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COMPONENT REDUCTION FOR REGULARITY CRITERIA OF THE THREE-DIMENSIONAL MAGNETOHYDRODYNAMICS SYSTEMS

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Abstract. We study the regularity of the three-dimensional magnetohydrodynamics system, and obtain criteria in terms of one velocity field component and two magnetic field components. In contrast to the previous results such as [22], we have eliminated the condition on the third component of the magnetic field completely while preserving the same upper bound on the integrability condition.

1. Introduction and statement of results

We study the three-dimensional magnetohydrodynamics (MHD) system

$$\begin{gathered}\frac{\partial u}{\partial t} + (u\cdot\nabla)u - (b\cdot\nabla)b + \nabla\pi = \nu\Delta u,\\ \frac{\partial b}{\partial t} + (u\cdot\nabla)b - (b\cdot\nabla)u = \eta\Delta b,\\ \nabla\cdot u = \nabla\cdot b = 0, \quad (u,b)(x,0) = (u_0,b_0)(x), \quad t\in\mathbb{R}^+\cup\{0\},\end{gathered} \tag{1.1}$$

where $u:\mathbb{R}^3\times\mathbb{R}^+\to\mathbb{R}^3$ represents the velocity field, $b:\mathbb{R}^3\times\mathbb{R}^+\to\mathbb{R}^3$ the magnetic field, $\pi:\mathbb{R}^3\times\mathbb{R}^+\to\mathbb{R}$ the pressure field, $\nu,\eta>0$ the viscosity and diffusivity constants respectively. Hereafter let us assume $\nu=\eta=1$ and write $\frac{\partial}{\partial t}=\partial_t$ and $\frac{\partial}{\partial x_i}=\partial_i$ and the components of u and b by

$$u=(u_1,u_2,u_3), \quad b=(b_1,b_2,b_3), \quad b_h:=(b_1,b_2,0).$$

Due to the works of [22], we know that (1.1) possesses at least one global L^2-weak solution pair for any initial data pair $(u_0,b_0)\in L^2$. However, whether the local solution pair remains smooth for all time remains open as in the case of the Navier-Stokes equations (NSE), the system (1.1) at $b\equiv 0$.

To show that the weak solution pair is actually strong, there has been a large amount of research conducted by many mathematicians to obtain a sufficient condition on (u,b) so that imposing such conditions lead to the H^1-norm bound on (u,b). We discuss some of them in particular.

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Following the pioneering work by Serrin [23], Beirão da Veiga [2] obtained regularity criteria on ∇u. Similar results followed for the MHD system; in particular, Zhou [34] showed that it suffices to bound only u dropping the conditions on b completely. For example, the following regularity criteria was obtained by He and Xin [11].

$$\int_0^T \|u\|_{L^p}^r d\tau < \infty, \quad \frac{3}{p} + \frac{2}{r} \le 1, \quad 3 < p.$$

In an accompanying paper [29], the author reduced this criteria to any two components of u. For the regularity criteria in terms of other quantities for the NSE such as vorticity, π, we refer readers to [1, 3, 7, 24, 33, 35].

Results related on the reduction of components appeared for example in Kukavica and Ziane [16]:

$$\int_0^T \|u_3\|_{L^p}^r d\tau < \infty, \quad \frac{3}{p} + \frac{2}{r} \le \frac{5}{8}, \quad \frac{24}{5} \le p,$$

(see also [17, 31, 32]). A few of the most recent results are the following:

$$\int_0^T \|u_3\|_{L^p}^r d\tau < \infty, \quad \frac{3}{p} + \frac{2}{r} < \frac{2(p+1)}{3p}, \quad \frac{7}{2} < p,$$

for the NSE see Cao and Titi [4] (also [5, 14, 20, 21, 36] followed by many in the case of the MHD system (e.g. [6, 13, 18, 25]). In particular, Jia and Zhou [12] showed that if

$$\int_0^T \|u_3\|_{L^p}^r + \|b\|_{L^p}^r d\tau < \infty, \quad \frac{3}{p} + \frac{2}{r} \le \frac{3}{4} + \frac{1}{2p}, \quad \frac{10}{3} < p, \tag{1.2}$$

then the solution pair (u, b) remains smooth (cf. [37] for the case of the NSE). More variations of (1.2) were also obtained in [12, 19]; however, in any case, if condition is given only on u_3 and no other component of u, then without a new idea, it seems we need to impose some integrability condition on every component of b. This is due to the difficulty in decomposing the four non-linear terms in the $\|\nabla_h u\|_{L^2}^2 + \|\nabla_h b\|_{L^2}^2$-estimates so that every term has either have u_3 or b_h, where $\nabla_h = (\partial_1, \partial_2, 0)$ (see the Appendix for details). Now we present our results.

Theorem 1.1. *Suppose (u, b) solves (1.1) in time interval $[0, T]$ and satisfies*

$$\int_0^T \|u_3\|_{L^\infty}^{8/3} + \|b_h\|_{L^p}^r d\tau < \infty, \quad b_h = (b_1, b_2, 0), \tag{1.3}$$

where $10/3 < p < \infty$ and $8 \le r$ satisfy

$$\frac{3}{p} + \frac{2}{r} \le \frac{3}{4} + \frac{1}{2p}, \quad \frac{10}{3} < p \le 5, \quad r = 8, \quad 5 \le p < \infty. \tag{1.4}$$

Then there is no singularity up to time T.

The boarder-line case of $p = 10/3$ may be obtained as well, via a slight modification of the proof for Theorem 1.1.

Theorem 1.2. *Suppose (u, b) solves (1.1) in time interval $[0, T]$ and*

$$\int_0^T \|u_3\|_{L^\infty}^{8/3} d\tau + \sup_{\tau \in [0,T]} \|b_h(\tau)\|_{L^{10/3}} < \infty, \quad b_h = (b_1, b_2, 0).$$

Then there is no singularity up to time T.

Theorem 1.3. *Suppose* (u, b) *solves* (1.1) *in time interval* $[0, T]$ *and satisfies*

$$\int_0^T \|u_2\|_{L^\infty}^{8/3} + \|u_3\|_{L^\infty}^{8/3} + \|b_1\|_{L^p}^r d\tau < \infty,$$

where $10/3 < p < \infty$*,* $8 \le r$ *satisfy* (1.4)*. Then there is no singularity up to time* T*.*

Theorem 1.4. *Suppose* (u, b) *solves* (1.1) *in time interval* $[0, T]$ *and*

$$\int_0^T \|u_2\|_{L^\infty}^{8/3} + \|u_3\|_{L^\infty}^{8/3} d\tau + \sup_{\tau \in [0,T]} \|b_1(\tau)\|_{L^{10/3}} < \infty.$$

Then there is no singularity up to time T*.*

Remark 1.5. (1) We may replace the role of u_3 with any other component of u as long as the two components of b will be the different two components.

