Electronic Journal of Differential Equations, Vol. 2014 (2014), No. 95, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

BIFURCATION OF LIMIT CYCLES FROM QUARTIC ISOCHRONOUS SYSTEMS

LINPING PENG, ZHAOSHENG FENG

Abstract. This article concerns the bifurcation of limit cycles for a quartic system with an isochronous center. By using the averaging theory, it shows that under any small quartic homogeneous perturbations, at most two limit cycles bifurcate from the period annulus of the considered system, and this upper bound can be reached. In addition, we study a family of perturbed isochronous systems and prove that there are at most three limit cycles bifurcating from the period annulus of the unperturbed one, and the upper bound is sharp.

1. Introduction

There has been a longstanding problem, called the Hilbert 16th problem, whose second part asks for the maximum $H(n)$ of the number of limit cycles and the relative positions for all planar polynomial differential systems of degree n. One of the most remarkable achievements, Ecalle-Ilyashenko Theorem, claims that the number of limit cycles is finite for any individual vector field [7, 21, 12, 22]. However, the existence of a uniform upper bound for the number even for quadratic vector fields is still an open problem.

To attack the Hilbert 16th problem, many researchers investigate the number of limit cycles of various planar polynomial differential systems. Among them, the problem of the number of limit cycles by perturbing the periodic orbits of a center has been extensively studied in the literatures [8, 14, 15, 20, 24, 25, 26] and the references therein. In general, some useful methods have been proposed based on the Poincaré map [6, 11, 23], the Poincaré-Pontryagin-Melnikov integrals or the Abelian integrals [1, 2, 3, 5, 10, 13, 30, 31], the inverse integrating factor [16, 17, 18, 29], and the averaging method which is equivalent to the Abelian integrals in the plane [4, 9, 19, 24, 25, 26].

Although in the plane the methods based on the Abelian integrals and the averaging theory are equivalent, each has its own advantages. For example, when the associated Abelian integrals are complicated or we need to study the periodic orbits of the non-autonomous differential systems, the averaging method displays

---

2000 *Mathematics Subject Classification.* 34C07, 37G15, 34C05.
Key words and phrases. Bifurcation; limit cycles; homogeneous perturbation; averaging method; isochronous center; period annulus.

Submitted December 2, 2013. Published April 10, 2014.

more flexibility. Roughly speaking, the averaging method gives a quantitative relation between the solutions of a non-autonomous periodic differential system and the solutions of its averaged differential system, which is autonomous. In particular, for the averaging method of the first order, the number of hyperbolic equilibrium points of the averaged differential system can give a lower bound of the maximal number of limit cycles of the non-autonomous periodic differential system [27, 28].

As mentioned above, by using the averaging method, the problem on the number of limit cycles of the non-autonomous periodic differential systems is equivalent to the exploration of the number of hyperbolic equilibrium points of the averaged differential systems. Hence, the averaging theory has played a crucial role in the study of limit cycles of differential systems. Now there are quite many important results on the number of limit cycles of the polynomial differential systems by the averaging method, such as Llibre [26], Buică and Llibre [4, 5], Gine and Llibre [19] and so on. It seems that among these results, more are focused on differential systems of lower degree. As far as we know, for the integrable systems of higher degree, in some cases the first integrals may have complicated expressions so that it is out of the reach to study the bifurcation of limit cycles of these systems under small perturbations.

In this article, we consider the quartic system

$$\begin{aligned} \dot{x} &= -y + x^3 y + x y^3, \\ \dot{y} &= x + x^2 y^2 + y^4, \end{aligned} \tag{1.1}$$

which has

$$H(x,y) = \frac{1}{3(x^2+y^2)^{3/2}} - \frac{x}{(x^2+y^2)^{1/2}} = c$$

as its first integral with the integrating factor $1/(x^2+y^2)^{5/2}$ and the unique finite singularity $(0,0)$ as its isochronous center. The period annulus, denoted by

$$\{(x,y) | H(x,y) = c,\ c \in (1, +\infty)\}$$

starts at the center $(0,0)$ and terminates with the separatrix passing the infinite degenerate singularity on the equator. The phase portrait of system (1.1) is shown in Fig.1.

By using the averaging method, we study the bifurcation of limit cycles from system (1.1) under any small perturbations, and prove the following main results.

Theorem 1.1. *For any sufficiently small parameter $|\varepsilon|$, and any real constants a_{ij} and b_{ij} $(i,j = 0,1,2,3,4)$, the following quartic perturbation of system* (1.1),

$$\begin{aligned} \dot{x} &= -y + x^3 y + x y^3 + \varepsilon \sum_{i+j=4} a_{ij} x^i y^j, \\ \dot{y} &= x + x^2 y^2 + y^4 + \varepsilon \sum_{i+j=4} b_{ij} x^i y^j, \end{aligned} \tag{1.2}$$

has at most two limit cycles bifurcating from the period annulus around the center $(0,0)$ of the unperturbed one, and this upper bound is sharp.

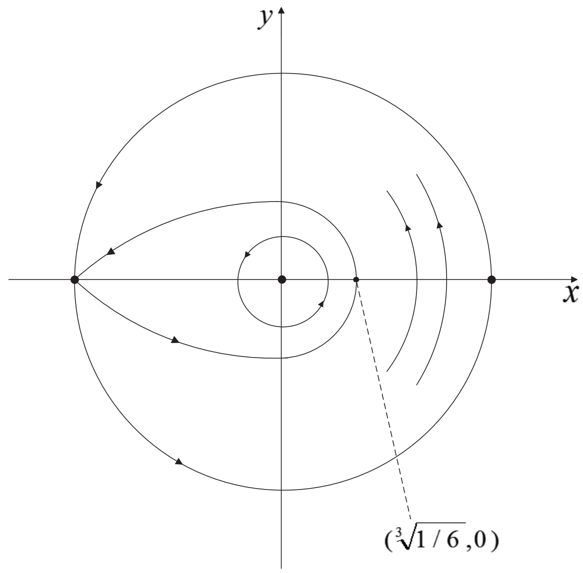

FIGURE 1. The phase portrait of system (1.1) in the Poincaré disk.

