac{du}{dt} - \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda u = f(x,u) + g(x), \quad x \in \mathbb{R}^n,\ t>0 \tag{5.3}$$

by u and Φ_0 respectively.

Note that the existence of the global attractor $\mathcal{A}_0$ in $L^2(\mathbb{R}^n)$ for the deterministic system (5.3) can similarly be achieved by the discussion in section 4.

Lemma 5.1. *Let $0 < \varepsilon \le 1, g \in L^2(\mathbb{R}^n)$ and (3.3)-(3.4) hold. Then the union $\cup_{0<\varepsilon\le 1}\mathcal{A}_\varepsilon(\omega)$ is precompact in $L^2(\mathbb{R}^n)$.*

Proof. Given $\eta > 0$, we want to show that the set $\cup_{0<\varepsilon\le 1}\mathcal{A}_\varepsilon(\omega)$ has a finite covering of balls of radii less than η. From Lemma 4.1

$$D_\varepsilon(\omega) = \{u \in L^2(\mathbb{R}^n) : \|u\|^2 \le 1 + ce^{2\varepsilon z(\omega)} M_\varepsilon(\omega)\} \tag{5.4}$$

is a closed and tempered random absorbing set for Φ_ε in $L^2(\mathbb{R}^n)$. Let

$$\begin{aligned} &D(\omega) \\ &= \Big\{ \quad u \in L^2(\mathbb{R}^n) : \|u\|^2 \le 1 + ce^{2|z(\omega)|} \int_{-\infty}^0 e^{2|z(\theta_s\omega)| + 2|\int_s^0 z(\theta_\tau\omega)d\tau| + \lambda s} ds < +\infty \Big\}. \end{aligned}$$

It is easy to show that $D(\omega)$ is also a tempered set, that is, $\{D(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$.

By Lemma 4.6, we find that, given $\eta > 0$ and P-a.e.$\omega \in \Omega$, there is $T(B,\omega,\eta) > 0$ and $R(\omega,\eta) > 0$, independent of ε, such that for all $u_0(\theta_{-t}(\omega)) \in D(\theta_{-t}(\omega))$ and $t \ge T(B,\omega,\eta)$

$$\int_{|x|\ge R} |u(t,\theta_{-t}\omega, u_0(\theta_{-t}\omega))|^2 dx \le \frac{\eta^2}{16}. \tag{5.5}$$

For $\forall u_0(\theta_{-t}(\omega)) \in \mathcal{A}_\varepsilon(\theta_{-t}(\omega))$, note that

$$\cup_{0<\varepsilon\le 1}\, \mathcal{A}_\varepsilon(\omega) \subset \cup_{0<\varepsilon\le 1} D_\varepsilon(\omega) \subset D(\omega), \tag{5.6}$$

which implies that $u_0(\theta_{-t}(\omega)) \in D(\theta_{-t}\omega)$. It follows that for every $0 < \varepsilon \le 1$, P-a.e. $\omega \in \Omega, t \ge T$ and for all $u_0(\theta_{-t}(\omega)) \in \mathcal{A}_\varepsilon(\theta_{-t}(\omega))$

$$\int_{|x|\ge R} |u(t,\theta_{-t}\omega, u_0(\theta_{-t}\omega))|^2 dx \le \frac{\eta^2}{16}, \tag{5.7}$$

which along with the invariance of $\{\mathcal{A}_\varepsilon(\omega)\}_{\omega\in\Omega}$ shows that for P-a.e. $\omega \in \Omega$

$$\int_{|x|\geq R} |\widetilde{u}(x)|^2 dx \leq \frac{\eta^2}{16}, \quad \forall\, \widetilde{u} \in \cup_{0<\varepsilon\leq 1}\mathcal{A}_\varepsilon(\omega). \tag{5.8}$$

Let

$$Q_R = \{x \in \mathbb{R}^n : |x| < R\}, \quad Q_R^c = \mathbb{R}^n - Q_R. \tag{5.9}$$

Then (5.8) implies that for P-a.e. $\omega \in \Omega$,

$$\|\widetilde{u}(x)\|_{L^2(Q_R^c)} \leq \frac{\eta}{4}, \quad \forall\, \widetilde{u} \in \cup_{0<\varepsilon\leq 1}\mathcal{A}_\varepsilon(\omega). \tag{5.10}$$

