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WEIGHTED ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO SEMILINEAR INTEGRO-DIFFERENTIAL EQUATIONS IN BANACH SPACES

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Abstract. In this article, we study weighted asymptotic behavior of solutions to the semilinear integro-differential equation

$$u'(t) = Au(t) + \alpha \int_{-\infty}^{t} e^{-\beta(t-s)} Au(s)ds + f(t, u(t)), \quad t \in \mathbb{R},$$

where $\alpha, \beta \in \mathbb{R}$, with $\beta > 0$, $\alpha \neq 0$ and $\alpha + \beta > 0$, A is the generator of an immediately norm continuous C_0-Semigroup defined on a Banach space $\mathbb{X}$, and $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is an S^p-weighted pseudo almost automorphic function satisfying suitable conditions. Some sufficient conditions are established by using properties of S^p-weighted pseudo almost automorphic functions combined with theories of uniformly exponentially stable and strongly continuous family of operators.

1. Introduction

In this article, we are mainly focused upon weighted asymptotic behavior of solutions to the semilinear integro-differential equation

$$u'(t) = Au(t) + \alpha \int_{-\infty}^{t} e^{-\beta(t-s)} Au(s)ds + f(t, u(t)), \quad t \in \mathbb{R}, \tag{1.1}$$

where $\alpha, \beta \in \mathbb{R}$ with $\beta > 0$, $\alpha \neq 0$ and $\alpha + \beta > 0$, $A : D(A) \subseteq \mathbb{X} \to \mathbb{X}$ is the generator of an immediately norm continuous C_0-semilinear defined on the Banach space $\mathbb{X}$, and $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is an S^p-weighted pseudo almost automorphic function satisfying suitable conditions given later.

The concept of almost automorphy was first introduced by Bochner in [5] in relation to some aspects of differential geometry, for more details about this topic we refer to [1, 2, 7, 8, 11, 12, 13, 14, 22, 23, 25] and references therein. Since then, this concept has undergone several interesting, natural and powerful generalizations. The concept of asymptotically almost automorphic functions was introduced by N'Guérékata in [21]. Liang, Xiao and Zhang [17, 27] presented the concept of pseudo almost automorhy. N'Guérékata and Pankov [24] introduced the concept of Stepanov-like almost automorphy and Blot et al. [6] introduced the

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notion of weighted pseudo almost automorphic functions with values in a Banach space. Xia and Fan [26] presented the notation of Stepanov-like (or S^p-) weighted pseudo almost automorphy, and Chang, N'Guérékata et al. [28, 29] investigated some properties and new composition theorems of Stepanov-like weighted pseudo almost automorphic functions.

Lizama and Ponce [18] studied systematically the existence and uniqueness of bounded solutions, such as almost periodic, almost automorphic and asymptotically almost periodic solutions, to the problem (1.1). However, few results are available for weighted asymptotic behavior of solutions to the problem (1.1). By the main theories developed in [18, 26, 28], the main aim of the present paper is to investigate weighted asymptotic behavior of solutions to the problem (1.1) with S^p-weighted pseudo almost automorphic coefficients. Some sufficient conditions are established via composition theorems of S^p-weighted pseudo almost automorphic functions combined with theories of uniformly exponentially stable and strongly continuous family of operators.

The rest of this paper is organized as follows. In section 2, we recall some preliminary results which will be used throughout this paper. In section 3, we establish some sufficient conditions for weighted pseudo almost automorphic solutions to the problem (1.1).

2. Preliminaries

Throughout this paper, we assume that $(\mathbb{X}, \|\cdot\|)$ and $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ are two Banach spaces. We let $C(\mathbb{R}, \mathbb{X})$ (respectively, $C(\mathbb{R}\times\mathbb{Y}, \mathbb{X})$) denote the collection of all continuous functions from $\mathbb{R}$ into $\mathbb{X}$ (respectively, the collection of all jointly continuous functions $f : \mathbb{R}\times\mathbb{Y} \to \mathbb{X}$). Furthermore, $BC(\mathbb{R}, \mathbb{X})$ (respectively, $BC(\mathbb{R}\times\mathbb{Y}, \mathbb{X})$) stands for the class of all bounded continuous functions from $\mathbb{R}$ into $\mathbb{X}$ (respectively, the class of all jointly bounded continuous functions from $\mathbb{R}\times\mathbb{Y}$ into $\mathbb{X}$). Note that $BC(\mathbb{R}, \mathbb{X})$ is a Banach space with the sup norm $\|\cdot\|_\infty$. Furthermore, we denote by $B(\mathbb{X})$ the space of bounded linear operators form $\mathbb{X}$ into $\mathbb{X}$ endowed with the operator topology.

First, we list some basic definitions, properties of some almost automorphic type functions in abstract spaces.

Definition 2.1 ([23])**.** A continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be almost automorphic if for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ such that

$$g(t) := \lim_{n\to\infty} f(t + s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} g(t - s_n) = f(t)$$

for each $t \in \mathbb{R}$. The collection of all such functions will be denoted by $AA(\mathbb{X})$.

Definition 2.2 ([23])**.** A continuous function $f : \mathbb{R}\times\mathbb{Y} \to \mathbb{X}$ is said to be almost automorphic if $f(t, x)$ is almost automorphic for each $t \in \mathbb{R}$ uniformly for all $x \in \mathbb{K}$, where $\mathbb{K}$ is any bounded subset of $\mathbb{Y}$. The collection of all such functions will be denoted by $AA(\mathbb{R}\times\mathbb{X}, \mathbb{X})$.

Let $\mathbb{U}$ denote the set of all functions $\rho : \mathbb{R} \to (0,\infty)$, which are locally integrable over $\mathbb{R}$ such that $\rho > 0$ almost everywhere. For a given $r > 0$ and for each $\rho \in \mathbb{U}$, we set $m(r,\rho) := \int_{-r}^{r} \rho(t)dt$. Thus the space of weights $\mathbb{U}_\infty$ is defined by

$$\mathbb{U}_\infty := \{\rho \in \mathbb{U} : \lim_{r\to\infty} m(r,\rho) = \infty\}.$$

For a given $\rho \in \mathbb{U}_\infty$, we define

$$PAA_0(\mathbb{X},\rho) := \Big\{f \in BC(\mathbb{R},\mathbb{X}) : \lim_{r\to\infty} \frac{1}{m(r,\rho)} \int_{-r}^{r} \|f(t)\|\rho(t)dt = 0\Big\};$$

$$PAA_0(\mathbb{Y},\mathbb{X},\rho) := \Big\{f \in C(\mathbb{R}\times\mathbb{Y},\mathbb{X}) : f(\cdot,y) \text{ is bounded for each } y \in \mathbb{Y} \text{ and } \lim_{r\to\infty} \frac{1}{m(r,\rho)} \int_{-r}^{r} \|f(t,y)\|\rho(t)dt = 0 \text{ uniformly in } y \in \mathbb{Y}\Big\}.$$

Definition 2.3 ([6]). Let $\rho \in \mathbb{U}_\infty$. A function $f \in BC(\mathbb{R},\mathbb{X})$ (respectively, $f \in BC(\mathbb{R}\times\mathbb{Y},\mathbb{X})$) is called weighted pseudo almost automorphic if it can be expressed as $f = g + \chi$, where $g \in AA(\mathbb{X})$ (respectively, $AA(\mathbb{R}\times\mathbb{Y},\mathbb{X})$) and $\chi \in PAA_0(\mathbb{X},\rho)$ (respectively, $PAA_0(\mathbb{Y},\mathbb{X},\rho)$). We denote by $WPAA(\mathbb{R},\mathbb{X})$ (respectively, $WPAA(\mathbb{R}\times\mathbb{Y},\mathbb{X})$) the set of all such functions.

Lemma 2.4 ([20, Theorem 2.15]). Let $\rho \in \mathbb{U}_\infty$. If $PAA_0(\mathbb{X},\rho)$ is translation invariant, then $\big(WPAA(\mathbb{R},\mathbb{X}), \|\cdot\|_\infty\big)$ is a Banach space.