Theorem 1.2. *For the family of quartic perturbations*

$$
\begin{aligned}
\dot{x} &= -y + x^3y + xy^3 + \varepsilon(a_{10}x + a_{01}y + a_{11}xy + a_{21}x^2y + a_{03}y^3 \\
&\quad + a_{40}x^4 + a_{31}x^3y + a_{22}x^2y^2 + a_{13}xy^3 + a_{04}y^4), \\
\dot{y} &= x + x^2y^2 + y^4 + \varepsilon(b_{10}x + b_{01}y + b_{20}x^2 + b_{02}y^2 + b_{30}x^3 + b_{12}xy^2 \\
&\quad + b_{40}x^4 + b_{31}x^3y + b_{22}x^2y^2 + b_{13}xy^3 + b_{04}y^4),
\end{aligned} \tag{1.3}
$$

where $|\varepsilon|$ is sufficiently small, $a_{i,j}$ and $b_{i,j}$ $(i, j = 0, 1, 2, 3, 4)$ are any real constants. Then there are at most three limit cycles bifurcating from the period annulus surrounding the center $(0, 0)$ of the unperturbed system, and this upper bound is sharp.

The rest of this paper is organized as follows. In Section 2, we give an introduction on the averaging theory, including some technical lemmas and methods employed in the averaging theory. Section 3 is dedicated to the proof of Theorem 1.1 by computing the averaged equations corresponding to the equivalent system of system (1.2) and exploring the number of its hyperbolic equilibriums. In Section 4, after making a transformation to system (1.3), theorem 1.2 is proven through analyzing an equivalent system and a corresponding averaged system. In addition, some examples are illustrated to verify the obtained results.

2. Preliminary results

In this section, we introduce some preliminary results on the averaging theory that will be used in our quartic polynomial systems.

The following lemma provides a first order approximation for the periodic solution of a periodic differential equation. For the proof, we refer the reader to [27, Theorem 2.6.1] and [28, Theorems 11.5 and 11.6].

Lemma 2.1. *Consider the two initial value problems*

$$\dot{x} = \varepsilon f(t,x) + \varepsilon^2 h(t,x,\varepsilon), \quad x(0) = x_0, \tag{2.1}$$

and

$$\dot{y} = \varepsilon f^0(y), \quad y(0) = x_0, \tag{2.2}$$

where $x, y, x_0 \in D$, *here* D *is an open subset of* $R, t \in [0, +\infty), \varepsilon \in (0, \varepsilon_0], f$ *and* h *are periodic with period* T *in* t, *and*

$$f^0(y) = \frac{1}{T}\int_0^T f(t,y)dt. \tag{2.3}$$

We suppose that

(1) f, $\partial f/\partial x$, $\partial^2 f/\partial x^2$ *and* $\partial h/\partial x$ *are continuous and bounded by a constant independent on* ε *in* $[0, +\infty) \times D$ *and* $\varepsilon \in (0, \varepsilon_0]$*;*
(2) T *is independent on* ε*; and*
(3) $y(t)$ *belongs to* D *on the time-scale* $1/\varepsilon$.

Then the following statements hold.

(a) *On the time-scale* $1/\varepsilon$, *we have that*
$$x(t) - y(t) = O(\varepsilon), \quad \text{as } \varepsilon \to 0.$$
(b) *If* p *is an equilibrium point of the averaged system* (2.2) *such that*
$$(df^0/dy)(p) \neq 0, \tag{2.4}$$
then there exists a T*-periodic solution* $\phi(t,\varepsilon)$ *of equation* (2.1) *which is close to* p *such that* $\phi(t,\varepsilon) \to p$ *as* $\varepsilon \to 0$.
(c) *If* (2.4) *is negative, then the corresponding periodic solution* $\phi(t,\varepsilon)$ *in the plane* (t,x) *is asymptotically stable for any sufficiently small* $|\varepsilon|$. *If* (2.4) *is positive, then it is unstable.*

Let us consider another integrable system of the form

$$\begin{aligned} \dot{x} &= P(x,y), \\ \dot{y} &= Q(x,y), \end{aligned} \tag{2.5}$$

with a first integral H and a continuous family of ovals

$$\{\gamma_h\} \subset \{(x,y) | H(x,y) = h, h_1 < h < h_2\}.$$

We consider a perturbed system:

$$\begin{aligned} \dot{x} &= P(x,y) + \varepsilon p(x,y), \\ \dot{y} &= Q(x,y) + \varepsilon q(x,y). \end{aligned} \tag{2.6}$$

To study the number of limit cycles for any sufficiently small $|\varepsilon|$ by using the above averaging theory, we need to transform system (2.6) to the canonical form in Lemma 2.1. The following lemma [4] provides us a useful transformation.

Lemma 2.2. *For system* (2.5), *assume* $xQ(x,y) - yP(x,y) \neq 0$ *for all* (x,y) *in the period annulus formed by the ovals* γ_h. *Let*

$$\rho : (\sqrt{h_1}, \sqrt{h_2}) \times [0, 2\pi) \to [0, +\infty)$$

be a continuous function such that

$$H(\rho(R,\varphi)\cos\varphi, \rho(R,\varphi)\sin\varphi) = R^2,$$

for all $R \in (\sqrt{h_1}, \sqrt{h_2})$ and $\varphi \in [0, 2\pi)$. Then the differential equation which describes the dependence between the square root of energy, $R = \sqrt{h}$, and the angle φ for system (2.6) *is*

$$\frac{dR}{d\varphi} = \varepsilon \frac{\mu(x^2+y^2)(Qp - Pq)}{2R(Qx - Py)} \Big(1 - \varepsilon \frac{qx - py}{Qx - Py}\Big) + O(\varepsilon^3), \tag{2.7}$$

where $x = \rho(R, \varphi)\cos\varphi$ and $y = \rho(R, \varphi)\sin\varphi$.