On the other hand, by Lemma 4.5, we find that, for every $0 < \varepsilon \leq 1$ and P-a.e. $\omega \in \Omega$, there exists $T(D,\omega) > 0$, independent of ε, such that for all $t \geq T(D,\omega)$,

$$\|\Phi_\varepsilon(t, \theta_{-t}\omega, D(\theta_{-t}\omega))\|^p_{W^{1,p}(Q_{R_3})} \leq c e^{c\varepsilon K(\omega)}(1 + \varepsilon K(\omega))(1 + M_\varepsilon(\omega)), \tag{5.11}$$

which together with (5.6) implies that the set $\cup_{0<\varepsilon\leq 1}\mathcal{A}_\varepsilon(\omega)$ is bounded in $W^{1,p}(Q_R)$ for P-a.e. $\omega \in \Omega$. By the compactness of embedding $W^{1,p}(Q_R) \hookrightarrow L^2(Q_R)$, we find that, for the given η, the set $\cup_{0<\varepsilon\leq 1}\mathcal{A}_\varepsilon(\omega)$ has a finite covering of balls of radii less than $\frac{\eta}{4}$ in $L^2(Q_R)$. This along with (5.10) shows that $\cup_{0<\varepsilon\leq 1}\mathcal{A}_\varepsilon(\omega)$ has a finite covering of balls of radii less than η in $L^2(\mathbb{R}^n)$. $\square$

Lemma 5.2. *Let $g \in L^2(\mathbb{R}^n)$, (3.3)-(3.4) and (5.1)-(5.2) hold. Given $0 < \varepsilon \leq 1$, let u^ε and u be the solutions of equation (3.1) and (5.3) with initial conditions u_0^ε and u_0, respectively. Then for P-a.e. $\omega \in \Omega, u_0^\varepsilon \to u_0$ $(\varepsilon \downarrow 0)$ and $t \geq 0$, we have*

$$\lim_{\varepsilon\downarrow 0} u^\varepsilon(t,\omega,u_0^\varepsilon) = u(t,\omega,u_0). \tag{5.12}$$

Proof. Let

$$v^\varepsilon(t,\omega,v_0^\varepsilon) = e^{-\varepsilon z(\theta_t\omega)} u^\varepsilon(t,\omega,u_0^\varepsilon), \quad W = v^\varepsilon(t,\omega,v_0^\varepsilon) - u(t,\omega,u_0).$$

Then together with (3.12) and (5.3), we obtain

$$\begin{aligned} &\frac{\partial W}{\partial t} + e^{\varepsilon(p-2)z(\theta_t\omega)} A v^\varepsilon - Au + \lambda W \\ &= e^{-\varepsilon z(\theta_t\omega)} f(x,u^\varepsilon) - f(x,u) + (e^{-\varepsilon z(\theta_t\omega)} - 1) g(x) + \varepsilon v^\varepsilon z(\theta_t\omega), \end{aligned} \tag{5.13}$$

where $A\zeta = -\operatorname{div}(|\nabla\zeta|^{p-2}\nabla\zeta)$.

Taking the inner product of (5.13) with W in $L^2(\mathbb{R}^n)$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|W\|^2 \\ &= -(e^{\varepsilon(p-2)z(\theta_t\omega)} A v^\varepsilon - Au, W) - \lambda\|W\|^2 + (e^{-\varepsilon z(\theta_t\omega)} f(x,u^\varepsilon) - f(x,u), W) \\ &\quad + ((e^{-\varepsilon z(\theta_t\omega)} - 1) g(x), W) + \varepsilon z(\theta_t\omega)(v^\varepsilon, W). \end{aligned} \tag{5.14}$$

By the property of p-Laplacian operator for $p \geq 2$ and the Young inequality, we obtain

$$\begin{aligned} &-(e^{\varepsilon(p-2)z(\theta_t\omega)} A v^\varepsilon - Au, W) \\ &= -e^{\varepsilon(p-2)z(\theta_t\omega)}(Av^\varepsilon - Au, v^\varepsilon - u) - (e^{\varepsilon(p-2)z(\theta_t\omega)} - 1)(Au, W) \\ &\leq |e^{\varepsilon(p-2)z(\theta_t\omega)} - 1||(Au, v^\varepsilon - u)| \\ &\leq c|e^{\varepsilon(p-2)z(\theta_t\omega)} - 1|(\|\nabla u\|_p^p + \|\nabla v^\varepsilon\|_p^p). \end{aligned} \tag{5.15}$$