Definition 2.5 ([9, 10]). The Bochner transform $f^b(t,s), t \in \mathbb{R}, s \in [0,1]$, of a function $f : \mathbb{R} \to \mathbb{X}$ is defined by

$$f^b(t,s) := f(t+s).$$

Definition 2.6 ([9, 10]). The Bochner transform $f^b(t,s,u), t \in \mathbb{R}, s \in [0,1], u \in \mathbb{X}$ of a functions $f : \mathbb{R}\times\mathbb{X} \longrightarrow \mathbb{X}$ is defined by

$$f^b(t,s,u) := f(t+s,u) \quad \text{for all } \ u \in \mathbb{X}.$$

Remark 2.7 ([10]). (i) A function $\varphi(t,s)$, $t \in \mathbb{R}$, $s \in [0,1]$, is the Bochner transform of a certain function f, $\varphi(t,s) = f^b(t,s)$, if and only if $\varphi(t+\tau, s-\tau) = \varphi(t,s)$ for all $t \in \mathbb{R}$, $s \in [0,1]$ and $\tau \in [s-1,s]$.

(ii) Note that if $f = g + \chi$, then $f^b = g^b + \chi^b$. Moreover, $(\lambda f)^b = \lambda f^b$ for each scalar λ.

Definition 2.8 ([9, 10]). Let $p \in [1,\infty)$. The space $BS^p(\mathbb{X})$ of all Stepanov-like bounded functions, with the exponent p, consists of all measurable functions $f : \mathbb{R} \to \mathbb{X}$ such that $f^b \in L^\infty(\mathbb{R}, L^p(0,1;\mathbb{X}))$. This is a Banach space with the norm

$$\|f\|_{S^p} = \|f^b\|_{L^\infty(\mathbb{R},L^p)} = \sup_{t\in\mathbb{R}} \Big(\int_t^{t+1} \|f(\tau)\|^p d\tau\Big)^{1/p}.$$

Definition 2.9 ([9, 10]). The space $AS^p(\mathbb{X})$ of Stepanov-like almost automorphic (or S^p-almost automorphic) functions consists of all $f \in BS^p(\mathbb{X})$ such that $f^b \in AA\,(L^p(0,1;\mathbb{X}))$. In other words, a function $f \in L^p_{\rm loc}(\mathbb{R},\mathbb{X})$ is said to be S^p-almost automorphic if its Bochner transform $f^b : \mathbb{R} \to L^p(0,1;\mathbb{X})$ is almost automorphic in the sense that for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exist a subsequence $\{s_n\}_{n\in\mathbb{N}}$ and a function $g \in L^p_{\rm loc}(\mathbb{R},\mathbb{X})$ such that

$$\lim_{n\to\infty} \Big(\int_0^1 \|f(t+s+s_n) - g(t+s)\|^p ds\Big)^{1/p} = 0,$$

$$\lim_{n\to\infty} \Big(\int_0^1 \|g(t+s-s_n) - f(t+s)\|^p ds\Big)^{1/p} = 0.$$

pointwise on $\mathbb{R}$.

Definition 2.10 ([9, 10])**.** A function $f : \mathbb{R}\times\mathbb{Y} \to \mathbb{X}$, $(t, u) \to f(t, u)$ with $f(\cdot, u) \in L^p_{\rm loc}(\mathbb{R}, \mathbb{X})$ for each $u \in \mathbb{Y}$, is said to be S^p-almost automorphic in $t \in \mathbb{R}$ uniformly in $u \in \mathbb{Y}$ if $t \to f(t, u)$ is S^p-almost automorphic for each $u \in \mathbb{Y}$. That means, for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exist a subsequence $\{s_n\}_{n\in\mathbb{N}}$ and a function $g(\cdot, u) \in L^p_{\rm loc}(\mathbb{R}, \mathbb{X})$ such that

$$\lim_{n\to\infty}\Big(\int_0^1 \|f(t+s+s_n,u)-g(t+s,u)\|^p ds\Big)^{1/p}=0,$$
$$\lim_{n\to\infty}\Big(\int_0^1 \|g(t+s-s_n,u)-f(t+s,u)\|^p ds\Big)^{1/p}=0,$$

pointwise on $\mathbb{R}$ and for each $u \in \mathbb{Y}$. We denote by $AS^p(\mathbb{R}\times\mathbb{Y}, \mathbb{X})$ the set of all such functions.

Definition 2.11 ([28])**.** Let $\rho \in \mathbb{U}_\infty$. A function $f \in BS^p(\mathbb{X})$ is said to be Stepanov-like weighted pseudo almost automorphic (or S^p-weighted pseudo almost automorphic) if it can be expressed as $f = g + \chi$, where $g \in AS^p(\mathbb{X})$ and $\chi^b \in PAA_0\left(L^p(0, 1; \mathbb{X}), \rho\right)$. In other words, a function $f \in L^p_{\rm loc}(\mathbb{R}, \mathbb{X})$ is said to be Stepanov-like weighted pseudo almost automorphic relatively to the weight $\rho \in \mathbb{U}_\infty$, if its Bochner transform $f^b : \mathbb{R} \to L^p(0, 1; \mathbb{X})$ is weighted pseudo almost automorphic in the sense that there exist two functions $g, \chi : \mathbb{R} \to \mathbb{X}$ such that $f = g + \chi$, where $g \in AS^p(\mathbb{X})$ and $\chi^b \in PAA_0\left(L^p(0, 1; \mathbb{X}), \rho\right)$. We denote by $WPAAS^p(\mathbb{R}, \mathbb{X})$ the set of all such functions.

Definition 2.12 ([28])**.** *Let $\rho \in \mathbb{U}_\infty$. A function $f : \mathbb{R}\times\mathbb{Y} \to \mathbb{X}$, $(t, u) \to f(t, u)$ with $f(\cdot, u) \in L^p_{\rm loc}(\mathbb{R}, \mathbb{X})$ for each $u \in \mathbb{Y}$, is said to be Stepanov-like weighted pseudo almost automorphic (or S^p-weighted pseudo almost automorphic) if it can be expressed as $f = g+\chi$, where $g \in AS^p(\mathbb{R}\times\mathbb{Y}, \mathbb{X})$ and $\chi^b \in PAA_0\left(\mathbb{Y}, L^p(0, 1; \mathbb{X}), \rho\right)$. We denote by $WPAAS^p(\mathbb{R}\times\mathbb{Y}, \mathbb{X})$ the set of all such functions.*

Remark 2.13 ([28])**.** It is clear that if $1 \le p < q < \infty$ and $f \in L^q_{\rm loc}(\mathbb{R}, \mathbb{X})$ is S^q-almost automorphic, then f is S^p almost automorphic. Also if $f \in AA(\mathbb{X})$, then f is S^p-almost automorphic for any $1 \le p < \infty$.

Lemma 2.14 ([28])**.** Let $\rho \in \mathbb{U}_\infty$. Assume that $PAA_0(L^p(0, 1; \mathbb{X}), \rho)$ is translation invariant. Then the decomposition of an S^p-weighted pseudo almost automorphic function is unique.

Lemma 2.15 ([26, 29])**.** Let $\rho \in \mathbb{U}_\infty$ be such that

$$\limsup_{t\to\infty}\frac{\rho(t+\iota)}{\rho(t)}<\infty \quad\text{and}\quad \limsup_{r\to\infty}\frac{m(r+\iota,\rho)}{m(r,\rho)}<\infty, \tag{2.1}$$

for every $\iota \in \mathbb{R}$, then spaces $WPAAS^p(\mathbb{R}, \mathbb{X})$ and $PAA_0(L^p(0, 1; \mathbb{X}), \rho)$ are translation invariant.

Lemma 2.16 ([26, 28])**.** If $f \in WPAA(\mathbb{R}, \mathbb{X})$, then $f \in WPAAS^p(\mathbb{R}, \mathbb{X})$ for each $1 \le p < \infty$. In other words, $WPAA(\mathbb{R}, \mathbb{X}) \subseteq WPAAS^p(\mathbb{R}, \mathbb{X})$. Moreover, $WPAAS^q(\mathbb{R}, \mathbb{X}) \subseteq WPAAS^p(\mathbb{R}, \mathbb{X})$ for $1 \le p < q < +\infty$.