(2) We emphasize that in particular in Theorem 1.1, we have eliminated the condition on b_3 completely while preserving the integrability condition on b_h and $p = \infty, r = 8/3$ also satisfies (1.2). Thus, it is clear that (1.4) is an improvement of the special case of (1.2). We were also able to obtain results in case when $p \neq \infty$ for u_3 in (1.3); however, the integrability conditions became worse; thus, for simplicity, we chose not to present those results. We also remark that in contrast to results from [12], Theorem 1.2 is not a smallness result.

(3) We also wish to emphasize that previously when the regularity criteria for the three-dimensional MHD system was obtained in terms of three terms, they have always been all from the velocity vector field; e.g. $\partial_3 u_1, \partial_3 u_2, \partial_3 u_3$ from [6] and [13], any three partial derivatives of u_1, u_2, u_3 from [15, 18, 25].

(4) The new idea in our proof is to make use of the structure of the magnetic vector field equation and estimate $\|b_3\|_{L^p}$ and obtain its bound in terms of b_h and u_3. Our proof was inspired by the others including [27], in particular [8, 9] concerning the [26, Theorems 1.3-1.4] and [28, Propositions 3.1-3.2]. Modification of Propositions 3.1-3.2 are possible indicating that in the future to obtain a regularity criteria of the MHD system in terms of one component of the velocity vector field, which has been done for the NSE but not for the MHD system, it suffices to discover a decomposition of the four non-linear terms that separate u_3 and b_3, not necessarily just u_3.

(5) After this manuscript was completed, the author discovered in [30] a new decomposition of the four non-linear terms of (4.1) which led to a regularity criteria of (1.1) in terms of u_3 and j_3 where j_3 is the third component of the current density $j := \nabla \times b$.

In the Preliminary section, we set notation. Thereafter, we prove two crucial propositions and then prove Theorems 1.1–1.4.

2. Preliminary

Let us denote a constant that depends on a, b by $c(a, b)$ and $A \lesssim B$ when there exists a constant $c \ge 0$ of no significance such that $A \le cB$. We shall also denote $\int f = \int_{\mathbb{R}^3} f(x)dx$,

$$\nabla_h = (\partial_1, \partial_2, 0), \quad \Delta_h = \partial_{11}^2 + \partial_{22}^2, \quad X(t) := \|\nabla u(t)\|_{L^2}^2 + \|\nabla b(t)\|_{L^2}^2,$$
$$Y(t) := \|\nabla_h u(t)\|_{L^2}^2 + \|\nabla_h b(t)\|_{L^2}^2, \quad Z(t) := \|\Delta u(t)\|_{L^2}^2 + \|\Delta b(t)\|_{L^2}^2,$$

$$M_1 := \int_0^T \|u_3\|_{L^\infty}^{8/3} d\tau, \quad M_2 := \|b_3(0)\|_{L^{10/3}} + \sup_{t\in[0,T]} \|b_h(t)\|_{L^{10/3}},$$
$$N_1 := \int_0^T \|u_{2,3}\|_{L^\infty}^{8/3} d\tau, \quad N_2 := \|b_{2,3}(0)\|_{L^{10/3}} + \sup_{t\in[0,T]} \|b_1(t)\|_{L^{10/3}},$$

where e.g. $b_{2,3}$ is a two dimensional vector of two entries b_2 and b_3.

We have the following special case of Troisi inequality (cf. [6, 10])

$$\|f\|_{L^6} \le c\|\partial_1 f\|_{L^2}^{1/3}\|\partial_2 f\|_{L^2}^{1/3}\|\partial_3 f\|_{L^2}^{1/3}. \tag{2.1}$$

Finally, we obtain the basic energy inequality by taking L^2-inner products of (1.1) with (u,b) respectively, integrating by parts and using the incompressibility of u and b to deduce after integrating in time

$$\sup_{t\in[0,T]} (\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2) + 2\int_0^T X(\tau)d\tau \lesssim 1. \tag{2.2}$$

3. Two propositions

Proposition 3.1. *Suppose (u,b) is the solution to (1.1) in time interval $[0,T]$. Then for any $p \in (2,\infty)$, the following inequality holds: for any distinct choices of $j_1, j_2, j_3 \in \{1,2,3\}$*

$$\begin{aligned}\sup_{t\in[0,T]} \|b_{j_1}(t)\|_{L^p}^2 &\le \|b_{j_1}(0)\|_{L^p}^2 e^{(p-1)\int_0^T \|u_{j_1}\|_{L^\infty}^2 d\lambda} \\ &\quad + 2(p-1)\int_0^T e^{(p-1)\int_\tau^T \|u_{j_1}\|_{L^\infty}^2 d\lambda} \|u_{j_1}\|_{L^\infty}^2 \|b_{j_2,j_3}\|_{L^p}^2 d\tau,\end{aligned} \tag{3.1}$$

where b_{j_2,j_3} is a two dimensional vector of two entries b_{j_2} and b_{j_3}.

Remark 3.2. We remark that we cannot obtain an analogous bound for u_3 due to the $\nabla\pi$-term in the first equation of (1.1). Moreover, we were able to obtain various modifications of this inequality; however, we emphasize that (3.1) in particular implies that if we have a sufficient bound on u_{j_1}, then we may bound the L^p-norm of b_{j_1} by the same L^p-norm of b_{j_2,j_3}.

Proof of Proposition 3.1. From the second equation of (1.1), we have the equation that governs the growth of b_{j_1} in time

$$\partial_t b_{j_1} + (u\cdot\nabla)b_{j_1} - (b\cdot\nabla)u_{j_1} = \Delta b_{j_1}. \tag{3.2}$$

We multiply by $|b_{j_1}|^{p-2}b_{j_1}$ and integrate in space to obtain

$$\frac{1}{p}\partial_t\|b_{j_1}\|_{L^p}^p - \int \Delta b_{j_1}|b_{j_1}|^{p-2}b_{j_1} = -\int (u\cdot\nabla)b_{j_1}|b_{j_1}|^{p-2}b_{j_1} + \int (b\cdot\nabla)u_{j_1}|b_{j_1}|^{p-2}b_{j_1}.$$

By the incompressibility condition, we see that the first term on the right hand side after integrating by parts equals zero. We compute the diffusive term after integrating by parts as follows:

$$-\int \Delta b_{j_1}|b_{j_1}|^{p-2}b_{j_1} = -\sum_{k=1}^3 \int (\partial_{kk}^2 b_{j_1})|b_{j_1}|^{p-2}b_{j_1} = (p-1)\sum_{k=1}^3 \int \big||\partial_k b_{j_1}||b_{j_1}|^{\frac{p-2}{2}}\big|^2.$$

