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ENTROPY SOLUTIONS FOR NONLINEAR ELLIPTIC EQUATIONS WITH VARIABLE EXPONENTS

CHAO ZHANG

Abstract. In this article we prove the existence and uniqueness of entropy solutions for $p(x)$-Laplace equations with a Radon measure which is absolutely continuous with respect to the relative $p(x)$-capacity. Moreover, the existence of entropy solutions for weighted $p(x)$-Laplace equation is also obtained.

1. Introduction

The study of partial differential equations and variational problems with non-standard growth conditions has been received considerable attention by many models coming from various branches of mathematical physics, such as elastic mechanics, image processing and electro-rheological fluid dynamics, etc. We refer the readers to [7, 10, 24, 26] and references therein.

Let Ω be a bounded open domain in $\mathbb{R}^N$ $(N \geq 2)$ with Lipschitz boundary $\partial\Omega$. In this article we consider the nonlinear elliptic problem

$$\begin{gathered} -\operatorname{div}\left(w(x)|\nabla u|^{p(x)-2}\nabla u\right) = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where the variable exponent $p : \overline{\Omega} \to (1, \infty)$ is a continuous function, w is a weight function and $f \in L^1(\Omega)$.

When dealing with the p-Laplacian type equations with L^1 or measure data, it is reasonable to work with entropy solutions or renormalized solutions, which need less regularity than the usual weak solutions. The notion of entropy solutions has been proposed by Bénilan et al. in [3] for the nonlinear elliptic problems. This framework was extended to related problems with constant p in [1, 5, 6, 23] and variable exponents $p(x)$ in [2, 25, 27, 28]. The interesting and difficult cases are those of $1 < p \leq N$, since the variational methods of Leray-Lions (see [21]) can be easily applied for $p > N$.

Recently, when $w(x) \equiv 1$, the existence and uniqueness of entropy solutions of $p(x)$-Laplace equation with L^1 data were proved in [27] by Sanchón and Urbano. The proofs rely crucially on *a priori* estimates in Marcinkiewicz spaces with variable exponents. Moreover, in [28] we extended the results in [27] to the case of a signed measure μ in $L^1(\Omega) + W^{-1,p'(\cdot)}(\Omega)$. In view of a refined method which is

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slightly different from [27], we obtained that the entropy solution of problem (1.1) is also a renormalized solution and proved the uniqueness of entropy solutions and renormalized solutions, and thus the equivalence of entropy solutions and renormalized solutions. Especially, when p is a constant function, w is an A_p weight and $f \in L^1(\Omega)$, Cavalheiro in [6] proved the existence of entropy solutions for the Dirichlet problem (1.1).

This work is a natural extension of the results in [6, 28]. The novelties in this paper are mainly two parts. First, when p is a constant function, we know from [5] that $\mu \in L^1(\Omega) + W^{-1,p'}(\Omega)$ if and only if $\mu \in \mathcal{M}_b^p(\Omega)$, i.e., every signed measure that is zero on the sets of zero p-capacity can be decomposed into the sum of a function in $L^1(\Omega)$ and an element in $W^{-1,p'}(\Omega)$, and conversely, every signed measure in $L^1(\Omega) + W^{-1,p'}(\Omega)$ has zero measure for the sets of zero p-capacity. In our previous paper [28], we proposed an open problem: what about the similar decomposition result for the variable exponent case? By using the similar arguments as in [5] and employing the properties of $L^{p(\cdot)}(\Omega)$ and the relative $p(\cdot)$-capacity (see [17]), we try to give a positive answer for this question. Although the proof follows basically the steps in [5], it is not a straightforward generalization of the same result for constant exponents which needs a more careful analysis to derive the conclusion. Second, as far as we know, there are no papers concerned with the entropy solutions for the weighted $p(x)$-Laplace equations. The main difficulty is that there are few results for the $A_{p(\cdot)}$-weight whenever p is not constant function. We refer the readers to paper [16] by Hästö and Diening for the latest results. The properties of weighted variable exponent Lebesgue-Sobolev spaces in [16, 19] provide a way to prove the existence of entropy solutions for problem (1.1).

Now we review the definitions and basic properties of the weighted generalized Lebesgue spaces $L^{p(x)}(\Omega, w)$ and weighted generalized Lebesgue-Sobolev spaces $W^{k,p(x)}(\Omega, w)$.

Let w be a measurable positive and a.e. finite function in $\mathbb{R}^N$. Set $C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} h(x) > 1\}$. For any $h \in C_+(\overline{\Omega})$ we define

$$h_+ = \sup_{x\in\Omega} h(x) \quad \text{and} \quad h_- = \inf_{x\in\Omega} h(x).$$

For any $p \in C_+(\overline{\Omega})$, we introduce the weighted variable exponent Lebesgue space $L^{p(\cdot)}(\Omega, w)$ to consist of all measurable functions such that

$$\int_\Omega w(x)|u(x)|^{p(x)}\,dx < \infty,$$

endowed with the Luxemburg norm

$$\|u\|_{L^{p(x)}(\Omega,w)} = \inf\Big\{\lambda > 0 : \int_\Omega w(x)\Big|\frac{u(x)}{\lambda}\Big|^{p(x)}\,dx \le 1\Big\}.$$

For any positive integer k, denote

$$W^{k,p(x)}(\Omega, w) = \{u \in L^{p(x)}(\Omega, w) : D^\alpha u \in L^{p(x)}(\Omega, w), |\alpha| \le k\},$$

with the norm

$$\|u\|_{W^{k,p(x)}(\Omega,w)} = \sum_{|\alpha|\le k} \|D^\alpha u\|_{L^{p(x)}(\Omega,w)}.$$

An interesting feature of a generalized Lebesgue-Sobolev space is that smooth functions are not dense in it without additional assumptions on the exponent $p(x)$. This was observed by Zhikov [29] in connection with Lavrentiev phenomenon. However, when the exponent $p(x)$ is *log-Hölder* continuous, i.e., there is a constant C such that

$$|p(x) - p(y)| \le \frac{C}{-\log|x-y|} \tag{1.2}$$

for every $x, y \in \Omega$ with $|x-y| \le 1/2$, then smooth functions are dense in variable exponent Sobolev spaces and there is no confusion in defining the Sobolev space with zero boundary values, $W_0^{1,p(\cdot)}(\Omega)$, as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{W^{1,p(\cdot)}(\Omega)}$ (see [15]).