Lemma 2.17 ([28])**.** *Let $\rho \in \mathbb{U}_\infty$ satisfy* (2.1)*. Then the space $WPAAS^p(\mathbb{R}, \mathbb{X})$ equipped with the norm $\|\cdot\|_{S^p}$ is a Banach space.*

Lemma 2.18 ([28]). *Let $\rho \in \mathbb{U}_\infty$ and let $f = g + \chi \in WPAAS^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ with $g \in AS^p(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$, $\chi^b \in PAA_0(\mathbb{Y}, L^p(0,1;\mathbb{X}), \rho)$. Assume that the following condition are satisfied:*

(i) *$f(t,x)$ is Lipschitzian in $x \in \mathbb{X}$ uniformly in $t \in \mathbb{R}$; that is, there exists a constant $L > 0$ such that*
$$\|f(t,x) - f(t,y)| \leq L\|x - y|$$
for all $x, y \in \mathbb{X}$ and $t \in \mathbb{R}$.

(ii) *$g(t,x)$ is uniformly continuous in any bounded subset $K' \subseteq \mathbb{X}$ uniformly for $t \in R$.*

If $u = u_1 + u_2 \in WPAAS^p(\mathbb{R}, \mathbb{X})$, with $u_1 \in AS^p(\mathbb{X})$, $u_2^b \in PAA_0(L^p(0,1;\mathbb{X}), p)$ and $K = \overline{\{u_1(t) : t \in \mathbb{R}\}}$ is compact, then $\Upsilon : \mathbb{R} \to \mathbb{X}$ defined by $\Upsilon(\cdot) = f(\cdot, u(\cdot))$ belongs to $WPAAS^p(\mathbb{R}, \mathbb{X})$.

Lemma 2.19 ([28]). *Let $\rho \in \mathbb{U}_\infty$ and let $f = g + \chi \in WPAAS^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ with $g \in AS^p(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$, $\chi^b \in PAA_0(\mathbb{Y}, L^p(0,1;\mathbb{X}), \rho)$. Assume that the following conditions are satisfied:*

(i) *there exists a nonnegative function $L_f(\cdot) \in BS^p(\mathbb{R})$ with $p > 1$ such that for all $u, v \in \mathbb{R}$ and $t \in \mathbb{R}$.*
$$\Big(\int_t^{t+1} \|f(s,u) - f(s,v)\|^p ds\Big)^{1/p} \leq L_f(t)\|u - v\|;$$

(ii) *$\rho \in L^q_{\rm loc}(\mathbb{R})$ satisfies $\lim_{T\to\infty} \sup \frac{T^{1/p} m_q(T,\rho)}{m(T,\rho)} < \infty$, where $\frac{1}{p} + \frac{1}{q} = 1$ and $m_q(T,\rho) = \big(\int_{-T}^T \rho^q(t)dt\big)^{1/q}$;*

(iii) *$g(t,x)$ is uniformly continuous in any bounded subset $K \subseteq \mathbb{X}$.*

If $u = u_1 + u_2 \in WPAAS^p(\mathbb{R}, \mathbb{X})$, with $u_1 \in AS^p(\mathbb{X})$, $u_2^b \in PAA_0(L^p(0,1;\mathbb{X}), p)$ and $K = \overline{\{u_1(t) : t \in \mathbb{R}\}}$ is compact, then $\Upsilon : \mathbb{R} \to \mathbb{X}$ defined by $\Upsilon(\cdot) = f(\cdot, u(\cdot))$ belongs to $WPAAS^p(\mathbb{R}, \mathbb{X})$.

Lemma 2.20 ([26]). *Let $\rho \in \mathbb{U}_\infty$, $p > 1$ and let $f = g + \chi \in WPAAS^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ with $g \in AS^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$, $\chi^b \in PAA_0(\mathbb{X}, L^p(0,1;\mathbb{X}), \rho)$. Assume that the following conditions are satisfied:*

(i) *there exist nonnegative functions $\mathcal{L}_f(\cdot)$ and $\mathcal{L}_g(\cdot)$ in $AS^r(\mathbb{R}, \mathbb{R})$ with $r \geq \max\{p, \frac{p}{p-1}\}$ such that for all $u, v \in \mathbb{X}$ and $t \in \mathbb{R}$,*
$$\|f(s,u) - f(s,v)\| \leq \mathcal{L}_f(t)\|u - v\|, \quad \|g(s,u) - g(s,v)\| \leq \mathcal{L}_g(t)\|u - v\|;$$

(ii) *$u = u_1 + u_2 \in WPAAS^p(\mathbb{R}, \mathbb{X})$, with $u_2^b \in PAA_0(L^p(0,1;\mathbb{X}), \rho)$, $u_1 \in AS^p(\mathbb{X})$, and $K = \overline{\{u_1(t) : t \in \mathbb{R}\}}$ compact in $\mathbb{X}$.*

Then there exists $\overline{q} \in [1, p)$ such that $\Upsilon : \mathbb{R} \longrightarrow \mathbb{X}$ defined by $\Upsilon(\cdot) = f(\cdot, u(\cdot))$ belongs to $WPAAS^{\overline{q}}(\mathbb{R}, \mathbb{X})$.

Lemma 2.21 ([28]). *Let $\rho \in \mathbb{U}_\infty$ and $f = g + \chi : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ be an S^p-weighted pseudo almost automorphic function with $g \in AS^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$, $\chi^b \in PAA_0(\mathbb{X}, L^p(0,1;\mathbb{X}), \rho)$. Suppose that f satisfies the following conditions:*

(i) *$f(t,x)$ is uniformly conditions in any bounded subset $K' \subseteq \mathbb{X}$ uniformly for $t \in \mathbb{R}$.*

(ii) *$g(t,x)$ is uniformly conditions in any bounded subset $K' \subseteq \mathbb{X}$ uniformly for $t \in \mathbb{R}$.*

(iii) *For every bounded subset $K' \subset \mathbb{X}$, the set of functions $f(\cdot, x) : x \in K'$ is bounded in $WPAAS^p(\mathbb{X})$.*

If $u = u_1 + u_2 \in WPAAS^p(\mathbb{X})$, with $u_1 \in AS^p(\mathbb{R}, \mathbb{X})$, $u_2^b \in PAA_0(L^p(0, 1; \mathbb{X}), p)$ and $K = \overline{\{u_1(t) : t \in \mathbb{R}\}}$ is compact, then $\Upsilon : \mathbb{R} \to \mathbb{X}$ defined by $\Upsilon(\cdot) = f(\cdot, u(\cdot))$ belongs to $WPAAS^p(\mathbb{R}, \mathbb{X})$.

Next, to establish an operator sketch to the problem (1.1), we recall some basic results which are main from the paper [18, 19]. Consider the following homogeneous abstract Volterra equation

$$\begin{gathered} u'(t) = Au(t) + \alpha \int_0^t e^{-\beta(t-s)} Au(s)ds, \quad t \geq 0 \\ u(0) = x. \end{gathered} \tag{2.2}$$

A solution of (2.2) is called to be uniformly exponentially bounded if for some $\omega \in \mathbb{R}$, there exists a constant $M > 0$ such that for each $x \in D(A)$, the corresponding solution $u(t)$ satisfies

$$\|u(t)\| \leq Me^{-\omega t}, \quad t \geq 0. \tag{2.3}$$

Particularly, the solutions of the equation (2.2) are said to be uniformly exponentially stable if the equation (2.3) holds for some $\omega > 0$ and $M > 0$.

Definition 2.22 ([18, Definition 2.3.]). Let $\mathbb{X}$ be a Banach space. A strongly continuous function $T : \mathbb{R}_+ \to B(\mathbb{X})$ is said to be immediately norm continuous if $T :\to B(\mathbb{X})$ is continuous.