Therefore, we obtain by integrating by parts and using the incompressibility condition of b,

$$\begin{aligned}
&\frac{1}{p}\partial_t\|b_{j_1}\|_{L^p}^p + (p-1)\sum_{k=1}^3\int \big||\partial_k b_{j_1}||b_{j_1}|^{\frac{p-2}{2}}\big|^2\\
&= -\sum_{k=1}^3\int \partial_k b_k u_{j_1}|b_{j_1}|^{p-2}b_{j_1} + b_k u_{j_1}\partial_k(|b_{j_1}|^{p-2}b_{j_1})\\
&= -\sum_{k=1}^3\int (p-1)^{1/2}b_k u_{j_1}|b_{j_1}|^{\frac{p-2}{2}}(p-1)^{1/2}|b_{j_1}|^{\frac{p-2}{2}}\partial_k b_{j_1}\\
&\le \big(\frac{p-1}{2}\big)\sum_{k=1}^3\int |b_k|^2|u_{j_1}|^2|b_{j_1}|^{p-2} + \frac{(p-1)}{2}\sum_{k=1}^3\int ||\partial_k b_{j_1}|b_{j_1}|^{\frac{p-2}{2}}|^2
\end{aligned}$$

by Young's inequality. Absorbing the diffusive term, we have

$$\frac{1}{p}\partial_t\|b_{j_1}\|_{L^p}^p + \frac{(p-1)}{2}\sum_{k=1}^3\int |(\partial_k b_{j_1})|b_{j_1}|^{\frac{p-2}{2}}|^2 \le \big(\frac{p-1}{2}\big)\sum_{k=1}^3\int |b_k|^2|u_{j_1}|^2|b_{j_1}|^{p-2}.$$

Therefore, Hölder's inequalities and then dividing by $\frac{1}{2}\|b_{j_1}\|_{L^p}^{p-2}$ lead to

$$\partial_t\|b_{j_1}\|_{L^p}^2 - (p-1)\|u_{j_1}\|_{L^\infty}^2\|b_{j_1}\|_{L^p}^2 \le 2(p-1)\|u_{j_1}\|_{L^\infty}^2\|b_{j_2,j_3}\|_{L^p}^2.$$

This leads to (3.1) completing the proof of Proposition 3.1. □

The next proposition may be obtained by an identical procedure.

Proposition 3.3. *Suppose (u,b) is the solution pair to (1.1) in time interval $[0,T]$. Then for any $p\in(2,\infty)$, the following inequality holds: for any distinct choices of $j_1,j_2,j_3\in\{1,2,3\}$*

$$\begin{aligned}
\sup_{t\in[0,T]}\|b_{j_1,j_2}(t)\|_{L^p}^2 &\le \|b_{j_1,j_2}(0)\|_{L^p}^2 e^{2(p-1)\int_0^T\|u_{j_1,j_2}\|_{L^\infty}^2 d\lambda}\\
&\quad + (p-1)\int_0^T e^{2(p-1)\int_\tau^T\|u_{j_1,j_2}\|_{L^\infty}^2 d\lambda}\|u_{j_1,j_2}\|_{L^\infty}^2\|b_{j_3}\|_{L^p}^2 d\tau.
\end{aligned} \tag{3.3}$$

4. Proof of Theorem 1.1

$\|\nabla_h u\|_{L^2}+\|\nabla_h b\|_{L^2}^2$-estimate. We now fix p and r that satisfy (1.4), take L^2-inner products of the first equation of (1.1) with $-\Delta_h u$ and the second with $-\Delta_h b$ to estimate

$$\begin{aligned}
&\frac{1}{2}\partial_t Y(t) + \|\nabla\nabla_h u\|_{L^2}^2 + \|\nabla\nabla_h b\|_{L^2}^2\\
&= \int (u\cdot\nabla)u\cdot\Delta_h u - (b\cdot\nabla)b\cdot\Delta_h u + (u\cdot\nabla)b\cdot\Delta_h b - (b\cdot\nabla)u\cdot\Delta_h b\\
&:= I_1 + I_2 + I_3 + I_4,
\end{aligned} \tag{4.1}$$

where $Y(t) := \|\nabla_h u(t)\|_{L^2}^2+\|\nabla_h b(t)\|_{L^2}^2$. The following decomposition was obtained in [12]; we provide details in the Appendix for convenience of readers:

$$I_1 \lesssim \int |u_3||\nabla u||\nabla\nabla_h u|, \quad I_2+I_3+I_4 \lesssim |b||\nabla b||\nabla\nabla_h u| + |b||\nabla u||\nabla\nabla_h b|. \tag{4.2}$$

Now by Hölder's and Young's inequalities we immediately obtain

$$I_1 \lesssim \|u_3\|_{L^\infty}\|\nabla u\|_{L^2}\|\nabla\nabla_h u\|_{L^2} \le \frac{1}{4}\|\nabla\nabla_h u\|_{L^2}^2 + c\|u_3\|_{L^\infty}^2\|\nabla u\|_{L^2}^2. \tag{4.3}$$

Next, by Hölder's inequalities and interpolation inequalities we estimate

$$\begin{aligned}
&I_2+I_3+I_4\\
&\lesssim \|b\|_{L^p}\|\nabla b\|_{L^{\frac{2p}{p-2}}}\|\nabla\nabla_h u\|_{L^2} + \|b\|_{L^p}\|\nabla u\|_{L^{\frac{2p}{p-2}}}\|\nabla\nabla_h b\|_{L^2}\\
&\lesssim \|b\|_{L^p}\|\nabla b\|_{L^2}^{\frac{p-3}{p}}\|\nabla b\|_{L^6}^{3/p}\|\nabla\nabla_h u\|_{L^2} + \|b\|_{L^p}\|\nabla u\|_{L^2}^{\frac{p-3}{p}}\|\nabla u\|_{L^6}^{3/p}\|\nabla\nabla_h b\|_{L^2}.
\end{aligned}$$