Let T_k denote the truncation function at height $k \ge 0$:

$$T_k(r) = \min\{k, \max\{r, -k\}\} = \begin{cases} k & \text{if } r \ge k, \\ r & \text{if } |r| < k, \\ -k & \text{if } r \le -k. \end{cases}$$

Denote

$$\mathcal{T}_0^{1,p(\cdot)}(\Omega) = \{u : u \text{ is measurable, } T_k(u) \in W_0^{1,p(\cdot)}(\Omega, w), \text{ for every } k > 0\}.$$

Next we define the very weak gradient of a measurable function $u \in \mathcal{T}_0^{1,p(\cdot)}(\Omega)$. As a matter of the fact, working as in [3, Lemma 2.1], we have the following result.

Proposition 1.1. *For every function $u \in \mathcal{T}_0^{1,p(\cdot)}(\Omega)$, there exists a unique measurable function $v : \Omega \to \mathbb{R}^N$, which we call the very weak gradient of u and denote $v = \nabla u$, such that*

$$\nabla T_k(u) = v\chi_{\{|u|<k\}} \quad \text{for a.e. } x \in \Omega \text{ and for every } k > 0,$$

where χ_E denotes the characteristic function of a measurable set E. Moreover, if u belongs to $W_0^{1,1}(\Omega, w)$, then v coincides with the weak gradient of u.

The notion of the very weak gradient allows us to give the following definition of entropy solutions for problem (1.1).

Definition 1.2. A function $u \in \mathcal{T}_0^{1,p(\cdot)}(\Omega)$ is called an entropy solution to problem (1.1) if

$$\int_\Omega w(x)|\nabla u|^{p(x)-2}\nabla u \cdot \nabla T_k(u-\phi)\,dx = \int_\Omega f T_k(u-\phi)\,dx, \tag{1.3}$$

for all $\phi \in W_0^{1,p(x)}(\Omega, w) \cap L^\infty(\Omega)$.

The rest of this paper is organized as follows. In Section 2, we prove the existence and uniqueness of entropy solutions for $p(x)$-Laplace equation with a Radon measure which is absolutely continuous with respect to the relative $p(\cdot)$-capacity. The existence of entropy solutions for weighted $p(x)$-Laplace equation will be considered in Section 3. In the following sections C will represent a generic constant that may change from line to line even if in the same inequality.

2. Unweighted case

In this section, we prove the existence and uniqueness of entropy solutions for the following problem

$$\begin{gathered} -\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = \mu \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{2.1}$$

where μ a Radon measure which is absolutely continuous with respect to the relative $p(\cdot)$-capacity. First we state some results that will be used later.

Lemma 2.1 ([13, 20]). (1) *The space $L^{p(\cdot)}(\Omega)$ is a separable, uniform convex Banach space, and its conjugate space is $L^{p'(\cdot)}(\Omega)$ where $1/p(x)+1/p'(x)=1$. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, we have*

$$\Big|\int_\Omega uv\,dx\Big| \le \big(\frac{1}{p_-}+\frac{1}{(p_-)'}\big)\|u\|_{L^{p(x)}(\Omega)}\|v\|_{L^{p'(x)}(\Omega)} \le 2\|u\|_{L^{p(x)}(\Omega)}\|v\|_{L^{p'(x)}(\Omega)};$$

(2) *If $p_1, p_2 \in C_+(\overline{\Omega}), p_1(x) \le p_2(x)$ for any $x \in \Omega$, then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$, whose norm does not exceed $|\Omega|+1$.*

Lemma 2.2 ([13]). *If we denote*

$$\rho(u) = \int_\Omega |u|^{p(x)}\,dx, \quad \forall u \in L^{p(x)}(\Omega),$$

then

$$\min\{\|u\|^{p_-}_{L^{p(x)}(\Omega)}, \|u\|^{p_+}_{L^{p(x)}(\Omega)}\} \le \rho(u) \le \max\{\|u\|^{p_-}_{L^{p(x)}(\Omega)}, \|u\|^{p_+}_{L^{p(x)}(\Omega)}\}.$$

Lemma 2.3 ([13]). *$W^{k,p(x)}(\Omega)$ is a separable and reflexive Banach space.*

Lemma 2.4 ([18, 20]). *Let $p \in C_+(\overline{\Omega})$ satisfy the log-Hölder continuity condition* (1.2). *Then, for $u \in W_0^{1,p(\cdot)}(\Omega)$, the $p(\cdot)$-Poincaré inequality*

$$\|u\|_{L^{p(x)}(\Omega)} \le C\|\nabla u\|_{L^{p(x)}(\Omega)}$$

holds, where the positive constant C depends on p, N and Ω.

Lemma 2.5 ([9, 12]). *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded set with Lipschitz boundary and $p(x) \in C_+(\overline{\Omega})$ with $1 < p_- \le p_+ < N$ satisfy the log-Hölder continuity condition* (1.2). *If $q \in L^\infty(\Omega)$ with $q_- > 1$ satisfies*

$$q(x) \le p^*(x) := \frac{Np(x)}{N-p(x)}, \quad \forall x \in \Omega,$$

then we have

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$$

and the imbedding is compact if $\inf_{x\in\Omega}(p^*(x)-q(x)) > 0$.