Lemma 2.23 ([18, Theorem 2.4.]). *Let $\beta > 0$, $\alpha \neq 0$ and $\alpha + \beta > 0$ be given. Assume that*

(A1) *A generates an immediately norm continuous C_0-semigroup on a Banach space $\mathbb{X}$;*

(A2) $\sup\{\Re\lambda, \lambda \in \mathbb{C} : \lambda(\lambda + \beta)(\lambda + \alpha + \beta)^{-1} \in \sigma(A)\} < 0$.

Then, the solutions of (2.2) *are uniformly exponentially stable.*

Lemma 2.24 ([18, Proposition 3.1.]). *Let $\beta > 0$, $\alpha \neq 0$ and $\alpha + \beta > 0$. Assume that conditions* (A1) *and* (A2) *in Lemma 2.23 hold, then there exists a uniformly exponentially stable and strongly continuous family of operators $\{S(t)\}_{t\geq 0} \subset B(\mathbb{X})$ such that $S(t)$ commutes with A, that is, $S(t)D(A) \subset D(A)$, $AS(t)x = S(t)Ax$ for all $x \in D(A)$, $t \geq 0$ and*

$$S(t)x = x + \int_0^t b(t-s)AS(t)xds, \quad \text{for all } x \in \mathbb{X},\ t \geq 0,$$

where $b(t) = 1 + \frac{\alpha}{\beta}[1 - e^{-\beta t}]$, $t \geq 0$.

Finally, we recall a useful compactness criterion and a well-known fixed point theorem. Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $h(t) \geq 1$ for all $t \in \mathbb{R}$ and $h(t) \to \infty$ as $|t| \to \infty$. We consider the space

$$C_h(\mathbb{X}) = \big\{u \in C(\mathbb{R}, \mathbb{X}) : \lim_{|t|\to\infty} \frac{u(t)}{h(t)}\big\}. \tag{2.4}$$

Endowed with the norm $\|u\|_h = \sup_{t\in\mathbb{R}} \frac{\|u(t)\|}{h(t)}$, it is a Banach space.

Lemma 2.25 ([16]). *A subset $\mathcal{R} \subseteq C_h(\mathbb{X})$ is a relatively compact set if it verifies the following conditions:*

(C1) *The set* $\mathcal{R}(t) = \{u(t) : u \in \mathcal{R}\}$ *is relatively compact in* $\mathbb{X}$ *for each* $t \in \mathbb{R}$.
(C2) *The* $\mathcal{R}$ *is equicontinuous.*
(C3) *For each* $\epsilon > 0$ *there exists* $\mathbb{L} > 0$ *such that* $\|u(t)\| \le \epsilon h(t)$ *for all* $u \in \mathcal{R}$ *and all* $|t| > \mathbb{L}$.

Lemma 2.26 ([15])**.** *Let* $\mathbb{D}$ *be a closed convex subset of a Banach space* $\mathbb{X}$ *such that* $0 \in D$. *Let* $\mathcal{F} : \mathbb{D} \to \mathbb{D}$ *be a completely continuous map. Then the set* $\{x \in \mathbb{D} : x = \lambda \mathcal{F}(x), 0 < \lambda < 1\}$ *is unbounded or the map* $\mathcal{F}$ *has a fixed point in* $\mathbb{D}$.

3. Main results

In this section, we study the weighted asymptotic behavior of solutions to (1.1).

Definition 3.1 ([18, Definition 4.1])**.** A function $u : \mathbb{R} \to \mathbb{X}$ is said to be a mild solution to (1.1) if

$$u(t) = \int_{-\infty}^{t} S(t-s) f(s, u(s)) ds,$$

for all $t \in \mathbb{R}$, where $\{S(t)\}_{t \ge 0}$ is given in Lemma 2.24.

In the sequel, we always assume that the weight ρ satisfies condition (2.1). And now, we list the following basic assumptions:

(H1) $f \in WPAAS^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$, and there exists a constant $L > 0$, such that
$$\|f(t,x) - f(t,y)\| \le L\|x - y\|$$
for all $t \in \mathbb{R}$ and each $x, y \in \mathbb{X}$.

(H2) $f \in WPAAS^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$, and there exists a nonnegative function $L_f(\cdot) \in BS^p(\mathbb{R})$, with $p > 1$ such that
$$\|f(t,x) - f(t,y)\| \le L_f(t)\|x - y\|$$
for all $t \in \mathbb{R}$ and each $x, y \in \mathbb{X}$.

(H3) $\rho \in L^q_{\rm loc}(\mathbb{R})$ satisfies
$$\lim_{T \to \infty} \frac{T^{1/p} m_q(T, \rho)}{m(T, \rho)} < \infty,$$
where $\frac{1}{p} + \frac{1}{q} = 1$ and $m_q(t) = (\int_{-T}^{T} \rho^q(t) dt)^{1/q}$.

(H4) $f = g + \chi \in WPAAS^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$, where $g \in AS^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ is uniformly continuous in any bounded subset $\mathbb{M} \subseteq \mathbb{X}$ in $t \in \mathbb{R}$ and $\chi^b \in PAA_0(\mathbb{Y}, L^p(0,1;\mathbb{X}), \rho)$.

(H5) The function $f = g + \chi \in WPAAS^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ $(p > 1)$ with $g \in AS^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$, $\chi^b \in PAA_0(\mathbb{X}, L^p(0,1;\mathbb{X}), \rho)$, and there exist nonnegative functions $\mathcal{L}_f(\cdot)$, $\mathcal{L}_g(\cdot)$ in $AS^r(\mathbb{R}, \mathbb{R})$ with $r \ge \max\{p, \frac{p}{p-1}\}$ such that for all $u, v \in \mathbb{X}$ and $t \in \mathbb{R}$
$$\|f(s,u) - f(s,v)\| \le \mathcal{L}_f(t)\|u - v\|, \quad \|g(s,u) - g(s,v)\| \le \mathcal{L}_g(t)\|u - v\|.$$

Lemma 3.2. *Let* $\beta > 0$, $\alpha \neq 0$ *with* $\alpha + \beta > 0$ *and conditions* (A1)–(A2) *in Lemma 2.23 hold. If* $f : \mathbb{R} \to \mathbb{X}$ *is a stepanov-like weighted pseudo almost automorphic function, and* $F(\cdot)$ *is given by*

$$F(t) = \int_{-\infty}^{t} S(t-s) f(s) ds, \quad t \in \mathbb{R},$$

then, $F(\cdot) \in WPAA(\mathbb{R} \times \mathbb{X})$, *where* $\{S(t)\}_{t \ge 0}$ *is given in Lemma 2.24.*

Proof. Since $f \in WPAAS^p(\mathbb{R} \times \mathbb{X})$, we have $f = g + \chi$ with $g \in AS^p(\mathbb{X})$, $\chi^b \in PAA_0(L^p(0,1;\mathbb{X}), p)$. Consider for each $n = 1, 2, \dots$, the integrals

$$\begin{aligned} F_n(t) &= \int_{t-n}^{t-n+1} S(t-s)f(s)ds \\ &= \int_{t-n}^{t-n+1} S(t-s)g(s)ds + \int_{t-n}^{t-n+1} S(t-s)\chi(s)ds \\ &= X_n(t) + Y_n(t), \end{aligned}$$

where

$$X_n(t) = \int_{t-n}^{t-n+1} S(t-s)g(s)ds, \quad Y_n(t) = \int_{t-n}^{t-n+1} S(t-s)\chi(s)ds.$$

To prove that each F_n is a weighted pseudo almost automorphic function, we only need to verify $X_n \in AA(\mathbb{X})$ and $Y_n \in PAA_0(\mathbb{X}, \rho)$ for each $n = 1, 2, \dots$.