By (2.1) and Young's inequalities we have

$$\begin{aligned}
I_2+I_3+I_4 &\lesssim \|b\|_{L^p}\|\nabla b\|_{L^2}^{\frac{p-3}{p}}\|\nabla\nabla_h b\|_{L^2}^{2/p}\|\Delta b\|_{L^2}^{1/p}\|\nabla\nabla_h u\|_{L^2}\\
&\quad + \|b\|_{L^p}\|\nabla u\|_{L^2}^{\frac{p-3}{p}}\|\nabla\nabla_h u\|_{L^2}^{2/p}\|\Delta u\|_{L^2}^{1/p}\|\nabla\nabla_h b\|_{L^2}\\
&\le \frac{1}{4}\|\nabla\nabla_h u\|_{L^2}^2 + \frac{1}{2}\|\nabla\nabla_h b\|_{L^2}^2 + c(\|b\|_{L^p}^{\frac{2p}{p-2}}\|\nabla b\|_{L^2}^{2(\frac{p-3}{p-2})}\|\Delta b\|_{L^2}^{\frac{2}{p-2}}\\
&\quad + \|b\|_{L^p}^{\frac{2p}{p-2}}\|\nabla u\|_{L^2}^{2(\frac{p-3}{p-2})}\|\Delta u\|_{L^2}^{\frac{2}{p-2}}).
\end{aligned} \tag{4.4}$$

Thus, with (4.3) and (4.4) applied to (4.2), absorbing the dissipative and diffusive terms, integrating in time we obtain

$$\begin{aligned}
&\sup_{\tau\in[0,t]} Y(\tau) + \int_0^t \|\nabla\nabla_h u\|_{L^2}^2 + \|\nabla\nabla_h b\|_{L^2}^2 d\tau\\
&\lesssim 1 + \int_0^t \|u_3\|_{L^\infty}^2\|\nabla u\|_{L^2}^2 + \|b\|_{L^p}^{\frac{2p}{p-2}} X^{\frac{p-3}{p-2}}(\tau) Z^{\frac{1}{p-2}}(\tau) d\tau.
\end{aligned} \tag{4.5}$$

$\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2$-estimate. For both equations in (1.1), we take the L^2-inner products with $-\Delta u$ and $-\Delta b$ respectively and integrate by parts to obtain

$$\begin{aligned}
&\frac{1}{2}\partial_t X(t) + Z(t)\\
&= \int (u\cdot\nabla)u\cdot\Delta u - (b\cdot\nabla)\cdot\Delta u + (u\cdot\nabla)b\cdot\Delta b - (b\cdot\nabla)u\cdot\Delta b\\
&:= \sum_{i=1}^4 II_i.
\end{aligned} \tag{4.6}$$

For II_2 and II_4 we have the estimate

$$\begin{aligned}
II_2 + II_4 &\le \|b\|_{L^p}\|\nabla b\|_{L^{\frac{2p}{p-2}}}\|\Delta u\|_{L^2} + \|b\|_{L^p}\|\nabla u\|_{L^{\frac{2p}{p-2}}}\|\Delta b\|_{L^2}\\
&\le c(\|b\|_{L^p}^2\|\nabla b\|_{L^{\frac{2p}{p-2}}}^2 + \|b\|_{L^p}^2\|\nabla u\|_{L^{\frac{2p}{p-2}}}^2) + \frac{1}{8}Z(t),
\end{aligned}$$

by Hölder's and Young's inequalities. Now we use a Gagliardo-Nirenberg inequality

$$\|f\|_{L^{\frac{2p}{p-2}}} \lesssim \|f\|_{L^2}^{\frac{p-3}{p}}\|\nabla f\|_{L^2}^{3/p} \tag{4.7}$$

and Young's inequalities to obtain

$$\begin{aligned} II_2 + II_4 &\le c(\|b\|_{L^p}^2 \|\nabla b\|_{L^2}^{2(\frac{p-3}{p})} \|\Delta b\|_{L^2}^{\frac{6}{p}} + \|b\|_{L^p}^2 \|\nabla u\|_{L^2}^{2(\frac{p-3}{p})} \|\Delta u\|_{L^2}^{\frac{6}{p}}) + \frac{1}{8}Z(t) \\ &\le c\|b\|_{L^p}^{\frac{2p}{p-3}} X + \frac{1}{4}Z(t). \end{aligned} \tag{4.8}$$

For II_3, we integrate by parts twice to deduce

$$\begin{aligned} II_3 &= -\sum_{i,j,k=1}^{3} \int \partial_k u_i \partial_i b_j \partial_k b_j + u_i \partial_{ik}^2 b_j \partial_k b_j \\ &= -\sum_{i,j,k=1}^{3} \int \partial_k u_i \partial_i b_j \partial_k b_j - \frac{1}{2}\partial_i u_i (\partial_k b_j)^2 \\ &= \sum_{i,j,k=1}^{3} \int \partial_k u_i b_j \partial_{ik}^2 b_j \lesssim \int |\nabla u||b||\nabla\nabla b| \end{aligned}$$

so that similarly as before, Hölder's and Young's inequalities, (4.7) and another Young's inequality lead to

$$\begin{aligned} II_3 &\lesssim \|b\|_{L^p} \|\nabla u\|_{L^{\frac{2p}{p-2}}} \|\Delta b\|_{L^2} \\ &\le c\|b\|_{L^p}^2 \|\nabla u\|_{L^2}^{2(\frac{p-3}{p})} \|\Delta u\|_{L^2}^{\frac{6}{p}} + \frac{1}{4}\|\Delta b\|_{L^2}^2 \\ &\le c\|b\|_{L^p}^{\frac{2p}{p-3}} X(t) + \frac{1}{4}Z(t). \end{aligned} \tag{4.9}$$

Finally, on II_1, we write

$$II_1 = \sum_{i=1}^{2} \int u_i \partial_i u \cdot \Delta u + u_3 \partial_3 u \cdot \Delta_h u + \frac{1}{2} u_3 \partial_3 (\partial_3 u)^2$$

and then integrate by parts on each to obtain

$$\begin{aligned} II_1 = &-\sum_{i=1}^{2}\sum_{k=1}^{3} \int \partial_k u_i \partial_i u \cdot \partial_k u + u_i \partial_{ik}^2 u \cdot \partial_k u \\ &+ \sum_{k=1}^{2} \partial_k u_3 \partial_3 u \cdot \partial_k u + u_3 \partial_{3k}^2 u \cdot \partial_k u + \frac{1}{2}\partial_3 u_3 (\partial_3 u)^2 \\ = &-\sum_{i=1}^{2}\sum_{k=1}^{3} \int \partial_k u_i \partial_i u \cdot \partial_k u - \frac{1}{2}\partial_i u_i (\partial_k u)^2 + \sum_{k=1}^{2} \partial_k u_3 \partial_3 u \cdot \partial_k u \\ &- \frac{1}{2}\partial_3 u_3 (\partial_k u)^2 - \frac{1}{2}(\partial_1 u_1 + \partial_2 u_2)(\partial_3 u)^2 \\ \lesssim &\int |\nabla_h u||\nabla u|^2. \end{aligned}$$