A variable exponent version of the relative $p(\cdot)$-capacity of the condenser has been used in [17]. This alternative capacity of a set is taken relative to a surrounding open subset of $\mathbb{R}^N$. Suppose that $p_+ < \infty$ and $p(x)$ satisfies the log-Hölder continuity condition (1.2). Let $K \subset \Omega$. The relative $p(\cdot)$-capacity of K in Ω is the number

$$\operatorname{cap}_{p(\cdot)}(K,\Omega) = \inf\Big\{\int_\Omega |\nabla\varphi|^{p(x)}\,dx : \varphi \in C_0^\infty(\Omega) \text{ and } \varphi \ge 1 \text{ in } K\Big\}.$$

For an open set $U \subset \Omega$ we define

$$\operatorname{cap}_{p(\cdot)}(U,\Omega) = \sup\big\{\operatorname{cap}_{p(\cdot)}(K,\Omega) : K \subset U \text{ compact}\big\}$$

and for an arbitrary $E \subset \Omega$,

$$\operatorname{cap}_{p(\cdot)}(E,\Omega) = \inf\big\{\operatorname{cap}_{p(\cdot)}(U,\Omega) : U \supset E \text{ open}\big\}.$$

Then

$$\operatorname{cap}_{p(\cdot)}(E,\Omega) = \sup\big\{\operatorname{cap}_{p(\cdot)}(K,\Omega) : K \subset E \text{ compact}\big\}$$

for all Borel sets $E \subset \Omega$. The number $\mathrm{cap}_{p(\cdot)}(E,\Omega)$ is called the variational $p(\cdot)$-capacity of E relative to Ω. We usually call it simply the relative $p(\cdot)$-capacity of the pair. The relative $p(\cdot)$-capacity is an outer capacity.

We say that a function $f : \Omega \to \overline{\mathbb{R}}$ is $p(\cdot)$-quasi continuous if for every $\varepsilon > 0$ there exists an open set $A \subset \Omega$ with $\mathrm{cap}_{p(\cdot)}(A,\Omega) \le \varepsilon$, such that $f|_{\Omega\setminus A}$ is continuous. Every $u \in W^{1,p(\cdot)}(\Omega)$ has a $p(\cdot)$-quasi continuous representative (see [5, 17]), always denoted in this paper by $\tilde{u}$, which is essentially unique.

Denote by $\mathcal{M}_b(\Omega)$ the space of all signed measures on Ω, i.e., the space of all σ-additive set functions μ with values in $\mathbb{R}$ defined on the Borel σ-algebra. If μ belongs to $\mathcal{M}_b(\Omega)$, then $|\mu|$ (the total variation of μ) is a bounded positive measure on Ω. We will denote by $\mathcal{M}_b^{p(\cdot)}(\Omega)$ the space of all measures μ in $\mathcal{M}_b(\Omega)$ such that $\mu(E) = 0$ for every set E satisfying $\mathrm{cap}_{p(\cdot)}(E,\Omega) = 0$. Examples of measures in $\mathcal{M}_b^{p(\cdot)}(\Omega)$ are the $L^1(\Omega)$ functions, or the measures in $W^{-1,p'(\cdot)}(\Omega)$.

Next we have a decomposition of a measure in $\mathcal{M}_b^{p(\cdot)}(\Omega)$.

Proposition 2.6. *Assume that $p(x)$ satisfies the log-Hölder condition* (1.2) *with $1 < p_- \le p_+ < +\infty$. Let μ be an element of $\mathcal{M}_b(\Omega)$. Then $\mu \in L^1(\Omega) + W^{-1,p'(\cdot)}(\Omega)$ if and only if $\mu \in \mathcal{M}_b^{p(\cdot)}(\Omega)$. Thus, if $\mu \in \mathcal{M}_b^{p(\cdot)}(\Omega)$, there exist f in $L^1(\Omega)$ and F in $(L^{p'(\cdot)}(\Omega))^N$, such that*

$$\mu = f - \operatorname{div} F,$$

in the sense of distributions.

Proof. Necessity. If μ belongs to $L^1(\Omega) + W^{-1,p'(\cdot)}(\Omega)$, then there exist $f \in L^1(\Omega)$ and $F \in L^{p'(\cdot)}(\Omega)$ such that $\mu = f - \operatorname{div} F$. We just need to show that $\mu(E) = 0$ for every set $E \subset \Omega$ such that $\mathrm{cap}_{p(\cdot)}(E,\Omega) = 0$. It is easy to see that $\mu \in \mathcal{M}_b(\Omega)$. From the definition of $p(\cdot)$-capacity and the similar arguments as in Lemma 2.4 of [22], there is a Borel set $E_0 \subset \Omega$ such that $E \subset E_0$ and $\mathrm{cap}_{p(\cdot)}(E_0,\Omega) = 0$. Let $K \subset E_0$ be compact and $\Omega' \subset \Omega$ an open set containing K. Then there is a sequence $(\varphi_j) \subset C_0^\infty(\Omega')$ such that $0 \le \varphi_j \le 1$, $\varphi_j = 1$ in K and $\int_{\Omega'} |\nabla \varphi_j|^{p(x)} dx \to 0$ as $j \to \infty$. Then we have

$$|\mu(K)| \le \Big| \int_{\Omega'} \varphi_j \, d\mu \Big| \le \Big| \int_{\Omega'} f \varphi_j \, dx + \int_{\Omega'} F \cdot \nabla \varphi_j \, dx \Big|.$$