Let us first show that $X_n \in AA(\mathbb{X})$. We have

$$\begin{aligned} \|X_n(t)\| &= \| \int_{t-n}^{t-n+1} S(t-s)g(s)ds\| \\ &\le \int_{t-n}^{t-n+1} Me^{-\omega(t-s)}\|g(s)\|ds \\ &\le Me^{-\omega(n-1)}\Big(\int_{t-n}^{t-n+1} \|g(s)\|^p ds\Big)^{1/p} \\ &\le Me^{-\omega(n-1)}\|g\|_{S^p}. \end{aligned}$$

Since $\sum_{n=1}^{\infty} e^{-\omega(n-1)} = \frac{1}{1-e^{-\omega}} < \infty$, we conclude that the series $\sum_{n=1}^{\infty} X_n(t)$ is uniformly convergent on $\mathbb{R}$. Furthermore,

$$X(t) := \int_{-\infty}^{t} S(t-s)g(s)ds = \sum_{n=1}^{\infty} X_n(t).$$

Clearly, $X(t) \in C(\mathbb{R}, \mathbb{X})$ and $\|X(t)\| \le \sum_{n=1}^{\infty} \|X_n(t)\| \le \sum_{n=1}^{\infty} Me^{-\omega(n-1)}\|g\|_{S^p}$.

Since $g \in AS^p(\mathbb{R}, \mathbb{X})$, then for every sequence of real numbers $\{S_{n'}\}_{n'\in\mathbb{N}}$ there exist a sequence $\{S_n\}_{n\in\mathbb{N}}$ and a function $g'(\cdot) \in L^p_{\rm loc}(\mathbb{R}, \mathbb{X})$ such that for each $t \in \mathbb{R}$, the following equalities hold:

$$\begin{aligned} \lim_{n\to\infty} \Big(\int_0^1 \|g(t+s+s_n) - g'(t+s)\|^p ds\Big)^{1/p} &= 0, \\ \lim_{n\to\infty} \Big(\int_0^1 \|g'(t+s-s_n) - g(t+s)\|^p ds\Big)^{1/p} &= 0. \end{aligned}$$

Let $X'_n(t) = \int_{t-n}^{t-n+1} S(t-s)g'(s)ds$. Then using the Hölder inequality, we have

$$\begin{aligned} &\|X_n(t+s_n) - X'_n(t)\| \\ &= \Big\| \int_{t+s_n-n}^{t+s_n-n+1} S(t+s_n-s)g(s)ds - \int_{t-n}^{t-n+1} S(t-s)g'(s)ds\Big\| \\ &\le \int_{t-n}^{t-n+1} \|S(t-s)\|\|g(s+s_n) - g'(s)\|ds \end{aligned}$$

$$
\begin{aligned}
&\le \int_{t-n}^{t-n+1} Me^{-\omega(t-s)}\|g(s+s_n)-g'(s)\|ds \\
&\le Me^{-\omega(n-1)}\Big(\int_0^1 \|g(s+t-n+s_n)-g'(s+t-n)\|^p ds\Big)^{1/p}.
\end{aligned}
$$

Obviously, $\|X_n(t+s_n)-X_n'(t)\| \to 0$ as $n\to\infty$. Similarly, we can prove that

$$
\lim_{n\to\infty} \|X_n'(t-s_n)-X_n(t)\| = 0.
$$

Thus, we conclude that each $X_n \in AA(\mathbb{X})$ and consequently their uniform limit $X(t) \in AA(\mathbb{X})$.

Next, we show that each $Y_n \in PAA_0(\mathbb{X},\rho)$. For this, we note that

$$
\begin{aligned}
\|Y_n(t)\| &\le \int_{t-n}^{t-n+1} \|S(t-s)\chi(s)\|ds \\
&\le \int_{t-n}^{t-n+1} \|S(t-s)\|\|\chi(s)\|ds \\
&\le \int_{t-n}^{t-n+1} Me^{-\omega(t-s)}\|\chi(s)\|ds \\
&\le Me^{-\omega(n-1)} \int_{t-n}^{t-n+1} \|\chi(s)\|ds \\
&\le Me^{-\omega(n-1)} \Big(\int_{t-n}^{t-n+1} \|\chi(s)\|^p ds\Big)^{1/p}.
\end{aligned}
$$

Then, for $T>0$, we see that

$$
\begin{aligned}
&\frac{1}{m(T,\rho)}\int_{-T}^{T} \|Y_n(t)\|\rho(t)dt \\
&\le Me^{-\omega(n-1)} \frac{1}{m(T,\rho)}\int_{-T}^{T}\Big(\int_{t-n}^{t-n+1} \|\chi(s)\|^p ds\Big)^{1/p}\rho(t)dt.
\end{aligned}
$$

Since $\chi^b \in PAA_0(L^p(0,1;\mathbb{X}),\rho)$, the above inequality leads to $Y_n \in PAA_0(\mathbb{X},\rho)$ for each $n = 1,2,\dots$. By a similar way, we deduce that the uniform limit $Y(\cdot) = \sum_{n=1}^{\infty} Y_n(t) \in PAA_0(\mathbb{X},\rho)$. Therefore, $F(t) := X(t)+Y(t)$ is weighted pseudo almost automorphic. The proof is complete. □

Now, we shall present and prove our main results.

Theorem 3.3. *Let $\beta > 0$, $\alpha \neq 0$ with $\alpha+\beta > 0$ and conditions* (A1)–(A2) *in Lemma 2.23 hold. If* (H1) *and* (H4) *are satisfied, then the problem* (1.1) *has a unique weighted pseudo almost automorphic mild solution on $\mathbb{R}$ provided that $L < \frac{\omega}{M}$.*

Proof. Let $\Gamma : WPAA(\mathbb{R},\mathbb{X}) \to WPAA(\mathbb{R},\mathbb{X})$ be the nonlinear operator defined by

$$
(\Gamma x)(t) = \int_{-\infty}^{t} S(t-s)f(s,x(s))ds, t\in\mathbb{R},
$$

where $\{S(t)\}_{t\ge 0}$ is given in Lemma 2.24. First, let us prove that $\Gamma(WPAA(\mathbb{R},\mathbb{X})) \subseteq WPAA(\mathbb{R},\mathbb{X})$. For each $x\in WPAA(\mathbb{R},\mathbb{X})$, by using Lemmas 2.16 and 2.18, one can easily see that $f(\cdot,x(\cdot)) \in WPAAS^p(\mathbb{R},\mathbb{X})$. Hence, from the proof of Lemma

3.2, we know that $(\Gamma x)(\cdot) \in WPAA(\mathbb{R},\mathbb{X})$. That is, Γ maps $WPAA(\mathbb{R},\mathbb{X})$ into $WPAA(\mathbb{R},\mathbb{X})$.

Next, we prove that Γ is a strict contraction mapping on $WPAA(\mathbb{R},\mathbb{X})$. To this end, for each $t \in \mathbb{R}$, $x, y \in WPAA(\mathbb{R},\mathbb{X})$, we have

$$\begin{aligned}
\|\Gamma x - \Gamma y\|_\infty &:= \sup_{t\in\mathbb{R}} \|\int_{-\infty}^{t} S(t-s)[f(s,x(s)) - f(s,y(s))]ds\| \\
&\le LM \sup_{t\in\mathbb{R}} \int_0^\infty e^{-\omega s}\|x(t-s) - y(t-s)\|ds \\
&\le LM\|x-y\|_\infty \int_0^\infty e^{-\omega s} ds \\
&= \frac{LM}{\omega}\|x-y\|_\infty,
\end{aligned}$$

which implies that Γ is a contraction on $WPAA(\mathbb{R},\mathbb{X})$. Therefore, by the Banach contraction principle, we draw a conclusion that there exists a unique fixed point $x(\cdot)$ for Γ in $WPAA(\mathbb{R},\mathbb{X})$ with $\Gamma x = x$. It is clear that x is the unique weighted pseudo almost automorphic mild solution of (1.1). This completes the proof. □

A different Lipschitz condition is considered in the following results.