Now Hölder's, interpolation inequalities, and (2.1) lead to

$$\begin{aligned} II_1 &\lesssim \|\nabla_h u\|_{L^2} \|\nabla u\|_{L^4}^2 \\ &\lesssim \|\nabla_h u\|_{L^2} \|\nabla u\|_{L^2}^{1/2} \|\nabla u\|_{L^6}^{3/2} \lesssim \|\nabla_h u\|_{L^2} \|\nabla u\|_{L^2}^{1/2} \|\nabla\nabla_h u\|_{L^2} \|\Delta u\|_{L^2}^{1/2}. \end{aligned} \tag{4.10}$$

We apply the bounds of (4.8)–(4.10) into (4.6) to obtain, after absorbing the dissipative and diffusive terms,

$$\partial_t X(t) + Z(t) \lesssim \|b\|_{L^p}^{\frac{2p}{p-3}} X(t) + \|\nabla_h u\|_{L^2} \|\nabla u\|_{L^2}^{1/2} \|\nabla\nabla_h u\|_{L^2} \|\Delta u\|_{L^2}^{1/2}. \tag{4.11}$$

Integrating in time, we obtain

$$\begin{aligned}
&X(t) + \int_0^t Z(\tau)d\tau \\
&\le X(0) + c\int_0^t \|b\|_{L^p}^{\frac{2p}{p-3}} X(\tau)d\tau + c\int_0^t \|\nabla_h u\|_{L^2} \|\nabla u\|_{L^2}^{1/2} \|\nabla\nabla_h u\|_{L^2} \|\Delta u\|_{L^2}^{1/2} d\tau \\
&\lesssim 1 + \int_0^t \|b\|_{L^p}^{\frac{2p}{p-3}} X(\tau)d\tau \\
&\quad + \sup_{\tau\in[0,t]} \|\nabla_h u(\tau)\|_{L^2} \Big(\int_0^t \|\nabla\nabla_h u\|_{L^2}^2 d\tau\Big)^{1/2} \Big(\int_0^t X(\tau)d\tau\Big)^{1/4} \Big(\int_0^t Z(\tau)d\tau\Big)^{1/4}
\end{aligned}$$

by Hölder's inequality. By (4.5) and (2.2) we obtain

$$\begin{aligned}
&X(t) + \int_0^t Z(\tau)d\tau \\
&\lesssim 1 + \int_0^t \|b\|_{L^p}^{\frac{2p}{p-3}} X(\tau)d\tau \\
&\quad + \Big(1 + \int_0^t \|u_3\|_{L^\infty}^2 \|\nabla u\|_{L^2}^2 + \|b\|_{L^p}^{\frac{2p}{p-2}} X^{\frac{p-3}{p-2}}(\tau) Z^{\frac{1}{p-2}}(\tau)d\tau\Big) \Big(\int_0^t Z(\tau)d\tau\Big)^{1/4} \\
&\le c_0 + \sum_{i=1}^4 III_i,
\end{aligned} \tag{4.12}$$

where

$$\begin{gathered}
III_1 = c_0 \int_0^t \|b\|_{L^p}^{\frac{2p}{p-3}} X(\tau)d\tau, \quad III_2 = c_0 \Big(\int_0^t Z(\tau)d\tau\Big)^{1/4}, \\
III_3 = c_0 \Big(\int_0^t \|u_3\|_{L^\infty}^2 \|\nabla u\|_{L^2}^2 d\tau\Big) \Big(\int_0^t Z(\tau)d\tau\Big)^{1/4}, \\
III_4 = c_0 \Big(\int_0^t \|b\|_{L^p}^{\frac{2p}{p-2}} X^{\frac{p-3}{p-2}}(\tau) Z^{\frac{1}{p-2}}(\tau)d\tau\Big) \Big(\int_0^t Z(\tau)d\tau\Big)^{1/4},
\end{gathered}$$

and c_0 does not depend on t. By (3.1) with $j_1 = 3, j_2 = 1, j_3 = 2$, we have

$$\sup_{t\in[0,T]} \|b_3(t)\|_{L^p}^2 \le c(p) e^{c\int_0^T \|u_3(\lambda)\|_{L^\infty}^2 d\lambda} \Big(1 + \int_0^T \|u_3(\lambda)\|_{L^\infty}^2 \|b_h(\lambda)\|_{L^p}^2 d\lambda\Big).$$

Using the elementary inequality

$$(a+b)^p \le 2^p(a^p + b^p) \quad p \ge 0, \quad a, b \ge 0, \tag{4.13}$$

we obtain

$$\begin{aligned}
\|b(\tau)\|_{L^p}^{\frac{2p}{p-3}} &\le c(p)\Big(\|b_h(\tau)\|_{L^p}^{\frac{2p}{p-3}} + \|b_3(\tau)\|_{L^p}^{\frac{2p}{p-3}}\Big) \\
&\le c(p)\|b_h\|_{L^p}^{\frac{2p}{p-3}} + e^{c(p)\int_0^T \|u_3\|_{L^\infty}^2 d\lambda} \Big(1 + \Big(\int_0^T \|u_3\|_{L^\infty}^2 \|b_h\|_{L^p}^2 d\lambda\Big)^{\frac{p}{p-3}}\Big).
\end{aligned} \tag{4.14}$$

Thus, by (2.2),

$$
\begin{aligned}
III_1 \le c(p) \int_0^t \|b_h(\tau)\|_{L^p}^{\frac{2p}{p-3}} X(\tau) d\tau \\
+ e^{c(p)\int_0^T \|u_3(\lambda)\|_{L^\infty}^2 d\lambda} \Big(1 + \Big(\int_0^T \|u_3(\lambda)\|_{L^\infty}^2 \|b_h(\lambda)\|_{L^p}^2 d\lambda\Big)^{\frac{p}{p-3}}\Big).
\end{aligned} \tag{4.15}
$$

The estimate on III_2 is immediate by Young's inequality

$$
III_2 = c_0 \Big(\int_0^t Z(\tau) d\tau\Big)^{1/4} \le c + \frac{1}{8}\int_0^t Z(\tau) d\tau. \tag{4.16}
$$

Next, by Young's and Hölder's inequalities and (2.2),

$$
III_3 \le c\int_0^t \|u_3(\tau)\|_{L^\infty}^{8/3} \|\nabla u(\tau)\|_{L^2}^2 d\tau + \frac{1}{8}\int_0^t Z(\tau) d\tau. \tag{4.17}
$$