Choosing the regular functions $\{f_n\}$ such that $\|f_n - f\|_{L^1(\Omega)} \to 0$ as $n \to \infty$ and applying Lemmas 2.1, 2.2 and 2.4 yield that

$$\begin{aligned}
|\mu(K)| &\le \int_{\Omega'} |f_n - f| \cdot |\varphi_j| \, dx + \int_{\Omega'} |f_n| \cdot |\varphi_j| \, dx + \int_{\Omega'} |F| \cdot |\nabla \varphi_j| \, dx \\
&\le \|\varphi_j\|_{L^\infty(\Omega')} \|f_n - f\|_{L^1(\Omega')} + 2\|f_n\|_{L^{p'(x)}(\Omega')} \|\varphi_j\|_{L^{p(x)}(\Omega')} \\
&\quad + 2\|F\|_{L^{p'(x)}(\Omega')} \|\nabla \varphi_j\|_{L^{p(x)}(\Omega')} \\
&\le \|\varphi_j\|_{L^\infty(\Omega')} \|f_n - f\|_{L^1(\Omega')} + C\|f_n\|_{L^{p'(x)}(\Omega')} \|\nabla \varphi_j\|_{L^{p(x)}(\Omega')} \\
&\quad + 2\|F\|_{L^{p'(x)}(\Omega')} \|\nabla \varphi_j\|_{L^{p(x)}(\Omega')} \\
&\le \|\varphi_j\|_{L^\infty(\Omega')} \|f_n - f\|_{L^1(\Omega')} + C\|f_n\|_{L^{p'(x)}(\Omega')} \Big(\int_{\Omega'} |\nabla \varphi_j|^{p(x)} \, dx \Big)^\gamma \\
&\quad + 2\|F\|_{L^{p'(x)}(\Omega')} \Big(\int_{\Omega'} |\nabla \varphi_j|^{p(x)} \, dx \Big)^\gamma,
\end{aligned}$$

where

$$\gamma = \begin{cases} 1/p_- & \text{if } \|\nabla\varphi_j\|_{L^{p(x)}(\Omega')} \geq 1, \\ 1/p_+ & \text{if } \|\nabla\varphi_j\|_{L^{p(x)}(\Omega')} \leq 1. \end{cases}$$

It follows that for all compact $K \subset E_0$,

$$|\mu(K)| \leq C\|f_n - f\|_{L^1(\Omega')} \quad \text{as} \quad j \to \infty,$$

where C is a positive constant that does not depend on n. Moreover, it implies that $\mu(K) = 0$ as $n \to \infty$, and then $\mu(E) \leq \mu(E_0) = \sup\{\mu(K) : K \subset E_0 \text{ compact}\} = 0$ by the regularity of μ.

Sufficiency. Motivated by the ideas developed in [5, 8, 11] with constant exponents, we sketch the proof. In the following we assume that μ is positive. (If not, we write $\mu = \mu^+ - \mu^-$.)

Step 1. First we prove that every measure μ in $\mathcal{M}_b^{p(\cdot)}(\Omega)$ can be decomposed as $\mu = f\gamma^{\rm meas}$, i.e., $d\mu = f d\gamma^{\rm meas}$, with f a positive Borel measurable function in $L^1(\Omega, \gamma^{\rm meas})$ and $\gamma^{\rm meas}$ a positive measure in $W^{-1,p'(\cdot)}(\Omega)$. Indeed, for any $u \in W_0^{1,p(\cdot)}(\Omega)$, let $\tilde{u}$ be the $p(\cdot)$-quasi continuous representative of u. Since $\tilde{u}$ is uniquely defined up to sets of zero $p(\cdot)$-capacity, we can define the functional $F : W_0^{1,p(\cdot)}(\Omega) \to [0, +\infty]$ by

$$F(u) = \int_\Omega \max\{\tilde{u}, 0\}\, d\mu.$$

Clearly, F is convex and lower semi-continuous on $W_0^{1,p(\cdot)}(\Omega)$. Since $W^{1,p(\cdot)}(\Omega)$ is separable from Lemma 2.3, the function F is the supremum of a countable family of continuous affine functions. Therefore, there exist a sequence $\{\lambda_n\}$ in $W^{-1,p'(\cdot)}(\Omega)$ and a sequence $\{a_n\}$ in $\mathbb{R}$ such that

$$F(u) = \sup_{n\in\mathbb{N}}\{\langle\lambda_n, u\rangle + a_n\}$$

for every $u \in W_0^{1,p(\cdot)}(\Omega)$. Since, for any positive t, $tF(u) = F(tu) \geq t\langle\lambda_n, u\rangle + a_n$ for every n, dividing by t and let $t \to +\infty$, we get $F(u) \geq \langle\lambda_n, u\rangle$ for all $u \in W_0^{1,p(\cdot)}(\Omega)$. For $u = 0$, we deduce that $a_n \leq 0$. Thus

$$F(u) \geq \sup_n\langle\lambda_n, u\rangle \geq \sup_n\{\langle\lambda_n, u\rangle + a_n\} = F(u), \tag{2.2}$$

which implies that

$$F(u) = \sup_{n\in\mathbb{N}}\langle\lambda_n, u\rangle. \tag{2.3}$$

In view of (2.3) and the definition of F, for all $\varphi \in C_0^\infty(\Omega)$, we have

$$\langle\lambda_n, \varphi\rangle \leq \sup_n\langle\lambda_n, \varphi\rangle = F(\varphi) = \int_\Omega \varphi^+\, d\mu \leq \|\mu\|_{\mathcal{M}_b(\Omega)}\|\varphi\|_{L^\infty(\Omega)}. \tag{2.4}$$

Thus, applying this inequality to φ and $-\varphi$, we obtain

$$|\langle\lambda_n, \varphi\rangle| \leq \|\mu\|_{\mathcal{M}_b(\Omega)}\|\varphi\|_{L^\infty(\Omega)},$$

which implies that $\lambda_n \in W^{-1,p'(\cdot)}(\Omega) \cap \mathcal{M}_b(\Omega)$. Moreover, since $F(-\varphi) = 0$ for any nonnegative $\varphi \in C_0^\infty(\Omega)$, we have $\langle\lambda_n, \varphi\rangle \geq 0$. By the Riesz representation theorem there exists a nonnegative measure on Ω, which we denote by $\lambda_n^{\rm meas}$, such that

$$\langle\lambda_n, \varphi\rangle = \int_\Omega \varphi\, d\lambda_n^{\rm meas}, \quad \text{for all such } \varphi,$$

which implies $\lambda_n^{\rm meas} \in \mathcal{M}_b^+(\Omega)$ (that is to say $\lambda_n \in W^{-1,p'(\cdot)}(\Omega) \cap \mathcal{M}_b^+(\Omega)$). Using again (2.4) to any nonnegative $\varphi \in C_0^\infty(\Omega)$, we obtain

$$\lambda_n^{\rm meas} \le \mu, \quad \|\lambda_n^{\rm meas}\|_{\mathcal{M}_b(\Omega)} \le \|\mu\|_{\mathcal{M}_b(\Omega)}. \tag{2.5}$$