The following lemma presents the version of the formula of the first order Melnikov integral associated with system (2.6) in the polar coordinates [4].

Lemma 2.3. *Under the conditions of Lemma 2.2, we define*

$$\begin{gathered} d(R, \varepsilon) = \int_0^{2\pi} \Big[\varepsilon \frac{\mu(x^2+y^2)(Qp - Pq)}{2R(Qx - Py)} \Big(1 - \varepsilon \frac{qx - py}{Qx - Py}\Big) + O(\varepsilon^3)\Big] d\varphi, \\ M_1(R) = \int_0^{2\pi} \frac{\mu(x^2+y^2)(Qp - Pq)}{2R(Qx - Py)} d\varphi, \end{gathered} \tag{2.8}$$

for system (2.6)*, where $\mu = \mu(x, y)$ is the integrating factor of system* (2.5) *corresponding to the first integral H, and $x = \rho\cos\varphi$ and $y = \rho\sin\varphi$. Then $d(R, \varepsilon)$ and $M_1(R)$ expressed by* (2.8) *are the displacement function and the first order Melnikov integral of system* (2.6)*, respectively.*

Based on Lemmas 2.1, 2.2 and 2.3, we can obtain

Corollary 2.4. *If $d^0(R)$ represents the averaged function of the first approximation in ε of the right side of system* (2.7)*, then the following relation holds,*

$$2\pi d^0(R) = M_1(R),$$

where $M_1(R)$ is defined by (2.8)*.*

Corollary 2.4 provides a relation between the averaged function and the first order Melnikov integral associated with the same differential system, which enables us to explore the maximal number of limit cycles of system (2.6) bifurcating from the period annulus of system (2.5) via the averaging method.

3. Proof of Theorem 1.1

For

$$H(x, y) = \frac{1}{3(x^2+y^2)^{3/2}} - \frac{x}{(x^2+y^2)^{1/2}},$$

we choose the function

$$\rho(R, \varphi) = \frac{1}{(R^2 + 3\cos\varphi)^{1/3}} \tag{3.1}$$

such that $H(\rho\cos\varphi, \rho\sin\varphi) = R^2/3$. Let

$$\begin{aligned} x &= \rho(R, \varphi)\cos\varphi, \\ y &= \rho(R, \varphi)\sin\varphi, \end{aligned} \tag{3.2}$$

for $\varphi \in [0, 2\pi)$ and $R > \sqrt{3}$. By using Lemma 2.2, we can transform system (1.2) as

$$\begin{aligned}\frac{dR}{d\varphi} &= \Big(\varepsilon \frac{3(Qp - Pq)}{2R(x^2+y^2)^{5/2}} - \varepsilon^2 \frac{3(Qp-Pq)(qx-py)}{2R(x^2+y^2)^{7/2}}\Big)\Big|_{x=\rho(R,\varphi)\cos\varphi, y=\rho(R,\varphi)\sin\varphi} \\ &\quad + O(\varepsilon^3),\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}Qp - Pq &= -b_{40}x^7 y + (a_{40} - b_{31})x^6y^2 + (a_{31} - b_{40} - b_{22})x^5y^3 \\ &\quad + (a_{40} + a_{22} - b_{31} - b_{13})x^4y^4 + (a_{31} + a_{13} - b_{22} - b_{04})x^3y^5 \\ &\quad + (a_{22} + a_{04} - b_{13})x^2y^6 + (a_{13} - b_{04})xy^7 + a_{04}y^8 \\ &\quad + a_{40}x^5 + (a_{31} + b_{40})x^4y + (a_{22} + b_{31})x^3y^2 \\ &\quad + (a_{13} + b_{22})x^2y^3 + (a_{04} + b_{13})xy^4 + b_{04}y^5,\end{aligned}$$

$$\begin{aligned}qx - py &= b_{40}x^4 + (b_{31} - a_{40})x^4y + (b_{22} - a_{31})x^3y^2 + (b_{13} - a_{22})x^2y^3 \\ &\quad + (b_{04} - a_{13})xy^4 - a_{04}y^5.\end{aligned}$$

The averaged equation corresponding to system (3.3) is

$$\dot{R} = \varepsilon f^0(R), \tag{3.4}$$

where

$$\begin{aligned}f^0(R) &= \frac{1}{2\pi}\int_0^{2\pi} \Big(\frac{3(Qp-Pq)}{2R(x^2+y^2)^{5/2}}\Big)\Big|_{x=\rho(R,\varphi)\cos\varphi, y=\rho(R,\varphi)\sin\varphi} d\varphi \\ &= \frac{1}{4\pi R}\Big[M_1 \int_0^{2\pi} \frac{\cos^6\varphi}{\cos\varphi + \frac{R^2}{3}} d\varphi + M_2 \int_0^{2\pi} \frac{\cos^4\varphi}{\cos\varphi + \frac{R^2}{3}} d\varphi \\ &\quad + M_3 \int_0^{2\pi} \frac{\cos^2\varphi}{\cos\varphi + \frac{R^2}{3}} d\varphi - (M_1 + M_2 + M_3) \int_0^{2\pi} \frac{1}{\cos\varphi + \frac{R^2}{3}} d\varphi\Big],\end{aligned} \tag{3.5}$$

and

$$\begin{gathered} M_1 = -a_{40} + a_{22} - a_{04} + b_{31} - b_{13}, \\ M_2 = a_{40} - 2a_{22} + 3a_{04} - b_{31} + 2b_{13}, \\ M_3 = a_{22} - 3a_{04} - b_{13}. \end{gathered} \tag{3.6}$$