Finally, by successive applications of Hölder's and Young's inequalities,

$$
\begin{aligned}
III_4 &\lesssim \Big(\int_0^t \|b\|_{L^p}^{\frac{2p}{p-3}} X(\tau) d\tau\Big)^{\frac{p-3}{p-2}} \Big(\int_0^t Z(\tau) d\tau\Big)^{\frac{p+2}{4(p-2)}} \\
&\le c\Big(\int_0^t \|b\|_{L^p}^{\frac{2p}{p-3}} X(\tau) d\tau\Big)^{(\frac{4(p-3)}{3p-10})} + \frac{1}{8}\int_0^t Z(\tau) d\tau \\
&\le c\Big(\int_0^t \|b\|_{L^p}^{\frac{8p}{3p-10}} X(\tau) d\tau\Big) + \frac{1}{8}\int_0^t Z(\tau) d\tau,
\end{aligned} \tag{4.18}
$$

where using (4.14), we may obtain

$$
\begin{aligned}
\|b(\tau)\|_{L^p}^{\frac{8p}{3p-10}} &= \Big(\|b(\tau)\|_{L^p}^{\frac{2p}{p-3}}\Big)^{\frac{4(p-3)}{3p-10}} \\
&\le c(p)\|b_h\|_{L^p}^{\frac{8p}{3p-10}} + e^{c(p)\int_0^T \|u_3\|_{L^\infty}^2 d\lambda} \\
&\quad \times \Big(1 + \Big(\int_0^T \|u_3\|_{L^\infty}^2 \|b_h\|_{L^p}^2 d\lambda\Big)^{\frac{4p}{3p-10}}\Big)
\end{aligned}
$$

and therefore,

$$
\begin{aligned}
\int_0^t \|b(\tau)\|_{L^p}^{\frac{8p}{3p-10}} X(\tau) d\tau \le c(p)\int_0^t \|b_h\|_{L^p}^{\frac{8p}{3p-10}} X d\tau + e^{c(p)\int_0^T \|u_3\|_{L^\infty}^2 d\lambda} \\
\times \Big(1 + \Big(\int_0^T \|u_3\|_{L^\infty}^2 \|b_h\|_{L^p}^2 d\lambda\Big)^{\frac{4p}{3p-10}}\Big)
\end{aligned} \tag{4.19}
$$

due to (2.2). We apply (4.19) into (4.18) and along with (4.15)-(4.17) applied to (4.12), obtain after absorbing dissipative and diffusive terms

$$
\begin{aligned}
& X(t) + \int_0^t Z(\tau) d\tau \\
& \lesssim 1 + \int_0^t \|b_h\|_{L^p}^{\frac{2p}{p-3}} X d\tau + e^{c(p)\int_0^T \|u_3\|_{L^\infty}^2 d\lambda} \Big(1 + \Big(\int_0^T \|u_3\|_{L^\infty}^2 \|b_h\|_{L^p}^2 d\lambda\Big)^{\frac{p}{p-3}}\Big) \\
& \quad + \int_0^t \|u_3\|_{L^\infty}^{8/3} \|\nabla u\|_{L^2}^2 d\tau + \int_0^t \|b_h\|_{L^p}^{\frac{8p}{3p-10}} X d\tau \\
& \quad + e^{c(p)\int_0^T \|u_3\|_{L^\infty}^2 d\lambda} \Big(1 + \Big(\int_0^T \|u_3\|_{L^\infty}^2 \|b_h\|_{L^p}^2 d\lambda\Big)^{\frac{4p}{3p-10}}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim 1+\int_0^t (1+\|b_h\|_{L^p}^{(\frac{2p}{p-3})(\frac{4(p-3)}{3p-10})})X d\tau \\
&\quad + e^{c(p)\int_0^T \|u_3\|_{L^\infty}^2 d\lambda}\Big(1+\Big(\int_0^T \|u_3\|_{L^\infty}^2 \|b_h\|_{L^p}^2 d\lambda\Big)^{(\frac{p}{p-3})(\frac{4(p-3)}{3p-10})}\Big) \\
&\quad + \int_0^t \|u_3\|_{L^\infty}^{8/3}\|\nabla u\|_{L^2}^2 d\tau \\
&\lesssim 1+\int_0^t (\|b_h\|_{L^p}^{\frac{8p}{3p-10}} + \|u_3\|_{L^\infty}^{8/3})X d\tau \\
&\quad + e^{c(p)T^{1/4}(\int_0^T \|u_3\|_{L^\infty}^{8/3} d\lambda)^{3/4}}\Big(1+\Big(\int_0^T \|u_3\|_{L^\infty}^2 \|b_h\|_{L^p}^2 d\lambda\Big)^{\frac{4p}{3p-10}}\Big) \\
&\lesssim 1+\int_0^t (\|b_h\|_{L^p}^{\frac{8p}{3p-10}} + \|u_3\|_{L^\infty}^{8/3})X d\tau + e^{c(p)T^{1/4}(\int_0^T \|u_3\|_{L^\infty}^{8/3} d\lambda)^{3/4}} \\
&\quad \times \Big(1+\Big(\Big(\int_0^T \|u_3\|_{L^\infty}^{8/3} d\lambda\Big)^{3/4}\Big(\int_0^T \|b_h\|_{L^p}^{8} d\lambda\Big)^{1/4}\Big)^{\frac{4p}{3p-10}}\Big) \\
&\approx \int_0^t (\|b_h\|_{L^p}^{\frac{8p}{3p-10}} + \|u_3\|_{L^\infty}^{8/3})X d\tau + c(p,M_1)\Big(1+\Big(\int_0^T \|b_h\|_{L^p}^{8} d\lambda\Big)^{\frac{p}{3p-10}}\Big).
\end{aligned}
$$

By Gronwall's inequality, the proof of Theorem 1.1 is complete if

$$
\int_0^T \|b_h(\tau)\|_{L^p}^{\frac{8p}{3p-10}} + \|u_3(\tau)\|_{L^\infty}^{8/3} + \|b_h(\tau)\|_{L^p}^{8} d\tau < \infty.
$$

For $p \in (10/3, 5)$, we use Hölder's inequality to obtain

$$
\int_0^T \|b_h(\tau)\|_{L^p}^{8} d\tau \le T^{\frac{10-2p}{p}}\Big(\int_0^T \|b_h(\tau)\|_{L^p}^{\frac{8p}{3p-10}} d\tau\Big)^{\frac{3p-10}{p}} < \infty
$$

by (1.4) whereas if $p \in (5,\infty)$, Hölder's inequality again by (1.4) implies

$$
\int_0^T \|b_h(\tau)\|_{L^p}^{\frac{8p}{3p-10}} d\tau \le T^{\frac{2p-10}{3p-10}}\Big(\int_0^T \|b_h(\tau)\|_{L^p}^{8} d\tau\Big)^{\frac{p}{3p-10}} < \infty.
$$