For the nonlinear term on the right-hand side of (5.14), by (3.4), (5.1) and (5.2),

$$
\begin{aligned}
&(e^{-\varepsilon z(\theta_t\omega)} f(x,u^\varepsilon) - f(x,u), W) \\
&= e^{-\varepsilon z(\theta_t\omega)} \int_{\mathbb{R}^n} (f(x,u^\varepsilon) - f(x,u)) W dx + (e^{-\varepsilon z(\theta_t\omega)} - 1) \int_{\mathbb{R}^n} f(x,u) W dx \\
&= e^{-\varepsilon z(\theta_t\omega)} \int_{\mathbb{R}^n} \frac{\partial f}{\partial s}(x,s)(u^\varepsilon - u) W dx + (e^{-\varepsilon z(\theta_t\omega)} - 1) \int_{\mathbb{R}^n} f(x,u) W dx \\
&= e^{-\varepsilon z(\theta_t\omega)} \int_{\mathbb{R}^n} \frac{\partial f}{\partial s}(x,s)[e^{\varepsilon z(\theta_t\omega)} v^\varepsilon - e^{\varepsilon z(\theta_t\omega)} u + (e^{\varepsilon z(\theta_t\omega)} - 1)u] W dx \\
&\quad + (e^{-\varepsilon z(\theta_t\omega)} - 1) \int_{\mathbb{R}^n} f(x,u) W dx \\
&= \int_{\mathbb{R}^n} \frac{\partial f}{\partial s}(x,s) W^2 dx + (1 - e^{-\varepsilon z(\theta_t\omega)}) \int_{\mathbb{R}^n} \frac{\partial f}{\partial s}(x,s) u W dx \\
&\quad + (e^{-\varepsilon z(\theta_t\omega)} - 1) \int_{\mathbb{R}^n} f(x,u) W dx \\
&\le \beta \|W\|^2 + |1 - e^{-\varepsilon z(\theta_t\omega)}| \int_{\mathbb{R}^n} [\alpha_3(|u^\varepsilon| + |u|)^{p-2}|u||W| + |\psi_3(x)||u||W| \\
&\quad + |f(x,u)||W|] dx \\
&\le \beta \|W\|^2 + c|1 - e^{-\varepsilon z(\theta_t\omega)}| \int_{\mathbb{R}^n} [|u^\varepsilon|^{p-2}|u||W| + |u|^{p-1}|W| + |\psi_3(x)||u||W| \\
&\quad + |u|^{p-1}|W| + |\psi_2(x)||W|] dx \\
&\le \beta \|W\|^2 + c|1 - e^{-\varepsilon z(\theta_t\omega)}| (\|u^\varepsilon\|_p^p + \|u\|_p^p + \|W\|_p^p + \|\psi_3\|_{\frac{p}{p-2}}^{\frac{p}{p-2}} + \|\psi_2\|^2 + \|W\|^2) \\
&\le \beta \|W\|^2 + c|1 - e^{-\varepsilon z(\theta_t\omega)}| (\|u^\varepsilon\|_p^p + \|u\|_p^p + \|W\|_p^p + \|W\|^2 + 1)
\end{aligned}
$$

By the Young inequality, the last two terms on the right-hand side of (5.14) is bounded by

$$
\begin{aligned}
&|e^{-\varepsilon z(\theta_t\omega)} - 1||(g(x), W)| + \varepsilon |z(\theta_t\omega)||(W + u, W)| \\
&\le |e^{-\varepsilon z(\theta_t\omega)} - 1|(\|g\|^2 + \|W\|^2) + c\varepsilon |z(\theta_t\omega)|(\|W\|^2 + \|u\|^2).
\end{aligned} \tag{5.16}
$$