Define

$$\gamma = \sum_{n=1}^\infty \frac{\lambda_n}{2^n(\|\lambda_n\|_{W^{-1,p'(\cdot)}(\Omega)} + 1)}. \tag{2.6}$$

It is obvious that the series is absolutely convergent in $W^{-1,p'(\cdot)}(\Omega)$. Then we have, for all $\varphi \in C_0^\infty(\Omega)$,

$$\begin{aligned} |\langle\gamma,\varphi\rangle| &= \Big|\sum_{n=1}^\infty \frac{\langle\lambda_n,\varphi\rangle}{2^n(\|\lambda_n\|_{W^{-1,p'(\cdot)}(\Omega)} + 1)}\Big| \\ &\le \sum_{n=1}^\infty \frac{\|\lambda_n^{\rm meas}\|_{\mathcal{M}_b(\Omega)}\|\varphi\|_{L^\infty(\Omega)}}{2^n} \\ &\le \|\mu\|_{\mathcal{M}_b(\Omega)}\|\varphi\|_{L^\infty(\Omega)}, \end{aligned}$$

and $\gamma \in W^{-1,p'(\cdot)}(\Omega) \cap \mathcal{M}_b(\Omega)$. Since the series $\sum_{n=1}^\infty \frac{\lambda_n^{\rm meas}}{2^n(\|\lambda_n\|_{W^{-1,p'(\cdot)}(\Omega)}+1)}$ strongly converges in $\mathcal{M}_b(\Omega)$. Applying (2.6) to functions of $C_0^\infty(\Omega)$, we can see that

$$\gamma^{\rm meas} = \sum_{n=1}^\infty \frac{\lambda_n^{\rm meas}}{2^n(\|\lambda_n\|_{W^{-1,p'(\cdot)}(\Omega)} + 1)}.$$

In particular, $\gamma^{\rm meas}$ is a nonnegative measure (each $\lambda_n^{\rm meas}$ is nonnegative).

Since $\lambda_n^{\rm meas} \ll \gamma^{\rm meas}$, there exists a nonnegative function $f_n \in L^1(\Omega, d\gamma^{\rm meas})$ such that $\lambda_n^{\rm meas} = f_n\gamma^{\rm meas}$. Thus (2.3) implies

$$\int_\Omega \varphi\, d\mu = \sup_n \int_\Omega f_n\varphi\, d\gamma^{\rm meas}, \tag{2.7}$$

for any nonnegative $\varphi \in C_0^\infty(\Omega)$. We also have, by (2.5), $f_n\gamma^{\rm meas} \le \mu$, that is

$$\int_B f_n\, d\gamma^{\rm meas} \le \mu(B), \tag{2.8}$$

for any Borelian subset $B \subset \Omega$ and every n.

Denote

$$B_s = \big\{x \in B : f_s(x) = \max\{f_1(x), \dots, f_k(x)\} \text{ and } f_s(x) > f_1(x), \dots, f_{s-1}(x)\big\}.$$

It is obvious that B_i $(i = 1, \dots, k)$ are disjoint and $B = \cup_{s=1}^k B_s$. Then by (2.8) we have

$$\int_{B_s} f_s\, d\gamma^{\rm meas} \le \mu(B_s);$$

that is,

$$\int_{B_s} \sup\{f_1, \dots, f_k\}\, d\gamma^{\rm meas} \le \mu(B_s).$$

Summing up the above inequalities for $s = 1, \dots, k$, we deduce that

$$\int_B \sup\{f_1, \dots, f_k\}\, d\gamma^{\rm meas} \le \mu(B),$$

for any Borelian subset $B \subset \Omega$ and any $k \geq 1$. Letting $k \to \infty$, we obtain from the monotone convergence theorem that

$$\int_B f \, d\gamma^{\text{meas}} \leq \mu(B),$$

where $f = \sup_n f_n$. Then from (2.7) we conclude that

$$\begin{aligned}\int_\Omega \varphi \, d\mu = \sup_n \int_\Omega f_n \varphi \, d\gamma^{\text{meas}} &\leq \sup_n \int_\Omega f \varphi \, d\gamma^{\text{meas}} \\ = \int_\Omega f \varphi \, d\gamma^{\text{meas}} &\leq \int_\Omega \varphi \, d\mu,\end{aligned}$$

for any nonnegative $\varphi \in C_0^\infty(\Omega)$, which yields that

$$\mu = f \gamma^{\text{meas}}.$$

Since $\mu(\Omega) < +\infty$, it follows that $f \in L^1(\Omega, d\gamma^{\text{meas}})$.

Step 2. Let K_n be an increasing sequence of compact sets contained in Ω such that $\cup_{n=1}^{+\infty} K_n = \Omega$. Denote $\mu_n^{(1)} = T_n(f\chi_{K_n})\gamma^{\text{meas}}$. It is obvious that $\{\mu_n^{(1)}\}$ is an increasing sequence of positive measure in $W^{-1,p'(\cdot)}(\Omega)$ with compact support in Ω. Set $\mu_0 = \mu_0^{(1)}$ and $\mu_n = \mu_n^{(1)} - \mu_{n-1}^{(1)}$. Then $\mu = \sum_{n=1}^{+\infty} \mu_n$, and the series converges strongly in $\mathcal{M}_b(\Omega)$. Since $\mu_n \geq 0$ and $\|\mu_n\|_{\mathcal{M}_b(\Omega)} = \mu_n(\Omega)$, we know that $\sum_{n=1}^{+\infty} \|\mu_n\|_{\mathcal{M}_b(\Omega)} < \infty$.