5. Proof of Theorem 1.2

$\|\nabla_h u\|_{L^2}^2 + \|\nabla_h b\|_{L^2}^2$-estimate. For fixed $T>0$, firstly, from (3.1) with $j_1 = 3, j_2 = 1, j_2 = 2$, by Hölder's inequalities we have

$$
\begin{aligned}
&\sup_{t\in[0,T]} \|b_3(t)\|_{L^{10/3}}^2 \\
&\le \|b_3(0)\|_{L^{10/3}}^2 e^{\frac{7}{3}T^{1/4}\big(\int_0^T \|u_3(\lambda)\|_{L^\infty}^{8/3} d\tau\big)^{3/4}} \\
&\quad + (\frac{14}{3})e^{\frac{7}{3}T^{1/4}\big(\int_0^T \|u_3(\lambda)\|_{L^\infty}^{8/3} d\tau\big)^{3/4}} \sup_{t\in[0,T]} \|b_h(t)\|_{L^{10/3}}^2 T^{1/4}\Big(\int_0^T \|u_3(\tau)\|_{L^\infty}^{8/3} d\tau\Big)^{3/4} \\
&\le M_2^2 e^{\frac{7}{3}T^{1/4}M_1^{3/4}} + (\frac{14}{3})e^{\frac{7}{3}T^{1/4}M_1^{3/4}} M_2^2 T^{1/4} M_1^{3/4}.
\end{aligned}
$$

Thus, using (4.13), we compute

$$\begin{aligned}\sup_{t\in[0,T]}\|b(t)\|^2_{L^{10/3}} &\le 2^{8/5}\Big(\sup_{t\in[0,T]}\|b_h(t)\|^2_{L^{10/3}}+\sup_{t\in[0,T]}\|b_3(t)\|^2_{L^{10/3}}\Big)\\ &\le 2^{8/5}\Big(M_2^2+M_2^2e^{\frac{7}{3}T^{1/4}M_1^{3/4}}+(\frac{14}{3})e^{\frac{7}{3}T^{1/4}M_1^{3/4}}M_2^2T^{1/4}M_1^{3/4}\Big).\end{aligned} \tag{5.1}$$

Next, we choose $t_1\in[0,T]$ to be specified subsequently and as before, we may obtain by (4.1)-(4.4) which only required $p>3$ and integrating in time over $[0,t_1]$ as in (4.5)

$$\begin{aligned}&\sup_{t\in[0,t_1]}Y(t)+\int_0^{t_1}\|\nabla\nabla_hu\|^2_{L^2}+\|\nabla\nabla_hb\|^2_{L^2}d\tau\\ &\lesssim 1+\int_0^{t_1}\|u_3\|^2_{L^\infty}\|\nabla u\|^2_{L^2}+\|b\|^5_{L^{10/3}}X^{1/4}(\tau)Z^{3/4}(\tau)d\tau.\end{aligned}$$

$\|\nabla u\|^2_{L^2}+\|\nabla b\|^2_{L^2}$-estimate. By (4.8)–(4.10), all of which only required $p>3$, applied to (4.6) we have

$$\partial_tX(t)+Z(t)\lesssim\|b\|^{20}_{L^{10/3}}X(t)+\|\nabla_hu\|_{L^2}\|\nabla u\|^{1/2}_{L^2}\|\nabla\nabla_hu\|_{L^2}\|\Delta u\|^{1/2}_{L^2}$$

as in (4.11). Integrating in time and by Hölder's inequality as before in (4.12), we have

$$\begin{aligned}&\sup_{t\in[0,t_1]}X(t)+\int_0^{t_1}Z(\tau)d\tau\\ &\le X(0)+c\int_0^{t_1}\|b\|^{20}_{L^{10/3}}X(\tau)d\tau+\sup_{t\in[0,t_1]}\|\nabla_hu(\tau)\|_{L^2}\Big(\int_0^{t_1}\|\nabla\nabla_hu\|^2_{L^2}d\tau\Big)^{1/2}\\ &\quad\times\Big(\int_0^{t_1}X(\tau)d\tau\Big)^{1/4}\Big(\int_0^{t_1}\|\Delta u\|^2_{L^2}d\tau\Big)^{1/4}\\ &\lesssim 1+\sup_{t\in[0,t_1]}\|b(t)\|^{20}_{L^{10/3}}\int_0^{t_1}X(\tau)d\tau\\ &\quad+\Big(1+\int_0^{t_1}\|u_3\|^2_{L^\infty}\|\nabla u\|^2_{L^2}+\|b\|^5_{L^{10/3}}X^{1/4}(\tau)Z^{3/4}(\tau)d\tau\Big)\Big(\int_0^{t_1}Z(\tau)d\tau\Big)^{1/4}\\ &\lesssim 1+\sum_{i=1}^4 IIII_i,\end{aligned} \tag{5.2}$$

where

$$\begin{gathered}IIII_1=c_1\sup_{t\in[0,t_1]}\|b(t)\|^{20}_{L^{10/3}},\quad IIII_2=c_1\Big(\int_0^{t_1}Z(\tau)d\tau\Big)^{1/4},\\ IIII_3=c_1\Big(\int_0^{t_1}\|u_3\|^2_{L^\infty}\|\nabla u\|^2_{L^2}d\tau\Big)\Big(\int_0^{t_1}Z(\tau)d\tau\Big)^{1/4},\\ IIII_4=c_1\Big(\int_0^{t_1}\|b\|^5_{L^{10/3}}X^{1/4}(\tau)Z^{3/4}(\tau)d\tau\Big)\Big(\int_0^{t_1}Z(\tau)d\tau\Big)^{1/4},\end{gathered}$$

for $c_1 \geq 0$ independent of time t_1. Now due to (2.2), we can choose $t_1 \in [0,T]$ so that

$$\begin{aligned} &c_1\Big(2^{8/5}\Big(M_2^2 + M_2^2 e^{\frac{7}{3}T^{1/4}M_1^{3/4}} + (\frac{14}{3})e^{\frac{7}{3}T^{1/4}M_1^{3/4}} M_2^2 T^{1/4} M_1^{3/4}\Big)\Big)^{5/2} \\ &\times \Big(\int_0^{t_1} X(\tau)d\tau\Big)^{1/4} \\ &\leq \frac{1}{8}. \end{aligned} \tag{5.3}$$

Then, by (5.1),

$$IIII_1 \leq c_1\Big(2^{8/5}\Big(M_2^2 + M_2^2 e^{\frac{7}{3}T^{1/4}M_1^{3/4}} + (\frac{14}{3})e^{\frac{7}{3}T^{1/4}M_1^{3/4}} M_2^2 T^{1/4} M_1^{3/4}\Big)\Big)^{10} \lesssim 1. \tag{5.4}$$