It follows from (5.14)-(5.16) that

$$
\begin{aligned}
&\frac{d}{dt}\|W\|^2 \\
&\le c|e^{\varepsilon(p-2)z(\theta_t\omega)} - 1|(\|\nabla u\|_p^p + \|\nabla v^\varepsilon\|_p^p) + c\|W\|^2 \\
&\quad + c|1 - e^{-\varepsilon z(\theta_t\omega)}|(\|u^\varepsilon\|_p^p + \|u\|_p^p + \|W\|_p^p + \|W\|^2 + 1) \\
&\quad + |e^{-\varepsilon z(\theta_t\omega)} - 1|(\|g\|^2 + \|W\|^2) + c\varepsilon |z(\theta_t\omega)|(\|W\|^2 + \|u\|^2) \\
&\le c|e^{\varepsilon(p-2)z(\theta_t\omega)} - 1|(\|\nabla u\|_p^p + \|\nabla v^\varepsilon\|_p^p) + c\|W\|^2 \\
&\quad + c|1 - e^{-\varepsilon z(\theta_t\omega)}|(\|u^\varepsilon\|_p^p + e^{-p\varepsilon z(\theta_t\omega)}\|u^\varepsilon\|_p^p + \|u\|_p^p + \|W\|^2 + 1) \\
&\quad + c\varepsilon |z(\theta_t\omega)|(\|W\|^2 + \|u\|^2) \\
&= c(1 + |1 - e^{-\varepsilon z(\theta_t\omega)}| + \varepsilon |z(\theta_t\omega)|)\|W\|^2 + c|e^{\varepsilon(p-2)z(\theta_t\omega)} - 1|(\|\nabla u\|_p^p \\
&\quad + \|\nabla v^\varepsilon\|_p^p) + c|1 - e^{-\varepsilon z(\theta_t\omega)}|[(e^{-p\varepsilon z(\theta_t\omega)} + 1)\|u^\varepsilon\|_p^p + \|u\|_p^p + 1] \\
&\quad + c\varepsilon |z(\theta_t\omega)|\|u\|^2
\end{aligned}
$$

$$
\begin{aligned}
&\le c(e^{|z(\theta_t\omega)|}+1+|z(\theta_t\omega)|)\|W\|^2+c|e^{\varepsilon(p-2)z(\theta_t\omega)}-1|(\|\nabla u\|_p^p+\|\nabla v^\varepsilon\|_p^p)\\
&\quad+c|1-e^{-\varepsilon z(\theta_t\omega)}|[(e^{-p\varepsilon z(\theta_t\omega)}+1)\|u^\varepsilon\|_p^p+\|u\|_p^p+1]+c\varepsilon|z(\theta_t\omega)|\|u\|^2,
\end{aligned}
$$

where we used $0<\varepsilon\le 1$. Then by the Gronwall Lemma, we have for $\forall$ fixed $t\ge 0$,

$$
\begin{aligned}
\|W(t)\|^2 &\le e^{c\int_0^t e^{(|z(\theta_s\omega)|}+1+|z(\theta_s\omega)|)ds}\|W(0)\|^2+c\int_0^t\Big\{|e^{\varepsilon(p-2)z(\theta_s\omega)}-1|(\|\nabla u\|_p^p\\
&\quad+\|\nabla v^\varepsilon\|_p^p)+|1-e^{-\varepsilon z(\theta_s\omega)}|[(e^{-p\varepsilon z(\theta_s\omega)}+1)\|u^\varepsilon(s)\|_p^p+\|u(s)\|_p^p+1]\\
&\quad+c\varepsilon|z(\theta_s\omega)|\|u(s)\|^2\Big\}e^{c\int_s^t e^{(|z(\theta_\tau\omega)|}+1+|z(\theta_\tau\omega)|)d\tau}ds\\
&\le e^{c\int_0^t e^{(|z(\theta_s\omega)|}+1+|z(\theta_s\omega)|)ds}[\|W(0)\|^2\\
&\quad+c\int_0^t|e^{\varepsilon(p-2)z(\theta_s\omega)}-1|(\|\nabla u\|_p^p+\|\nabla v^\varepsilon\|_p^p)ds\\
&\quad+c\int_0^t|1-e^{-\varepsilon z(\theta_s\omega)}|(e^{-p\varepsilon z(\theta_s\omega)}+1)\|u^\varepsilon(s)\|_p^pds\\
&\quad+c\int_0^t|1-e^{-\varepsilon z(\theta_s\omega)}|(\|u(s)\|_p^p+1)ds+c\varepsilon\int_0^t|z(\theta_s\omega)|\|u(s)\|^2ds].
\end{aligned}
\tag{5.17}
$$