Straightforward computations give

$$\begin{gathered} \int_0^{2\pi} \frac{\cos^6\varphi}{\cos\varphi + \frac{R^2}{3}} d\varphi = -\frac{\pi R^2}{4} - \frac{\pi R^6}{27} - \frac{2\pi R^{10}}{243} + \frac{2\pi R^{12}}{243\sqrt{R^4 - 9}}, \\ \int_0^{2\pi} \frac{\cos^4\varphi}{\cos\varphi + \frac{R^2}{3}} d\varphi = -\frac{\pi R^2}{3} - \frac{2\pi R^6}{27} + \frac{2\pi R^8}{27\sqrt{R^4 - 9}}, \\ \int_0^{2\pi} \frac{\cos^2\varphi}{\cos\varphi + \frac{R^2}{3}} d\varphi = -\frac{2\pi R^2}{3} + \frac{2\pi R^4}{3\sqrt{R^4 - 9}}. \end{gathered}$$

From these expressions, we obtain

$$
\begin{aligned}
f^0(R) &= \frac{1}{4R}\Big\{\Big[-\frac{2M_1}{243}R^{10}-\frac{M_1+2M_2}{27}R^6-\frac{3M_1+4M_2+8M_3}{12}R^2\Big]\\
&\quad+\Big[\frac{2M_1}{243}R^{12}+\frac{2M_2}{27}R^8+\frac{2M_3}{3}R^4-6(M_1+M_2+M_3)\Big]\frac{1}{\sqrt{R^4-9}}\Big\}\\
&= \frac{1}{4R}\Big\{\Big[-\frac{2M_1}{243}S^5-\frac{M_1+2M_2}{27}S^3-\frac{3M_1+4M_2+8M_3}{12}S\Big]\\
&\quad+\Big[\frac{2M_1}{243}S^6+\frac{2M_2}{27}S^4+\frac{2M_3}{3}S^2-6(M_1+M_2+M_3)\Big]\frac{1}{\sqrt{S^2-9}}\Big\},
\end{aligned}
\tag{3.7}
$$

where $S=R^2$. Let

$$S=\frac{3(1+w^2)}{1-w^2}.$$

For $0<w<1$, formula (3.7) becomes

$$
\begin{aligned}
f^0(R) &= \frac{(w-1)}{16R(w+1)^5}g(w)\\
&= -\frac{\sqrt{3}(1-w)^{3/2}}{48(1+w^2)^{1/2}(1+w)^{9/2}}[N_1w^4+N_2w^3+N_3w^2+N_2w+N_1],
\end{aligned}
$$

where

$$
\begin{gathered}
g(w)=N_1w^4+N_2w^3+N_3w^2+N_2w+N_1,\\
N_1=15M_1+12M_2+8M_3,\quad N_2=42M_1+40M_2+32M_3,\\
N_3=62M_1+56M_2+48M_3.
\end{gathered}
$$

As a result of the symmetry of coefficients of $g(w)$, we know that if $w_0\neq 0$ is one root of $g(w)=0$, so is $1/w_0$. Hence, the fact that $g(w)$ has at most two zeros in $w\in(0,1)$ implies that there exist at most two zeros for $f^0(R)$ in $R\in(\sqrt{3},+\infty)$. By Lemma 2.1 and Corollary 2.4, we get that system (3.3) has at most two periodic solutions which tend to the corresponding hyperbolic equilibriums, respectively. That is, for system (1.2) with any sufficiently small $|\varepsilon|$, at most two limit cycles bifurcate from the period annulus around the center $(0,0)$ of system (1.1).

In fact, there exist many systems expressed like (1.2) which have exactly two limit cycles emerging from the period annulus of the unperturbed system. In the following, we not only provide some examples satisfying this property, but also introduce a method to construct such systems.

Suppose that

$$
\begin{aligned}
\tilde{g}(w) &= \big(w-\frac{1}{10}\big)\big(w-\frac{1}{5}\big)(w-10)(w-5)\\
&= w^4-\frac{153}{10}w^3+\frac{1363}{25}w^2-\frac{153}{10}w+1.
\end{aligned}
$$

Take the constants

$$C_1=1,\quad C_2=-\frac{153}{10},\quad C_3=\frac{1363}{25},$$

then we can choose

$$M_1=\frac{1089}{100},\quad M_2=-\frac{2209}{100},\quad M_3=\frac{10273}{800}, \tag{3.8}$$

such that

$$
\begin{gathered}
15M_1+12M_2+8M_3=1,\\
42M_1+40M_2+32M_3=-\frac{153}{10},\\
62M_1+56M_2+48M_3=\frac{1363}{25}.
\end{gathered}
$$

From (3.6) and (3.8), we have

$$
a_{40}=b_{31}-\frac{213}{160},\quad a_{22}=b_{13}+\frac{3167}{400},\quad a_{04}=-\frac{1313}{800}.
$$

Hence, for the sufficiently small $|\varepsilon|$, we obtain a family of systems

$$
\begin{gathered}
\dot{x}=-y+x^3y+xy^3+\varepsilon\Big[(b_{31}-\frac{213}{160})x^4+a_{31}x^3y\\
+(b_{13}+\frac{3167}{400})x^2y^2+a_{13}xy^3-\frac{1313}{800}y^4\Big],\\
\dot{y}=x+x^2y^2+y^4+\varepsilon\big[b_{40}x^4+b_{31}x^3y+b_{22}x^2y^2+b_{13}xy^3+b_{04}y^4\big],
\end{gathered}
\tag{3.9}
$$

where $a_{13},\ a_{31}$ and b_{ij} $(i,j=0,1,2,3,4)$ are any real constants.