Step 3. Let $\rho \geq 0$ be a function in $C_0^\infty(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} \rho(x) \, dx = 1$. Let $\{\rho_n\}$ be a sequence of mollifiers associated to ρ; i.e., $\rho_n(x) = n^N \rho(nx)$ for every $x \in \mathbb{R}^N$. For $n \in \mathbb{N}$, if μ_n is the measure defined in Step 2, the log-Hölder continuity condition (1.2) implies that $\{\mu_n * \rho_m\}$ converges to μ_n in $W^{-1,p'(\cdot)}(\Omega)$ as m tends to infinity. By the properties of μ_n and ρ_m, $\mu_n * \rho_m$ belongs to $C_0^\infty(\Omega)$ if m is large enough.

Choose $m = m_n$ such that $\mu_n * \rho_{m_n}$ belongs to $C_0^\infty(\Omega)$ and $\|\mu_n * \rho_{m_n} - \mu_n\|_{W^{-1,p'(\cdot)}(\Omega)} \leq 2^{-n}$. Then $\mu_n = f_n + g_n$, where $f_n = \mu_n * \rho_{m_n}$ and $g_n = \mu_n - \mu_n * \rho_{m_n}$. The choice of m_n implies that the series $\sum_{n=1}^{+\infty} g_n$ converges in $W^{-1,p'(\cdot)}(\Omega)$ and $g = \sum_{n=1}^{+\infty} g_n$ belongs to $W^{-1,p'(\cdot)}(\Omega)$. Since $\|f_n\|_{L^1(\Omega)} = \|\mu_n * \rho_{m_n}\|_{L^1(\Omega)} \leq \|\mu_n\|_{\mathcal{M}_b(\Omega)}$, by Step 2 the series $\sum_{n=1}^{+\infty} f_n$ is absolutely convergent in $L^1(\Omega)$, and $f_0 = \sum_{n=1}^{+\infty} f_n$ belongs to $L^1(\Omega)$. Therefore, the three series $\sum_{n=1}^{+\infty} \mu_n, \sum_{n=1}^{+\infty} g_n$ and $\sum_{n=1}^{+\infty} f_n$ converge in the sense of distributions. Then $\mu = f_0 + g$. This completes the proof. $\square$

Remark 2.7. From Proposition 2.6, we can conclude that $\mu \in M_b^{p(\cdot)}(\Omega)$ is a signed measure in $L^1(\Omega) + W^{-1,p'(\cdot)}(\Omega)$; i.e.,

$$\mu = f - \operatorname{div} F \quad \text{in the sense of distributions,}$$

where $f \in L^1(\Omega)$ and $F \in (L^{p'(\cdot)}(\Omega))^N$. Therefore, the equality (1.3) can be written as

$$\begin{aligned}&\int_\Omega |\nabla u|^{p(x)-2} \nabla u \cdot \nabla T_k(u - \phi) \, dx \\ &= \int_\Omega f T_k(u - \phi) \, dx + \int_\Omega F \cdot \nabla T_k(u - \phi) \, dx,\end{aligned} \tag{2.9}$$

for all $\phi \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$.

Based on the decomposition of a measure in $\mathcal{M}_b^{p(\cdot)}(\Omega)$, we have the following result, whose proof can be found in [28].

Theorem 2.8. *Assume that $p(x)$ satisfies the log-Hölder condition* (1.2) *and $\mu \in M_b^{p(\cdot)}(\Omega)$. Then there exists a unique entropy solution $u \in \mathcal{T}_0^{1,p(\cdot)}(\Omega)$ for problem* (2.1).

3. Weighted case

In this section, we are ready to prove the existence of entropy solutions for weighted $p(x)$-Laplace problem (1.1).

3.1. **Preliminaries.** Let w be a weight function satisfying that

- (W1) $w \in L^1_{\rm loc}(\Omega)$ and $w^{-1/(p(x)-1)} \in L^1_{\rm loc}(\Omega)$;
- (W2) $w^{-s(x)} \in L^1(\Omega)$ with $s(x) \in \big(\frac{N}{p(x)}, \infty\big) \cap [\frac{1}{p(x)-1}, \infty)$.

Lemma 3.1 ([16, 19])**.** *If we denote*

$$\rho(u) = \int_\Omega w(x)|u|^{p(x)}\,dx, \quad \forall u \in L^{p(x)}(\Omega, w),$$

then

$$\min\{\|u\|^{p_-}_{L^{p(x)}(\Omega,w)}, \|u\|^{p_+}_{L^{p(x)}(\Omega,w)}\} \le \rho(u) \le \max\{\|u\|^{p_-}_{L^{p(x)}(\Omega,w)}, \|u\|^{p_+}_{L^{p(x)}(\Omega,w)}\}.$$

Lemma 3.2 ([19])**.** *If* (W1) *holds, $W^{1,p(x)}(\Omega, w)$ is a separable and reflexive Banach space.*

For $p, s \in C_+(\overline{\Omega})$, set

$$p_s(x) := \frac{p(x)s(x)}{1+s(x)} < p(x),$$

where $s(x)$ is given in (W2). Assume that we fix the variable exponent restrictions

$$p_s^*(x) := \begin{cases} \frac{p(x)s(x)N}{(s(x)+1)N-p(x)s(x)} & \text{if } N > p_s(x),\\ \text{arbitrary} & \text{if } N \le p_s(x), \end{cases} \tag{3.1}$$

for almost all $x \in \Omega$.

Next we state a continuous imbedding theorem for the weighted variable exponent Sobolev space.