Theorem 3.4. *Let $\beta > 0$, $\alpha \neq 0$ with $\alpha + \beta > 0$ and conditions* (A1)–(A2) *in Lemma 2.23 hold. If* (H2)–(H4) *hold, then* (1.1) *has a unique weighted pseudo almost automorphic mild solution whenever $\frac{M}{1-e^{-\omega}}\|L_f\|_{S^p} < 1$.*

Proof. Consider the nonlinear operator Γ is given by

$$(\Gamma x)(t) = \int_{-\infty}^{t} S(t-s)f(s,x(s))ds, t \in \mathbb{R}.$$

For given $x \in WPAA(\mathbb{R},\mathbb{X})$, it follows from Lemmas 2.16 and 2.19 that the function $s \to f(s,x(s))$ is in $WPAAS^p(\mathbb{R},\mathbb{X})$. Moreover, from Lemma 3.2, we infer that $\Gamma x \in WPAA(\mathbb{R},\mathbb{X})$, that is, Γ maps $WPAA(\mathbb{R},\mathbb{X})$ into itself. Next, we prove that the operator Γ has a unique fixed point in $WPAA(\mathbb{R},\mathbb{X})$. Indeed, for each $t \in \mathbb{R}$,$x, y \in WPAA(\mathbb{R},\mathbb{X})$, we have

$$\begin{aligned}
\|\Gamma x(t) - \Gamma y(t)\| &= \Big\|\int_{-\infty}^{t} S(t-s)[f(s,x(s)) - f(s,y(s))]ds\Big\| \\
&\le \int_{-\infty}^{t} Me^{-\omega(t-s)}L_f(s)\|x(s) - y(s)\|ds \\
&\le \sum_{n=1}^{\infty} Me^{-\omega(n-1)} \int_{t-n}^{t-n+1} L_f(s)\|x-y\|_\infty ds \\
&\le \sum_{n=1}^{\infty} Me^{-\omega(n-1)} \Big(\int_{t-n}^{t-n+1} \|L_f(s)\|^p ds\Big)^{1/p} \|x-y\|_\infty \\
&\le \frac{M}{1-e^{-\omega}}\|L_f\|_{S^p}\|x-y\|_\infty.
\end{aligned}$$

Hence

$$\|\Gamma x - \Gamma y\|_\infty \le \frac{M}{1-e^{-\omega}}\|L_f\|_{S^p}\|x-y\|_\infty.$$

Since $\frac{M}{1-e^{-\omega}}\|L_f\|_{S^p} < 1$, Γ has a unique fixed point $x \in WPAA(\mathbb{R}, \mathbb{X})$. The proof is complete. □

Theorem 3.5. *Let $\beta > 0$, $\alpha \neq 0$ with $\alpha + \beta > 0$ and conditions* (A1)–(A2) *in Lemma 2.23 hold. If* (H5) *holds, then* (1.1) *admits a unique weighted pseudo almost automorphic mild solution whenever $\frac{M}{1-e^{-\omega}}\|\mathcal{L}_f\|_{S^r} < 1$.*

Proof. Consider the nonlinear operator Γ given by

$$(\Gamma x)(t) = \int_{-\infty}^{t} S(t-s)f(s, x(s))ds, t \in \mathbb{R}.$$

For given $x \in WPAA(\mathbb{R}, \mathbb{X})$, it follows from Lemmas 2.16 and 2.20 that the function $s \to f(s, x(s))$ is in $WPAAS^{\overline{q}}(\mathbb{R}, \mathbb{X})$. Moreover, from Lemma 3.2, we infer that $\Gamma x \in WPAA(\mathbb{R}, \mathbb{X})$, that is, Γ maps $WPAA(\mathbb{R}, \mathbb{X})$ into itself. Next, we prove that the operator Γ has a unique fixed point in $WPAA(\mathbb{R}, \mathbb{X})$. Indeed, for each $t \in \mathbb{R}$,$x, y \in WPAA(\mathbb{R}, \mathbb{X})$, we have

$$\begin{aligned}
\|\Gamma x(t) - \Gamma y(t)\| &= \Big\| \int_{-\infty}^{t} S(t-s)[f(s, x(s)) - f(s, y(s))]ds \Big\| \\
&\leq \int_{-\infty}^{t} Me^{-\omega(t-s)}\mathcal{L}_f(s)\|x(s) - y(s)\|ds \\
&\leq \sum_{n=1}^{\infty} Me^{-\omega(n-1)} \int_{t-n}^{t-n+1} \mathcal{L}_f(s)\|x - y\|_\infty ds \\
&\leq \sum_{n=1}^{\infty} Me^{-\omega(n-1)} \Big(\int_{t-n}^{t-n+1} \|\mathcal{L}_f(s)\|^r ds \Big)^{1/r} \|x - y\|_\infty \\
&\leq \frac{M}{1-e^{-\omega}}\|\mathcal{L}_f\|_{S^r}\|x - y\|_\infty.
\end{aligned}$$

Hence

$$\|\Gamma x - \Gamma y\|_\infty \leq \frac{M}{1-e^{-\omega}}\|\mathcal{L}_f\|_{S^r}\|x - y\|_\infty.$$

Since $\frac{M}{1-e^{-\omega}}\|\mathcal{L}_f\|_{S^r} < 1$, Γ has a unique fixed point $x \in WPAA(\mathbb{R}, \mathbb{X})$. The proof is complete. □

In the following, we investigate the existence of weighted pseudo almost automorphic solutions to the problem (1.1) when f is not necessarily Lipschitz continuous. For that, we require the following assumptions:

- (H6) $f \in WPAAS^p(\mathbb{R}\times\mathbb{X}, \mathbb{X})$ and $f(t, x)$ is uniformly continuous in any bounded subset $\mathbb{M} \subseteq \mathbb{X}$ uniformly for $t \in \mathbb{R}$ and for every bounded subset $\mathbb{M} \subseteq \mathbb{X}$, $\{f(\cdot, x) : x \in \mathbb{M}\}$ is bounded in $WPAAS^p(\mathbb{X})$.
- (H7) There exists a continuous nondecreasing function $W : [0, \infty) \to (0, \infty)$ such that
$$\|f(t, x)\| \leq W(\|x\|) \quad \text{for all } t \in \mathbb{R} \text{ and } x \in \mathbb{R}.$$

Remark 3.6. Let $f(t, x) = \sin(t) + \sin(\pi t) + \sin\big(\frac{x}{\phi(t)}\big)$, where $\phi(t) = \max\{1, |t|\}$, $t \in \mathbb{R}$. According to [3, Remark 3.4], this defined function also satisfies the condition (H6) with $\rho(t) = 1 + t^2$.

Remark 3.7. For condition (H7), an interesting result (see Corollary 3.9) is given for the perturbation f satisfying the Hölder type condition.

Theorem 3.8. *Assume that $\beta > 0$, $\alpha \neq 0$ with $\alpha + \beta > 0$, and conditions* (A1)–(A2) *in Lemma 2.23 hold. Let $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ be a function satisfying assumptions* (H6), (H7), *and the following additional conditions:*

(i) *For each $\mathbb{C} \geq 0$, the function $t \to \int_{-\infty}^{t} Me^{-\omega(t-s)} W(\mathbb{C}h(s))ds$ belongs to $BC(\mathbb{R})$, where $h(\cdot)$ is defined in* (2.4). *Let*
$$\beta(\mathbb{C}) = \| \int_{-\infty}^{t} Me^{-\omega(t-s)} W(\mathbb{C}h(s))ds \|_h.$$

(ii) *For each $\varepsilon > 0$, there exists $\delta > 0$ such that for each $u, v \in C_h(\mathbb{X})$, $\|u - v\|_h \leq \delta$ implies that*
$$\int_{-\infty}^{t} Me^{-\omega(t-s)} \|f(s, x(s)) - f(s, y(s))\| ds \leq \varepsilon$$
for all $t \in \mathbb{R}$.

(iii) $\liminf_{\xi \to \infty} \xi / \beta(\xi) > 1$.

(iv) *For all $a, b \in \mathbb{R}$, $a < b$, and $\Lambda > 0$, the set $\{f(s, x) : a \leq s \leq b,\ x \in C_h(\mathbb{X}),\ \|x\|_h \leq \Lambda\}$ is relatively compact in $\mathbb{X}$.*

Then (1.1) *admits at least one weighted pseudo almost autorphic mild solution on* $\mathbb{R}$.

Proof. We define the nonlinear operator $\Gamma : C_h(\mathbb{X}) \to C_h(\mathbb{X})$ by
$$(\Gamma x)(t) := \int_{-\infty}^{t} S(t - s)f(s, x(s))ds, t \in \mathbb{R}.$$
We will show that Γ has a fixed point in $WPAA(\mathbb{R}, \mathbb{X})$. For the sake of convenience, we divide the proof into several steps.