By using polar coordinates $x=\rho\cos\varphi$ and $y=\rho\sin\varphi$, system (3.9) can be rewritten as

$$
\frac{dR}{d\varphi}=\varepsilon F(\varphi,R)+O(\varepsilon^2),
\tag{3.10}
$$

where

$$
\begin{aligned}
F(\varphi,R)=\rho^3\Big[&-b_{40}\cos^7\varphi\sin\varphi-\frac{213}{160}\cos^6\varphi\sin^2\varphi+(a_{31}-b_{40}-b_{22})\cos^5\varphi\sin^3\varphi\\
&+\frac{5269}{800}\cos^4\varphi\sin^4\varphi+(a_{31}+a_{13}-b_{22}-b_{04})\cos^3\varphi\sin^5\varphi\\
&+\frac{5021}{800}\cos^2\varphi\sin^6\varphi+(a_{13}-b_{04})\cos\varphi\sin^7\varphi-\frac{1313}{800}sin^8\varphi\Big]\\
&+\Big[(b_{31}-\frac{213}{160})\cos^5\varphi+(a_{31}+b_{40})\cos^4\varphi\sin\varphi\\
&+\big(b_{31}+b_{13}+\frac{3167}{400}\big)\cos^3\varphi\sin^2\varphi+\big(a_{13}+b_{22}\big)\cos^2\varphi\sin^3\varphi\\
&+\big(b_{13}-\frac{1313}{800}\big)\cos\varphi\sin^4\varphi+b_{04}\sin^5\varphi\Big].
\end{aligned}
$$

The averaged equation of system (3.10) is given by

$$
\frac{dR}{d\varphi}=\varepsilon f_*^0(R)+O(\varepsilon^2),
\tag{3.11}
$$

where

$$
\begin{aligned}
f_*^0(R) &= \frac{1}{2\pi}\int_0^{2\pi} F(\varphi,R)d\varphi \\
&= \frac{1}{4\pi R}\Big[\frac{1089}{100}\int_0^{2\pi}\frac{\cos^6\varphi}{\cos\varphi+\frac{R^2}{3}}d\varphi-\frac{2209}{100}\int_0^{2\pi}\frac{\cos^4\varphi}{\cos\varphi+\frac{R^2}{3}}d\varphi \\
&\quad +\frac{10273}{800}\int_0^{2\pi}\frac{\cos^2\varphi}{\cos\varphi+\frac{R^2}{3}}d\varphi-\frac{1313}{800}\int_0^{2\pi}\frac{1}{\cos\varphi+\frac{R^2}{3}}d\varphi\Big] \\
&= -\frac{\sqrt{3}(1-w)^{3/2}}{48(1+w^2)^{1/2}(1+w)^{9/2}}(w-\frac{1}{10})(w-\frac{1}{5})(w-10)(w-5).
\end{aligned}
\tag{3.12}
$$

Apparently, $f_*^0(R)$ has exactly two positive zeros, denoted by

$$
R_1=\frac{\sqrt{29997}}{99}\approx 1.749458791,\quad R_2=\frac{\sqrt{1872}}{24}\approx 1.802775638,
$$

corresponding to $w_1=1/10$ and $w_2=1/5$ in $R\in(\sqrt{3},+\infty)$. Moreover, we have

$$
\begin{aligned}
\frac{df_*^0(R_1)}{dR} &= \frac{107163}{387200}\approx 0.5260835926>0, \\
\frac{df_*^0(R_2)}{dR} &= -\frac{49}{675}\approx -0.07259259259<0.
\end{aligned}
$$

It follows from Lemma 2.1 and Corollary 2.4 that for the sufficiently small $|\varepsilon|$, system (3.9) has just two limit cycles emerging from the period annulus of the corresponding unperturbed system: one is unstable and the other is stable. This completes the proof of Theorem 1.1.

As a byproduct, we obtain

Theorem 3.1. *For the sufficiently small $|\varepsilon|$, system* (3.10) *has exactly two periodic solutions, denoted by l_1 and l_2 respectively, such that l_1 shrinks to R_1 and l_2 shrinks to R_2 as ε goes to* 0. *Moreover, l_1 is unstable while l_2 is stable.*

4. Proof of Theorem 1.2

After using the transformation (3.2), system (1.3) can be re-expressed as

$$
\frac{dR}{d\varphi}=\varepsilon\Big(\frac{3}{2R}\frac{Q\tilde{p}-P\tilde{q}}{\rho^5}\Big)\Big|_{x=\rho\cos\varphi,y=\rho\sin\varphi}+O(\varepsilon^2), \tag{4.1}
$$

where ρ is defined as (3.1), and

$$
\begin{aligned}
&Q\tilde{p}-P\tilde{q} \\
&=[a_{10}x^2+(a_{01}+b_{10})xy+b_{01}y^2]+[(a_{11}+b_{20})x^2y+b_{02}y^3] \\
&\quad +[(a_{21}+b_{30})x^3y+(a_{03}+b_{12})xy^3]+[a_{40}x^5+(a_{31}+b_{40}-b_{10})x^4y \\
&\quad +(a_{22}+a_{10}+b_{31}-b_{01})x^3y^2+(a_{13}+a_{01}+b_{22}-b_{10})x^2y^3 \\
&\quad +(a_{04}+a_{10}+b_{13}-b_{01})xy^4+(a_{01}+b_{04})y^5] \\
&\quad +[-b_{20}x^5y+(a_{11}-b_{20}-b_{02})x^3y^3+(a_{11}-b_{02})xy^5] \\
&\quad +[-b_{30}x^6y+(a_{21}-b_{30}-b_{12})x^4y^3+(a_{21}+a_{03}-b_{12})x^2y^5+a_{03}y^7] \\
&\quad +[-b_{40}x^7y+(a_{40}-b_{31})x^6y^2+(a_{31}-b_{40}-b_{22})x^5y^3 \\
&\quad +(a_{40}+a_{22}-b_{31}-b_{13})x^4y^4+(a_{31}+a_{13}-b_{22}-b_{04})x^3y^5
\end{aligned}
$$