Lemma 3.3 ([19])**.** *Let $p, s \in C_+(\overline{\Omega})$ and let* (W1) *and* (W2) *be satisfied. Then we have the continuous imbedding*

$$W^{1,p(x)}(\Omega, w) \hookrightarrow L^{r(x)}(\Omega)$$

provided that $r \in C_+(\overline{\Omega})$ and $r(x) \le p_s^*(x)$ for all $x \in \Omega$ and the embedding is compact if $\inf_{x\in\Omega}(p_s^*(x) - r(x)) > 0$.

We conclude this subsection by proving *a priori* estimate for entropy solutions of problem (1.1), which plays a key role in proving our main result.

Proposition 3.4. *If u is an entropy solution of problem* (1.1)*, then there exists a positive constant C such that for all $k > 1$,*

$$\operatorname{meas}\{|u| > k\} \le \frac{C(M+1)^{\frac{(p_s^*)_-}{p_-}}}{k^{(p_s^*)_-(1-\frac{1}{p_-})}},$$

where

$$M = \|f\|_{L^1(\Omega)}, \quad (p_s^*)_- := \frac{p_- s_- N}{(s_- + 1)N - p_- s_-}.$$

Proof. Choosing $\phi = 0$ in the entropy equality (1.3), we obtain

$$\int_\Omega w(x)|\nabla T_k(u)|^{p(x)}\,dx = \int_{\{|u|\le k\}} w(x)|\nabla u|^{p(x)}\,dx \le k\|f\|_{L^1(\Omega)},$$

which implies that for all $k > 1$,

$$\frac{1}{k}\int_\Omega w(x)|\nabla T_k(u)|^{p(x)}\,dx \le M, \tag{3.2}$$

where $M = \|f\|_{L^1(\Omega)}$.

Recalling Sobolev embedding theorem in Lemma 3.3, we have the following continuous embedding

$$W_0^{1,p(x)}(\Omega, w) \hookrightarrow L^{p_s^*(x)}(\Omega) \hookrightarrow L^{(p_s^*)_-}(\Omega),$$

where $p_s^*(x) := \frac{p(x)s(x)N}{(s(x)+1)N - p(x)s(x)}$ and $(p_s^*)_- := \frac{p_- s_- N}{(s_-+1)N - p_- s_-}$. It follows from Lemma 3.1 and (2.2) that for every $k > 1$,

$$\begin{aligned}\|T_k(u)\|_{L^{(p_s^*)_-}(\Omega)} &\le C\|\nabla T_k(u)\|_{L^{p(x)}(\Omega,w)}\\ &\le C\Big(\int_\Omega w(x)|\nabla T_k(u)|^{p(x)}\,dx\Big)^\beta \le C(Mk)^\beta,\end{aligned}$$

where

$$\beta = \begin{cases} \frac{1}{p_-} & \text{if } \|\nabla T_k(u)\|_{L^{p(x)}(\Omega,w)} \ge 1,\\ \frac{1}{p_+} & \text{if } \|\nabla T_k(u)\|_{L^{p(x)}(\Omega,w)} \le 1.\end{cases}$$

Noting that $\{|u| \ge k\} = \{|T_k(u)| \ge k\}$, we have

$$\operatorname{meas}\{|u| > k\} \le \Big(\frac{\|T_k(u)\|_{L^{(p_s^*)_-}(\Omega)}}{k}\Big)^{(p_s^*)_-} \le \frac{CM^{\beta(p_s^*)_-}}{k^{(p_s^*)_-(1-\beta)}} \le \frac{C(M+1)^{\frac{(p_s^*)_-}{p_-}}}{k^{(p_s^*)_-(1-\frac{1}{p_-})}}.$$

This completes the proof. □

3.2. **Main result.**

Theorem 3.5. *Let* (W1) *and* (W2) *be satisfied. Then there exists an entropy solution for problem* (1.1).

Proof. We first introduce the approximation problems. Find a sequence of $C_0^\infty(\Omega)$ functions $\{f_n\}$ strongly converging to f in $L^1(\Omega)$ such that

$$\|f_n\|_{L^1(\Omega)} \le C\big(\|f\|_{L^1(\Omega)} + 1\big). \tag{3.3}$$

Then we consider approximate problems of (1.1)

$$\begin{gathered} -\operatorname{div}\big(w(x)|\nabla u_n|^{p(x)-2}\nabla u_n\big) = f_n \quad \text{in } \Omega,\\ u_n = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{3.4}$$

Then from the result in [14], we can easily find a unique weak solution $u_n \in W_0^{1,p(\cdot)}(\Omega, w)$ of problem (3.4), which is obviously an entropy solution, satisfying that for all $\phi \in W_0^{1,p(x)}(\Omega, w) \cap L^\infty(\Omega)$,

$$\int_\Omega w(x)|\nabla u_n|^{p(x)-2}\nabla u_n \cdot \nabla T_k(u_n - \phi)\,dx = \int_\Omega f_n T_k(u_n - \phi)\,dx.$$

Following the same arguments as in Proposition 3.4 and (1.2), we have

$$\int_\Omega w(x)|\nabla T_k(u_n)|^{p(x)}\,dx \le Ck(\|f\|_{L^1(\Omega)}+1). \tag{3.5}$$

Our aim is to prove that a subsequence of these approximate solutions $\{u_n\}$ converges to a measurable function u, which is an entropy solution of problem (1.1). We will divide the proof into several steps.