(I) For $x \in C_h(\mathbb{X})$, we have that
$$\|(\Gamma x)(t)\| \leq \int_{-\infty}^{t} Me^{-\omega(t-s)} W(\|x(s)\|)ds \leq \int_{-\infty}^{t} Me^{-\omega(t-s)} W(\|x\|_h h(s))ds.$$
It follows from condition (i) that Γ is well defined.

(II) The operator Γ is continuous. In fact, for any $\varepsilon > 0$, we take $\delta > 0$ involved in condition (ii). If $x, y \in C_h(\mathbb{X})$ and $\|x - y\|_h \leq \delta$, then
$$\|(\Gamma x)(t) - (\Gamma y)(t)\| \leq \int_{-\infty}^{t} Me^{-\omega(t-s)} \|f(s, x(s)) - f(s, y(s))\| ds \leq \varepsilon,$$
which shows the assertion.

(III) The operator Γ is completely continuous. Set $B_\Lambda(\mathbb{X})$ for the closed ball with center at 0 and radius Λ in the space $\mathbb{X}$. Let $V = \Gamma(B_\Lambda(C_h(\mathbb{X})))$ and $v = \Gamma(x)$ for $x \in B_\Lambda(C_h(\mathbb{X}))$. First, we will prove that $V(t)$ is a relatively compact subset of $\mathbb{X}$ for each $t \in \mathbb{R}$. It follows form condition (i) that the function $s \to Me^{-\omega s} W(\Lambda h(t - s))$ is integrable on $[0, \infty)$. Hence, for $\varepsilon > 0$,we can choose $a \geq 0$ such that $\int_a^\infty Me^{-\omega s} W(\Lambda h(t - s)ds) \leq \varepsilon$. Since
$$v(t) = \int_0^a S(s)f(t - s, x(t - s))ds + \int_a^\infty S(s)f(t - s, x(t - s))ds$$
and
$$\| \int_a^\infty S(s)f(t - s, x(t - s))ds\| \leq \int_a^\infty Me^{-\omega s} W(\Lambda h(t - s)ds) \leq \varepsilon,$$

we obtain that $v(t) \in \overline{ac_0(\mathcal{N})} + B_\varepsilon(\mathbb{X})$, where $c_0(\mathcal{N})$ denotes the convex hull of $\mathcal{N}$ and $\mathcal{N} := \{S(s)f(\xi, x) : 0 \le s \le a, t-a \le \xi \le t, \|x\|_h \le \Lambda\}$. Just as the proofs in [4, Theorem 3.5(ii)] and [16, Theorem 4.9(iii)], in view of the strong continuity of $S(\cdot)$ and property (iv) of f, we infer that $\mathcal{N}$ is a relatively compact set, and $V(t) \subseteq \overline{ac_0(\mathcal{N})} + B_\varepsilon(\mathbb{X})$, which establishes our assertion.

Second, we show that the set V is equicontinuous. In fact, we can decompose

$$\begin{aligned} v(t+s) - v(t) \le & \int_0^s S(\sigma) f(t+s-\sigma, x(t+s-\sigma)) d\sigma \\ & + \int_0^a [S(\sigma+s) - S(\sigma)] f(t-\sigma, x(t-\sigma)) d\sigma \\ & + \int_a^\infty [S(\sigma+s) - S(\sigma)] f(t-\sigma, x(t-\sigma)) d\sigma. \end{aligned}$$

For each $\varepsilon > 0$, we can choose $a > 0$ and $\delta_1 > 0$ such that

$$\begin{aligned} & \| \int_0^s S(\sigma) f(t+s-\sigma, x(t+s-\sigma)) d\sigma + \int_a^\infty [S(\sigma+s) - S(\sigma)] f(t-\sigma, x(t-\sigma)) d\sigma \| \\ & \le \int_0^s M e^{-\omega\sigma} W(\Lambda h(t+s-\sigma)) d\sigma + \int_a^\infty (M e^{-\omega(\sigma+s)} + M e^{-\omega\sigma}) W(\Lambda h(t-\sigma)) d\sigma \\ & \le \frac{\varepsilon}{2} \end{aligned}$$

for $s \le \delta_1$. Moreover, since $\{f(t-\sigma, x(t-\sigma)) : 0 \le \sigma \le a, x \in B_\Lambda(C_h(\mathbb{X}))\}$ is a relatively compact set and $S(\cdot)$ is strongly continuous, we can choose $\delta_2 > 0$ such that $\|[S(\sigma+s) - S(\sigma)] f(t-\sigma, x(t-\sigma))\| \le \frac{\varepsilon}{2a}$ for $s \le \delta_2$. Combining these estimates, we get $\|v(t+s) - v(t)\| \le \varepsilon$ for s small enough and independent of $x \in B_\Lambda(C_h(\mathbb{X}))$.

Finally, applying condition (i) in Theorem 3.8, we can see that

$$\frac{\|v(t)\|}{h(t)} \le \frac{1}{h(t)} \int_{-\infty}^t M e^{-\omega(t-s)} W(\Lambda h(s)) ds \to 0, \quad |t| \to \infty,$$

and this convergence is independent of $x \in B_\Lambda(C_h(\mathbb{X}))$. Hence, by Lemma 2.25, V is a relatively compact set in $C_h(\mathbb{X})$.

(IV) The set $\{x^\lambda : x^\lambda = \lambda\Gamma(x^\lambda), \lambda \in (0,1)\}$ is bounded. Let us show assume that $x^\lambda(\cdot)$ is a solution of equation $x^\lambda = \lambda\Gamma(x^\lambda)$ for some $0 < \lambda < 1$. We can estimate

$$\begin{aligned} \|x^\lambda(t)\| &= \lambda \| \int_{-\infty}^t S(t-s) f(s, x^\lambda(s)) ds \| \\ &\le \lambda \int_{-\infty}^t M e^{-\omega(t-s)} W(\|x^\lambda\|_h h(s)) ds \\ &\le \beta(\|x^\lambda\|_h) h(t). \end{aligned}$$

Hence, we obtain

$$\frac{\|x^\lambda\|_h}{\beta(\|x^\lambda\|_h)} \le 1$$

and combining with condition (iii), we conclude that the set $\{x^\lambda : x^\lambda = \lambda\Gamma(x^\lambda), \lambda \in (0,1)\}$ is bounded.

(V) It follows from Lemma 2.16, (H4), (H6) and Lemma 2.21 that the function $t \to f(t, x(t))$ belongs to $WPAAS^p(\mathbb{R}, \mathbb{X})$, whenever $x \in WPAA(\mathbb{R}, \mathbb{X})$. Moreover, from Lemma 3.2 we infer that $\Gamma(WPAA(\mathbb{R}, \mathbb{X})) \subseteq WPAA(\mathbb{R}, \mathbb{X})$ and nothing that $WPAA(\mathbb{R}, \mathbb{X})$ is a closed subspace of $C_h(\mathbb{X})$, consequently, we can consider Γ :

$WPAA(\mathbb{R}, \mathbb{X}) \to WPAA(\mathbb{R}, \mathbb{X})$. Using properties (I)–(III), we deduce that this map is completely continuous. Applying Lemma 2.26, we infer that Γ has a fixed point $x \in WPAA(\mathbb{R}, \mathbb{X})$, which completes the proof. □

Taking into account Lemma 2.21 and Theorem 3.8, and using the same argument as in [16, Corollary 4.10], we can obtain the following result.