$$+ (a_{22} + a_{04} - b_{13})x^2y^6 + (a_{13} - b_{04})xy^7 + a_{04}y^8].$$

The averaged equation associated with system (4.1) is

$$\frac{dR}{d\varphi} = \varepsilon g^0(R) + O(\varepsilon^2), \tag{4.2}$$

where

$$\begin{aligned} g^0(R) &= \frac{1}{2\pi}\int_0^{2\pi} \frac{3}{2R}\Big(\frac{Q\tilde{p} - P\tilde{q}}{\rho^5}\Big)\Big|_{x=\rho\cos\varphi, y=\rho\sin\varphi} d\varphi \\ &= \frac{3}{4\pi R}\int_0^{2\pi} \Big\{\frac{a_{10}\cos^2\varphi + b_{01}\sin^2\varphi}{\rho^3} \\ &\quad + \Big[a_{40}\cos^5\varphi + (a_{22} + a_{10} + b_{31} - b_{01})\cos^3\varphi\sin^2\varphi \\ &\quad + (a_{04} + a_{10} + b_{13} - b_{01})\cos\varphi\sin^4\varphi\Big] \\ &\quad + \rho^3\Big[(a_{40} - b_{31})\cos^6\varphi\sin^2\varphi + (a_{40} + a_{22} - b_{31} - b_{13})\cos^4\varphi\sin^4\varphi \\ &\quad + (a_{22} + a_{04} - b_{13})\cos^2\varphi\sin^6\varphi + a_{04}\sin^8\varphi\Big]\Big\} d\varphi. \end{aligned} \tag{4.3}$$

Using a similar transformation as in the preceding section to (4.3), the function $g^0(R)$ can be simplified as

$$\begin{aligned} g^0(R) &= \frac{3}{4R}\Big[-\frac{2M_1}{729}R^{10} - \frac{M_1 + 2M_2}{81}R^6 + \Big(-\frac{3M_1 + 4M_2 + 8M_3}{36} + M_4\Big)R^2 \\ &\quad + \Big(\frac{2M_1}{729}R^{12} + \frac{2M_2}{81}R^8 + \frac{2M_3}{9}R^4 - 2(M_1 + M_2 + M_3)\Big)\frac{1}{\sqrt{R^4 - 9}}\Big] \\ &= \frac{3}{4R}\Big[-\frac{2M_1}{729}S^5 - \frac{M_1 + 2M_2}{81}S^3 + \Big(-\frac{3M_1 + 4M_2 + 8M_3}{36} + M_4\Big)S \\ &\quad + \Big(\frac{2M_1}{729}S^6 + \frac{2M_2}{81}S^4 + \frac{2M_3}{9}S^2 - 2(M_1 + M_2 + M_3)\Big)\frac{1}{\sqrt{S^2 - 9}}\Big] \\ &= -\frac{\sqrt{3}}{48(1 - w^2)^{1/2}(1 + w^2)^{1/2}(1 + w)^4} \\ &\quad \times [\tilde{N}_1w^6 + \tilde{N}_2w^5 + \tilde{N}_3w^4 + \tilde{N}_4w^3 + \tilde{N}_3w^2 + \tilde{N}_2w + \tilde{N}_1], \end{aligned} \tag{4.4}$$

where M_i $(i = 1, 2, 3)$ are defined as (3.6), and

$$\begin{gathered} M_4 = a_{10} + b_{01}, \\ \tilde{N}_1 = 15M_1 + 12M_2 + 8M_3 - 36M_4, \\ \tilde{N}_2 = 12M_1 + 16M_2 + 16M_3 - 144M_4, \\ \tilde{N}_3 = -7M_1 - 12M_2 - 8M_3 - 252M_4, \\ \tilde{N}_4 = -40M_1 - 32M_2 - 32M_3 - 288M_4. \end{gathered}$$

Similarly, from (4.4), we get that $g^0(R)$ has at most three zeros in $R \in (\sqrt{3}, +\infty)$. Using this fact together with Lemma 2.1 and Corollary 2.4, it follows that system (4.1) has at most three periodic solutions tending to the corresponding hyperbolic equilibriums, respectively. This means that the maximal number of limit cycles of system (1.3) emerging from the period annulus of the unperturbed one is three. Moreover, the upper bound can be reached.

As an example, we consider the system

$$
\begin{aligned}
\dot{x} &= -y + x^3y + xy^3 + \varepsilon\Big[\big(-b_{01} + \frac{9}{800}\big)x + a_{01}y + a_{11}xy + a_{21}x^2y + a_{03}y^3 \\
&\quad + \big(b_{31} - \frac{109}{80}\big)x^4 + a_{31}x^3y + \big(b_{13} + \frac{28279}{3200}\big)x^2y^2 + a_{13}xy^3 - \frac{1313}{640}y^4\Big], \\
\dot{y} &= x + x^2y^2 + y^4 + \varepsilon\Big[b_{10}x + b_{01}y + b_{20}x^2 + b_{02}y^2 + b_{30}x^3 + b_{12}xy^2 + b_{40}x^4 \\
&\quad + b_{31}x^3y + b_{22}x^2y^2 + b_{13}xy^3 + b_{04}y^4\Big],
\end{aligned}
\tag{4.5}
$$

where $|\varepsilon|$ is sufficiently small, a_{ij} $(i = 0, 1, 2, 3, j = 1, 3)$ and b_{ij} $(i, j = 0, 1, 2, 3, 4)$ are any real constants.

By polar coordinates in (3.2), system (4.5) is equivalent to

$$
\frac{dR}{d\varphi} = \varepsilon G(\varphi, R) + O(\varepsilon^2), \tag{4.6}
$$