Since $u_0^\varepsilon\to u_0$ (as $\varepsilon\downarrow 0$), we have

$$
\begin{aligned}
\|W(0)\|^2&=\|v_0^\varepsilon-u_0\|^2=\|e^{-\varepsilon z(\omega)}u_0^\varepsilon-e^{-\varepsilon z(\omega)}u_0+(e^{-\varepsilon z(\omega)}-1)u_0\|^2\\
&\le c(|e^{-\varepsilon z(\omega)}|^2\|u_0^\varepsilon-u_0\|^2+|e^{-\varepsilon z(\omega)}-1|^2\|u_0\|^2)\to 0\quad\text{as }\varepsilon\downarrow 0.
\end{aligned}
\tag{5.18}
$$

According to (4.15) with T replaced by 0, we have

$$
\begin{aligned}
&\int_0^t e^{-2\varepsilon z(\theta_s\omega)+2\varepsilon\int_s^t z(\theta_\tau\omega)d\tau+\lambda(s-t)}\|u^\varepsilon(s,\omega,u_0^\varepsilon(\omega))\|_p^pds\\
&+\int_0^t e^{(p-2)\varepsilon z(\theta_s\omega)+2\varepsilon\int_s^t z(\theta_\tau\omega)d\tau+\lambda(s-t)}\|\nabla v^\varepsilon(s,\omega,v_0^\varepsilon(\omega))\|_p^pds\\
&\le ce^{2\varepsilon\int_0^t z(\theta_s\omega)ds-\lambda t}\|v_0^\varepsilon(\omega)\|^2+c\int_0^t e^{-2\varepsilon z(\theta_s\omega)+2\varepsilon\int_s^t z(\theta_\tau\omega)d\tau+\lambda(s-t)}\\
&=ce^{2\varepsilon\int_0^t z(\theta_s\omega)ds-\lambda t-2\varepsilon z(\omega)}\|u_0^\varepsilon(\omega)\|^2+c\int_0^t e^{-2\varepsilon z(\theta_s\omega)+2\varepsilon\int_s^t z(\theta_\tau\omega)d\tau+\lambda(s-t)}\\
&\le ce^{2\int_0^t|z(\theta_s\omega)|ds-\lambda t+2|z(\omega)|}\|u_0^\varepsilon(\omega)\|^2+c\int_0^t e^{2|z(\theta_s\omega)|+2\int_0^t|z(\theta_\tau\omega)|d\tau}.
\end{aligned}
\tag{5.19}
$$

Thus

$$
\begin{aligned}
&\int_0^t(e^{-p\varepsilon z(\theta_s\omega)}+1)\|u^\varepsilon(s)\|_p^pds\\
&=\int_0^t(e^{-p\varepsilon z(\theta_s\omega)}+1)e^{2\varepsilon z(\theta_s\omega)-2\varepsilon\int_s^t z(\theta_\tau\omega)d\tau-\lambda(s-t)}e^{-2\varepsilon z(\theta_s\omega)+2\varepsilon\int_s^t z(\theta_\tau\omega)d\tau+\lambda(s-t)}\\
&\quad\times\|u^\varepsilon(s,\omega,u_0^\varepsilon(\omega))\|_p^pds\\
&\le(e^{p\max_{0\le s\le t}|z(\theta_s\omega)|}+1)e^{2\max_{0\le s\le t}|z(\theta_s\omega)|+2\int_0^t|z(\theta_\tau\omega)|d\tau+\lambda t}\\
&\quad\times\Big(ce^{2\int_0^t|z(\theta_s\omega)|ds-\lambda t+2|z(\omega)|}\|u_0^\varepsilon(\omega)\|^2+c\int_0^t e^{2|z(\theta_s\omega)|+2\int_0^t|z(\theta_\tau\omega)|d\tau}ds\Big)
\end{aligned}
$$

$$\leq ce^{c(1+t)\max_{0\leq s\leq t}|z(\theta_s\omega)|}\|u_0^\varepsilon(\omega)\|^2 + cte^{c(1+t)\max_{0\leq s\leq t}|z(\theta_s\omega)|+\lambda t}.$$