Corollary 3.9. *Let $\beta > 0$, $\alpha \neq 0$ with $\alpha + \beta > 0$ and conditions* (A1)–(A2) *in Lemma 2.23 hold. Assume that $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ satisfies assumption* (H6) *and the Hölder type condition*

$$\|f(t, x) - f(t, y)\| \leq \zeta \|x - y\|^{\gamma}, \ 0 < \gamma < 1,$$

for all $t \in \mathbb{R}$ and x, $y \in \mathbb{X}$, where ζ is a positive constant. Moreover, assume the following conditions:

(a1) $f(t, 0) = \eta$.
(a2) $M \sup_{t \in \mathbb{R}} \int_{-\infty}^{t} e^{-\omega(t-s)} h(s)^{\gamma} ds = \vartheta < \infty$.
(a3) *For all a, $b \in \mathbb{R}$, $a < b$, and $r > 0$, the set $\{f(s, x) : a \leq s \leq b, x \in \mathbb{X}, \|x\|_h \leq r\}$ is relatively compact in $\mathbb{X}$.*

Then (1.1) *has a weighted pseudo almost automorphic mild solution on $\mathbb{R}$.*

Example 3.10. Consider the problem

$$\begin{gathered} \frac{\partial u}{\partial t}(t, x) = \frac{\partial^2 u}{\partial x^2}(t, x) + \int_{-\infty}^{t} e^{-(t-s)} \frac{\partial^2 u}{\partial x^2}(s, x) ds + f(t, u(t, x)), \quad t \in \mathbb{R}, \\ u(0, t) = u(\pi, t) = 0, \end{gathered} \tag{3.1}$$

where $f \in L^2[0, \pi] \to L^2[0, \pi]$ is given by

$$f(t, u(t, x)) = u(t, x) \sin \frac{1}{2 + \cos t + \cos \sqrt{2} t} + e^{-|t|} \sin u(t, x).$$

We set $X := L^2[0, \pi]$ and define $A := \frac{\partial^2 u}{\partial x^2}$, with domain of the operator A as

$$D(A) := \{u \in L^2[0, \pi] : u(0) = u(\pi) = 0, u'' \in L^2[0, \pi]\}.$$

From the computation of [18, Example 4.10], we can see the operator A satisfies condition (A2) in Lemma 2.23 and generates an immediately norm continuous and compact C_0-semigroup $T(t)$ on $\mathbb{X}$. Thus (3.1) can be converted into the abstract system (1.1) with $\alpha = \beta = 1$.

Note that the function f defined above is an S^p-weighted almost automorphic function with weight $\rho(t) = |t|$ for $t \in \mathbb{R}$, and

$$\|f(t, u) - f(t, v)\| \leq 2\|u - v\|.$$

The following corollary is a consequence of Theorem 3.3.

Corollary 3.11. *Problem* (3.1) *admits a unique weighted pseudo almost automorphic solution with weight $\rho(t) = |t|$ provided that $\frac{M}{\omega} < 1/2$.*

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References

[1] S. Abbas; Pseudo almost automorphic solutions of some nonlinear integro-differential equations, *Comp. Math. Appl.*, **62** (2011), 2259-2272.

[2] S. Abbas, Y. Xia; Existence and attractivity of k-almost automorphic solutions of a model of cellular neural network with delay, *Acta Mathematica Scientia*, **1** (2013), 290-302.

[3] R. P. Agarwal, B. de Andrade, C. Cuevas; Weighted pseudo-almost periodic solutions of a class of semilinear fractional differential equations, *Nonlinear Anal. RWA* **11** (2010), 3532-3554.

[4] R. P. Agarwal, B. de Andrade, C. Cuevas; On type of periodicity and ergodicity to a class of fractional order differential equations, *Adv. Difference Equ. Article*, ID 179750, 1-25, 2010.

[5] S. Bochner; Continuous mappings of almost automorphic and almost periodic functions, *Proc. Natl. Acad. Sci. USA*, **52** (1964), 907-910.

[6] J. Blot, G. M. Mophou, G. M. N'Guérékata, D. Pennequin; Weighted pseudo almost automorphic functions and applications to abstract differential equations, *Nonlinear Anal.*, **71** (2009), 903-909.

[7] Y. K. Chang, Z. H. Zhao, J. J. Nieto; Pseudo almost automorphic and weighted pseudo almost automorphic mild solutions to semi-linear differential equations in Hilbert spaces, *Rev. Mat. Complut.* **24** (2011), 421-438.

[8] Y. K. Chang, R. Zhang, G. M. N'Guérékata; Weighted pseudo almost automorphic mild solutions to fractional differential equations, *Comp. Math. Appl.*, **64** (2012), 3160-3170.

[9] T. Diagana, G. M. N'Guérékata; Stepanov-like almost automorphic functions and applications to some semilinear equations, *Appl. Anal.*, **86** (2007), 723-733.

[10] T. Diagana; *Almost Automorphic Type and Almost Periodic Type Functions in Abstract Spaces*, Springer, New York, 2013.

[11] H. S. Ding, J. Liang, G. M. N'Guérékata, T. J. Xiao; Existence of positive almost automorphic solutions to neutral nonlinear integral equations, *Nonlinear Anal.*, **69** (2008), 1188-1199.

[12] H. S. Ding, W. Long, G. M. N'Guérékata; Almost automorphic solutions of nonautonomous evolution equations, *Nonlinear Anal.*, **70** (2009), 4158-4164.

[13] H. S. Ding, J. Liang, T. J. Xiao; Almost automorphic solutions to nonautonomous semilinear evolution equations in Banach spaces, *Nonlinear Anal.*, **73** (2010), 1426-1438.

[14] H. S. Ding, T. J. Xiao, J. Liang; Existence of positive almost automorphic solutions to nonlinear delay integral equations, *Nonlinear Anal.*, **70** (2009), 2216-2231.

[15] A. Granas, J. Dugundji; *Fixed Point Theory*, Springer, New York, 2003.

[16] H. R. Henríquez, C. Lizama; Compoct almost automorphic solutions to integral equations with infinite delay, *Nonlinear Anal.*, **71** (2009), 6029-6037.

[17] J. Liang, J. Zhang, T. J.Xiao; Composition of pseudo almost automorphic and asymptotically almost automorphic functions, *J. Math. Anal. Appl.*, **340** (2008), 1493-1499.

[18] C. Lizama, R. Ponce; Bounded solutions to a class of semilinear integro-differential equations in Banach spaces, *Nonlinear Anal.*, **74** (2011), 3397-3406.

[19] C. Lizama, G. M. N'Guérékata; Mild solutions for abstract fractional differential equations, *Appl. Anal.*, **92** (2013), 1731-1754.

[20] G. M. Mophou; Weighted pseudo almost automorphic mild solutions to semilinear fractional differential equations, *Appl. Math. Comp.*, **217** (2011), 7579-7587.

[21] G. M. N'Guérékata; Sue les solutions presqu'automorphes d'équations différentielles abstraites, *Annales des Sciences Mathématqiues du Québec*, **1** (1981), 69-79.

[22] G. M. N'Guérékata; *Almost Automorphic and Almost Periodic Functions in Abstract Spaces*, Kluwer Academic, New York, 2001.

[23] G. M. N'Guérékata; *Topics in Almost Automorphy*, Springer, New York, 2005.

[24] G. M. N'Guérékata, A. Pankov, Stepanov-like almost automorphic functions and monotone evolution; *Nonlinear Anal.*, **68** (2008), 2658-2667.

[25] R. Sakthivel, P. Revathi, S. Marshal Anthoni; Existence of pseudo almost automorphic mild solutions to stochastic fractional differential equations, *Nonlinear Anal.*, **75** (2012), 3339-3347.

[26] Z. N. Xia, M. Fan; Weighted Stepanov-like pseudo almost automorphy and applications, *Nonlinear Anal.*, **75** (2012), 2378-2397.

[27] T. J. Xiao, J. Liang, J. Zhang; Pseudo almost automorphic solutions to semilinear differential equations in Banach spaces, *Semigroup Forum*, **76** (2008), 518-524.

[28] R. Zhang, Y. K. Chang, G. M. N'Guérékata; New composition theorems of Stepanov-like almost automorphic functions and applications to nonautonomous evolution equations, *Nonlinear Anal. RWA*, **13** (2012), 2866-2879.

[29] R. Zhang, Y. K. Chang, G. M. N'Guérékata; Weighted pseudo almost automorphic solutions for non-autonomous neutral functional differential equations with infinite delay (in Chinese), *Sci. Sin. Math.*, **43** (2013), 273-292, doi: 10.1360/012013-9.

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