Step 1. We shall prove the convergence in measure of $\{u_n\}$ and we shall find a subsequence which is almost everywhere convergent in Ω. For every fixed $\epsilon > 0$, and every positive integer k, we know that

$$\{|u_n-u_m|>\epsilon\} \subset \{|u_n|>k\}\cup\{|u_m|>k\}\cup\{|T_k(u_n)-T_k(u_m)|>\epsilon\}.$$

Using Sobolev embedding theorem in Lemma 3.3, we find that $W^{1,p(x)}(\Omega,w)$ can embed into $L^q(\Omega)$ with $q<(p_s^*)_-$ compactly. Then we know $\{T_k u_n\}$ is convergent in $L^q(\Omega)$ with $q<(p_s^*)_-$. It follows from Proposition 3.4 that

$$\limsup_{n,m\to\infty} \operatorname{meas}\{|u_n-u_m|>\epsilon\} \le Ck^{-\alpha},$$

where $\alpha=(p_s^*)_-(1-\frac{1}{p_-})>0$ and the constant C depends on $p(\cdot)$, $s(\cdot)$ and $\|f\|_{L^1(\Omega)}$.

Because of the arbitrariness of k, we prove that

$$\limsup_{n,m\to\infty} \operatorname{meas}\{|u_n-u_m|>\epsilon\} = 0,$$

which implies the convergence in measure of $\{u_n\}$, and then we find an a.e. convergent subsequence (still denoted by $\{u_n\}$) in Ω such that

$$u_n \to u \quad \text{a.e. in } \Omega. \tag{3.6}$$

Step 2. We shall prove that

$$\nabla T_k(u_n) \to \nabla T_k(u) \quad \text{strongly in } W_0^{1,p(x)}(\Omega,w), \tag{3.7}$$

for every $k>0$. Let $h>k$. We choose

$$w_n = T_{2k}\big(u_n - T_h(u_n) + T_k(u_n) - T_k(u)\big)$$

as a test function in (3.4). If we set $M=4k+h$, then it is easy to see that $\nabla w_n = 0$ where $\{|u_n|>M\}$. Therefore, we may write the weak form of (3.4) as

$$\int_\Omega w(x)|\nabla T_M(u_n)|^{p(x)-2}\nabla T_M(u_n)\cdot\nabla w_n\,dx = \int_\Omega f_n w_n\,dx.$$

Splitting the integral in the left-hand side on the sets where $\{|u_n|\le k\}$ and where $\{|u_n|>k\}$ and discarding some nonnegative terms, we find

$$\begin{aligned}
&\int_\Omega w(x)|\nabla T_M(u_n)|^{p(x)-2}\nabla T_M(u_n)\cdot\nabla T_{2k}(u_n-T_h(u_n)+T_k(u_n)-T_k(u))\,dx\\
&\ge \int_\Omega w(x)|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n)\cdot\nabla(T_k(u_n)-T_k(u))\,dx\\
&\quad - \int_{\{|u_n|>k\}} w(x)\big||\nabla T_M(u_n)|^{p(x)-2}\nabla T_M(u_n)\big||\nabla T_k(u)|\,dx.
\end{aligned}$$

It follows from the above inequality that

$$
\begin{aligned}
&\int_\Omega w(x)\Big(|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n)-|\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)\Big)\\
&\quad\cdot\nabla(T_k(u_n)-T_k(u))\,dx\\
&\le\int_{\{|u_n|>k\}} w(x)\big||\nabla T_M(u_n)|^{p(x)-2}\nabla T_M(u_n)\big||\nabla T_k(u)|\,dx\\
&\quad+\int_\Omega f_nT_{2k}(u_n-T_h(u_n)+T_k(u_n)-T_k(u))\,dx\\
&\quad-\int_\Omega w(x)|\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)\cdot\nabla(T_k(u_n)-T_k(u))\,dx\\
&:=I_1+I_2+I_3.
\end{aligned}
\tag{3.8}
$$

Using the properties of $L^{p(x)}(\Omega,w)$ and the similar estimates as in [6], we can show the limits of I_1, I_2 and I_3 are zeros when n, and then h tend to infinity, respectively.

Therefore, passing to the limits in (3.8) as n, and then h tend to infinity, we deduce that

$$
\lim_{n\to+\infty}E(n)=0,
$$

where

$$
\begin{aligned}
E(n)=\int_\Omega & w(x)(|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n)\\
&-|\nabla T_k(u)|^{p(x)-2}\nabla T_k(u))\nabla(T_k(u_n)-T_k(u))\,dx.
\end{aligned}
$$

Applying [6, Lemma 3.1], we conclude that

$$
T_k(u_n)\to T_k(u)\quad\text{strongly in } W_0^{1,p(x)}(\Omega,w)
$$

for every $k>0$, which also implies that

$$
|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n)\to|\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)\quad\text{strongly in } (L^{p'(\cdot)}(\Omega,w))^N.
$$

Step 3. We shall prove that u is an entropy solution. Set $L=k+\|\phi\|_{L^\infty(\Omega)}$. Observe that

$$
\begin{aligned}
&\int_\Omega w(x)|\nabla u_n|^{p(x)-2}\nabla u_n\cdot\nabla T_k(u_n-\phi)\,dx\\
&=\int_\Omega|\nabla T_L(u_n)|^{p(x)-2}\nabla T_L(u_n)\cdot\nabla T_k(u_n-\phi)\,dx.
\end{aligned}
$$

Then we have

$$
\int_\Omega w(x)|\nabla T_L(u_n)|^{p(x)-2}\nabla T_L(u_n)\cdot\nabla T_k(u_n-\phi)\,dx=\int_\Omega f_nT_k(u_n-\phi)\,dx.
$$

Using (3.6) and (3.7), we can pass to the limits as n tends to infinity to conclude that

$$
\int_\Omega w(x)|\nabla u|^{p(x)-2}\nabla u\cdot\nabla T_k(u-\phi)\,dx=\int_\Omega fT_k(u-\phi)\,dx,
$$

for every $k>0$ and every $\phi\in W_0^{1,p(x)}(\Omega,w)\cap L^\infty(\Omega)$. This finishes the proof. $\square$

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Chao Zhang

Department of Mathematics, Harbin Institute of Technology, Harbin 150001, China

E-mail address: `czhangmath@hit.edu.cn`