where

$$
\begin{aligned}
&G(\varphi, R) \\
&= \frac{\big(-b_{01} + \frac{9}{800}\big)\cos^2\varphi + (a_{01} + b_{10})\cos\varphi\sin\varphi + b_{01}\sin^2\varphi}{\rho^3} \\
&\quad + \frac{(a_{11} + b_{20})\cos^2\varphi\sin\varphi + b_{02}\sin^3\varphi}{\rho^2} \\
&\quad + \frac{(a_{21} + b_{30})\cos^3\varphi\sin\varphi + (a_{03} + b_{12})\cos\varphi\sin^3\varphi}{\rho} \\
&\quad + \Big[\big(b_{31} - \frac{109}{80}\big)\cos^5\varphi + (a_{31} + b_{40} - b_{10})\cos^4\varphi\sin\varphi \\
&\quad + \big(b_{31} + b_{13} - 2b_{01} + \frac{5663}{640}\big)\cos^3\varphi\sin^2\varphi \\
&\quad + (a_{13} + a_{01} + b_{22} - b_{10})\cos^2\varphi\sin^3\varphi \\
&\quad + \big(b_{13} - 2b_{01} - \frac{6529}{3200}\big)\cos\varphi\sin^4\varphi + (a_{01} + b_{04})\sin^5\varphi\Big] \\
&\quad + \rho\Big[-b_{20}\cos^5\varphi\sin\varphi + (a_{11} - b_{02})\cos\varphi\sin^5\varphi \\
&\quad + (a_{11} - b_{20} - b_{02})\cos^3\varphi\sin^3\varphi\Big] \\
&\quad + \rho^2\Big[-b_{30}\cos^6\varphi\sin\varphi + (a_{21} - b_{30} - b_{12})\cos^4\varphi\sin^3\varphi \\
&\quad + (a_{21} + a_{03} - b_{12})\cos^2\varphi\sin^5\varphi + a_{03}\sin^7\varphi\Big] \\
&\quad + \rho^3\Big[-b_{40}\cos^7\varphi\sin\varphi - \frac{109}{80}\cos^6\varphi\sin^2\varphi + (a_{31} - b_{40} - b_{22})\cos^5\varphi\sin^3\varphi \\
&\quad + \frac{23919}{3200}\cos^4\varphi\sin^4\varphi + (a_{31} + a_{13} - b_{22} - b_{04})\cos^3\varphi\sin^5\varphi \\
&\quad + \frac{10857}{1600}\cos^2\varphi\sin^6\varphi + (a_{13} - b_{04})\cos\varphi\sin^7\varphi - \frac{1313}{640}\sin^8\varphi\Big].
\end{aligned}
$$

The averaged equation of system (4.6) is

$$\frac{dR}{d\varphi} = \varepsilon g_*^0(R) + O(\varepsilon^2), \tag{4.7}$$

where

$$\begin{aligned}
g_*^0(R) &= \frac{1}{2\pi}\int_0^{2\pi} G(\varphi,R)d\varphi \\
&= \frac{3}{4\pi R}\int_0^{2\pi}\Big\{\rho^3\Big[-\frac{109}{80}\cos^6\varphi\sin^2\varphi + \frac{23919}{3200}\cos^4\varphi\sin^4\varphi \\
&\quad + \frac{10857}{1600}\cos^2\varphi\sin^6\varphi - \frac{1313}{640}\sin^8\varphi\Big] \\
&\quad + \Big[\big(b_{31}-\frac{109}{80}\big)\cos^5\varphi + \big(b_{31}+b_{13}-2b_{01}+\frac{5663}{640}\big)\cos^3\varphi\sin^2\varphi \\
&\quad + \big(b_{13}-2b_{01}-\frac{6529}{3200}\big)\cos\varphi\sin^4\varphi\Big] \\
&\quad + \frac{1}{\rho^3}\Big[\big(-b_{01}+\frac{9}{800}\big)\cos^2\varphi + b_{01}\sin^2\varphi\Big]\Big\}d\varphi \\
&= -\frac{\sqrt{3}}{48(1-w^2)^{1/2}(1+w^2)^{1/2}(1+w)^4} \\
&\quad \times (w-\frac{1}{10})(w-\frac{1}{5})(w-\frac{1}{2})(w-10)(w-5)(w-2).
\end{aligned} \tag{4.8}$$

Hence, $g_*^0(R)$ has exactly three positive zeros, denoted by

$$\tilde{R}_1 = \frac{\sqrt{29997}}{99} \approx 1.749458791, \quad \tilde{R}_2 = \frac{\sqrt{1872}}{24} \approx 1.802775638, \\ \tilde{R}_3 = \sqrt{5} \approx 2.236067977,$$

which correspond to

$$\tilde{w}_1 = \frac{1}{10}, \quad \tilde{w}_2 = \frac{1}{5}, \quad \tilde{w}_3 = \frac{1}{2}$$

in $R \in (\sqrt{3}, +\infty)$, respectively. Moreover, we have

$$\begin{gathered}
\frac{dg_*^0(\tilde{R}_1)}{dR} = \frac{25137}{96800} \approx 0.2596797521 > 0, \\
\frac{dg_*^0(\tilde{R}_2)}{dR} = -\frac{49}{800} \approx -0.06125 < 0, \\
\frac{dg_*^0(\tilde{R}_3)}{dR} = \frac{19}{800} \approx 0.02375 > 0.
\end{gathered}$$

According to Lemma 2.1 and Corollary 2.4, we obtain that for the sufficiently small $|\varepsilon|$, system (4.5) has exactly three limit cycles emerging from the period annulus of the unperturbed system. Hence, we completes the proof of Theorem 1.2.

Theorem 4.1. *For the sufficiently small $|\varepsilon|$, system (4.6) has just three periodic solutions, denoted by $\tilde{l}_1, \tilde{l}_2$ and $\tilde{l}_3$ respectively, such that $\tilde{l}_1$ shrinks to $\tilde{R}_1$, $\tilde{l}_2$ shrinks to $\tilde{R}_2$ and $\tilde{l}_3$ shrinks to $\tilde{R}_3$ as ε goes to 0. Moreover, $\tilde{l}_1$ and $\tilde{l}_3$ are unstable while $\tilde{l}_2$ is stable.*

Acknowledgments. This work is supported by the National Science Foundation of China under contracts No. 11371046 and No.11290141. The first author would like to thank the Department of Mathematics at the University of Texas-Pan American for its hospitality and generous support during her visiting from January 2013 to January 2014.

References

[1] V. I. Arnold, Y. S. Ilyashenko; *Dynamical systems I: Ordinary differential equations, Encyclopaedia Math. Sci., Vol. 1*, Springer, Berlin, 1986.