Then it follows that

$$\begin{aligned}
&c\int_0^t |1-e^{-\varepsilon z(\theta_s\omega)}|(e^{-p\varepsilon z(\theta_s\omega)}+1)\|u^\varepsilon(s)\|_p^p ds \\
&= c\max_{0\leq s\leq t}|1-e^{-\varepsilon z(\theta_s\omega)}|\int_0^t (e^{-p\varepsilon z(\theta_s\omega)}+1)\|u^\varepsilon(s)\|_p^p ds \to 0 \quad \text{as } \varepsilon\downarrow 0.
\end{aligned} \tag{5.20}$$

Similarly, according to (5.19), we can get

$$c\int_0^t |e^{\varepsilon(p-2)z(\theta_s\omega)}-1|\|\nabla v^\varepsilon(s)\|_p^p ds \to 0 \quad \text{as } \varepsilon\downarrow 0\,. \tag{5.21}$$

And by (5.3) for $\varepsilon = 0$, we can also get that

$$c\int_0^t |e^{\varepsilon(p-2)z(\theta_s\omega)}-1|\|\nabla u(s)\|_p^p ds + c\int_0^t |1-e^{-\varepsilon z(\theta_s\omega)}|\|u(s)\|_p^p ds \to 0 \tag{5.22}$$

as $\varepsilon\downarrow 0$. Furthermore, it is easy to see

$$\lim_{\varepsilon\downarrow 0}\Big(c\int_0^t |1-e^{-\varepsilon z(\theta_s\omega)}|ds + c\varepsilon\int_0^t |z(\theta_s\omega)|\|u(s)\|^2 ds\Big) = 0. \tag{5.23}$$

Thus, together (5.17), (5.18), (5.20)–(5.23), imply that for P-a.e. $\in \Omega, \forall$ fixed $t\geq 0, u_0^\varepsilon \to u_0$ $(\varepsilon\downarrow 0)$,

$$\|W(t)\|^2 = \|v^\varepsilon - u\|^2 \to 0 \quad \text{as } \varepsilon\downarrow 0. \tag{5.24}$$

Finally, by (5.13) and (5.24), we obtain

$$\begin{aligned}
\|u^\varepsilon - u\|^2 &= \|e^{\varepsilon z(\theta_t\omega)}v^\varepsilon - u\|^2 = \|e^{\varepsilon z(\theta_t\omega)}v^\varepsilon - e^{\varepsilon z(\theta_t\omega)}u + (e^{\varepsilon z(\theta_t\omega)}-1)u\|^2 \\
&\leq c(e^{2\varepsilon z(\theta_t\omega)}\|v^\varepsilon - u\|^2 + |e^{\varepsilon z(\theta_t\omega)}-1|^2\|u\|^2) \to 0 \quad \text{as } \varepsilon\downarrow 0.
\end{aligned}$$

This completes the proof. □

Theorem 5.3. *Let $g\in L^2(\mathbb{R}^n)$, (3.3)-(3.4) and (5.1)-(5.2) hold. Then for P-a.e. $\omega\in\Omega$*

$$\lim_{\varepsilon\downarrow 0}\operatorname{dist}_{L^2(\mathbb{R}^n)}(\mathcal{A}_\varepsilon(\omega),\mathcal{A}_0) = 0. \tag{5.25}$$

Proof. It suffices to show that conditions (i)–(iii) of Proposition 2.9 are all satisfied. Note that $\{D_\varepsilon(\omega)\}_{\omega\in\Omega}$ is a closed absorbing set for Φ_ε in $\mathcal{D}$, where $D_\varepsilon(\omega)$ is given by (5.3). And it is easy to get

$$\limsup_{\varepsilon\downarrow 0}\|D_\varepsilon(\omega)\| \leq M = \sqrt{1+\frac{1}{\lambda}}, \tag{5.26}$$

where $\sqrt{1+\frac{1}{\lambda}}$ is a positive deterministic constant. Thus condition (ii) of Proposition 2.9 is satisfied. Lemma 5.1 and Lemma 5.2 show that condition (iii) and (i) of Proposition 2.9 are satisfied respectively. Hence, (5.25) follows from Proposition 2.9 immediately. □

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