[2] A. Atabaigi, N. Nyamoradi, H. R. Z. Zangeneh; *The number of limit cycles of a quintic polynomial system*, Comput. Math. Appl., 57 (2009), 677-684.

[3] A. Atabaigi, N. Nyamoradi, H. R. Z. Zangeneh; *The number of limit cycles of a quintic polynomial system with a center, Nonlinear Anal.*, 71 (2009), 3008-3017.

[4] A. Buică, J. Llibre; *Averaging methods for finding periodic orbits via Brouwer degree*, Bull. Sci. Math., 128 (2004), 7-22.

[5] A. Buică, J. Llibre; Limit cycles of a perturbed cubic polynominal differential center, Chaos Solitons Fractals, 32 (2007), 1059-1069.

[6] T. R. Blows, L. M. Perko; Bifurcation of limit cycles from centers and separatrix cycles of planar analytic systems, SIAM Rev., 36 (1994), 341-376.

[7] S. Benditkis, D. Novikov; On the number of zeros of Melnikov fuctions, arXiv: 1007.0672vl, [math. DS] 5, July, 2010.

[8] F. D. Chen, C. Li, J. Llibre, Z. H. Zhang; *A unified proof on the weak Hilbert 16th problem for $n = 2$*, J. Differential Equations, 221 (2006), 309-342.

[9] B. Coll, J. Llibre, R. Prohens; *Limit cycles bifurcating from a perturbed quartic center*, Chaos Solitons Fractals, 44 (2011), 317-334.

[10] B. Coll, A. Gasull, R. Prohens; *Bifurcation of limit cycles from two families of centers*, Dyn. Contin. Discrete Impuls. Syst., Ser. A (Math. Anal.), 12 (2005), 275-287.

[11] C. Chicone, M. Jacobs; *Bifurcation of limit cycles from quadratic isochrones*, J. Differential Equations, 91 (1991), 268-326.

[12] J. Écalle; *Introduction aux fonctions analysables et preuve constructive de la conjecture de Dulac*, Actualitiées Math., Hermann, Paris, 1992.

[13] J. Guckenheimer, P. Holmes; *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields*, Appl. Math. Sci., Vol. 42, New York, Springer-Verlag, 1986.

[14] L. Gavrilov, I. D. Iliev; *Quadratic perturbations of quadratic codimension-four centers*, J. Math. Anal. Appl., 357 (2009), 69-76.

[15] S. Gautier, L. Gavrilov, I. D. Iliev; *Perturbations of quadratic center of genus one*, Discrete Contin. Dyn. Syst., 25 (2009), 511-535.

[16] H. Giacomini, J. Llibre, M. Viano; *On the nonexistence, existence and uniqueness of limit cycles*, Nonlinearity, 9(1996), 501-516.

[17] H. Giacomini, J. Llibre, M. Viano; *On the shape of limit cycles that bifurcate from Hamiltonian centers*, Nonlinear Anal., 41 (2000), 523-537.

[18] H. Giacomini, J. Llibre, M. Viano; *On the shape of limit cycles that bifurcate from non-Hamiltonian centers*, Nonlinear Anal., 43 (2001), 837-859.

[19] J. Giné, J. Llibre; *Limit cycles of cubic polynomial vector fields via the averaging theory*, Nonlinear Anal., 66 (2007), 1707-1721.

[20] I. D. Iliev; *Perturbations of quadratic centers*, Bull. Sci. Math., 122 (1998), 107-161.

[21] Y. S. Ilyashenko; *Finiteness theorems for limit cycles*, Uspekhi Mat. Nauk, 45 (1990), 143-200 (Russian); English transl. Russian Math. Survey, 45 (1990), 129-203.

[22] Y. S. Ilyashenko; *Centennial history Hilbert's 16th problem*, Bull. Amer. Math. Soc. (N.S.), 39 (2002), 301-354.

[23] C. Li, J. Llibre, Z. Zhang; *Weak focus, limit cycles and bifurcations for bounded quadrtic systems*, J. Differential Equations, 115 (1995), 193-223.

[24] C. Li, J. Llibre; *Quadratic perturbations of a quadratic reversible Lotka-Volterra system*, Qual. Theory Dyn. Syst., 9 (2010), 235-249.

[25] J. Llibre, J. S. Pérez del Río, J. A. Rodríguez; *Averaging analysis of a perturbed quadratic center*, Nonlinear Anal., 46 (2001), 45-51.

[26] J. Llibre; *Averaging theory and limit cycles for quadratic systems*, Radovi Matematički, 11 (2002), 1-14.

[27] J. A. Sanders, F. Verhulst; *Averaging methods in nonlinear dynamical systems*, Appl. Math. Sci., Vol. 59, New York, Springer-Verlag, 1985.

[28] F. Verhulst; *Nonlinear differential equations and dynamical systems*, Universitext, Berlin, Springer-Verlag, 1996.

[29] M. Viano, J. Llibre, H. Giacomini; *Arbitrary order bifurcations for perturbed Hamiltonian planar systems via the reciprocal of an integrating factor*, Nonlinear Anal., 48(2002), 117-136.

[30] G. Xiang, M. Han; *Global bifurcation of limit cycles in a family of polynomial systems*, J. Math. Anal. Appl., 295(2004), 633-644.

[31] G. Xiang, M. Han; *Global bifurcation of limit cycles in a family of multiparameter systems*, Int. J. Bifur. Chaos Appl. Sci. Engrg., 14 (2004), 3325-3335.

Linping Peng

School of Mathematics and System Sciences, Beihang University, LIMB of the Ministry of Education, Beijing, 100191, China

E-mail address: penglp@buaa.edu.cn, fax (86-10) 8231-7933

Zhaosheng Feng

Department of Mathematics, University of Texas-Pan American, Edinburg, Texas 78539, USA

E-mail address: zsfeng@utpa.edu