The estimate of $IIII_2$ is same as before in (4.16) and the estimate of $IIII_3$ is also same as (4.17). Finally,

$$\begin{aligned} &IIII_4 \\ &\leq c_1\Big(\int_0^{t_1} \|b\|_{L^{10/3}}^{20} X(\tau)d\tau\Big)^{1/4}\Big(\int_0^{t_1} Z(\tau)d\tau\Big) \\ &\leq c_1 \sup_{t\in[0,t_1]} \|b(t)\|_{L^{10/3}}^5 \Big(\int_0^{t_1} X(\tau)d\tau\Big)^{1/4}\Big(\int_0^{t_1} Z(\tau)d\tau\Big) \\ &\leq c_1\Big(2^{8/5}\Big(M_2^2 + M_2^2 e^{\frac{7}{3}T^{1/4}M_1^{3/4}} + (\frac{14}{3})e^{\frac{7}{3}T^{1/4}M_1^{3/4}} M_2^2 T^{1/4} M_1^{3/4}\Big)\Big)^{5/2} \\ &\quad \times \Big(\int_0^{t_1} Z(\tau)d\tau\Big) \\ &\leq \frac{1}{8}\int_0^{t_1} Z(\tau)d\tau, \end{aligned} \tag{5.5}$$

by Hölder's inequality, (5.1) and (5.3). Using (5.4) and (5.5) in (5.2), absorbing the dissipative and diffusive terms, we have

$$\sup_{t\in[0,t_1]} X(t) + \frac{1}{2}\int_0^{t_1} Z(\tau)d\tau \lesssim 1 + \int_0^{t_1} \|u_3\|_{L^\infty}^{8/3} \|\nabla u\|_{L^2}^2 d\tau.$$

By Gronwall's inequality, we have the bound

$$\sup_{t\in[0,t_1]} X(t) + \frac{1}{2}\int_0^{t_1} Z(\tau)d\tau.$$

We restart on time interval $[t_1, 2t_1]$ and after finite number of repetitions obtain the same bound on $[0,T]$. This completes the proof of Theorem 1.2.

6. Proof of Theorem 1.3

This proof is similar to that of Theorem 1.1. We sketch it for completeness.

$\|\nabla_h u\|_{L^2}^2 + \|\nabla_h b\|_{L^2}^2$ - estimate. As before, from (4.1)–(4.4) which only required $p > 3$ leading to (4.5), we have

$$\sup_{\tau\in[0,t]} Y(\tau) + \int_0^t \|\nabla\nabla_h u\|_{L^2}^2 + \|\nabla\nabla_h b\|_{L^2}^2 d\tau$$

$$\lesssim 1 + \int_0^t \|u_3\|_{L^\infty}^2 \|\nabla u\|_{L^2}^2 + \|b\|_{L^p}^{\frac{2p}{p-2}} X^{\frac{p-3}{p-2}}(\tau) Z^{\frac{1}{p-2}}(\tau) d\tau.$$

$\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2$-estimate. As in the proof of Theorem 1.1, from (4.6), (4.8)–(4.10) and using $\|\nabla_h u\|_{L^2}^2 + \|\nabla_h b\|_{L^2}^2$-estimate leading to (4.12), we have

$$X(t) + \int_0^t Z(\tau) d\tau \le c_0 + \sum_{i=1}^4 III_i. \tag{6.1}$$

By (3.3) with $j_1 = 2, j_2 = 3, j_3 = 1$, we have

$$\sup_{t\in[0,T]} \|b_{2,3}(t)\|_{L^p}^2 \le e^{c(p)\int_0^T \|u_{2,3}(\lambda)\|_{L^\infty}^2 d\lambda}\Big(1 + \int_0^T \|u_{2,3}(\lambda)\|_{L^\infty}^2 \|b_1(\lambda)\|_{L^p}^2 d\lambda\Big).$$

so that by (4.13), similarly to (4.14),

$$\|b(\tau)\|_{L^p}^{\frac{2p}{p-3}} \le c(p)\|b_1(\tau)\|_{L^p}^{\frac{2p}{p-3}} + e^{c(p)\int_0^T \|u_{2,3}\|_{L^\infty}^2 d\lambda}\Big(1+\Big(\int_0^T \|u_{2,3}\|_{L^\infty}^2 \|b_1\|_{L^p}^2 d\lambda\Big)^{\frac{p}{p-3}}\Big) \tag{6.2}$$

and hence

$$\begin{aligned} III_1 &\le c(p)\int_0^t \|b_1(\tau)\|_{L^p}^{\frac{2p}{p-3}} X(\tau) d\tau \\ &\quad + e^{c(p)\int_0^T \|u_{2,3}(\lambda)\|_{L^\infty}^2 d\lambda}\Big(1 + \Big(\int_0^T \|u_{2,3}(\lambda)\|_{L^\infty}^2 \|b_1(\lambda)\|_{L^p}^2 d\lambda\Big)^{\frac{p}{p-3}}\Big) \end{aligned} \tag{6.3}$$

by (6.2) and (2.2). We take the identically same estimates on III_2 and III_3 as before from (4.16) and (4.17). On III_4, from (4.18), we have

$$III_4 \le c\Big(\int_0^t \|b\|_{L^p}^{\frac{8p}{3p-10}} X(\tau) d\tau\Big) + \frac{1}{8}\int_0^t Z(\tau) d\tau, \tag{6.4}$$

where due to (6.2) and (4.13) we have

$$\begin{aligned} &\|b(\tau)\|_{L^p}^{\frac{8p}{3p-10}} \\ &\le c(p)\|b_1(\tau)\|_{L^p}^{\frac{8p}{3p-10}} + e^{c(p)\int_0^T \|u_{2,3}\|_{L^\infty}^2 d\lambda}\Big(1 + \Big(\int_0^T \|u_{2,3}\|_{L^\infty}^2 \|b_1\|_{L^p}^2 d\lambda\Big)^{\frac{4p}{3p-10}}\Big). \end{aligned}$$

Thus, by (2.2) similarly to (4.19),

$$\begin{aligned} \int_0^t \|b(\tau)\|_{L^p}^{\frac{8p}{3p-10}} X(\tau) d\tau &\le c(p)\int_0^t \|b_1\|_{L^p}^{\frac{8p}{3p-10}} X d\tau + e^{c(p)\int_0^T \|u_{2,3}\|_{L^\infty}^2 d\lambda} \\ &\quad \times \Big(1 + \Big(\int_0^T \|u_{2,3}\|_{L^\infty}^2 \|b_1\|_{L^p}^2 d\lambda\Big)^{\frac{4p}{3p-10}}\Big). \end{aligned} \tag{6.5}$$

With (6.5) applied to (6.4), along with (4.16), (4.17) and (6.3) applied to (6.1), absorbing dissipative and diffusive terms, we have by Hölder's inequalities

$$\begin{aligned} &X(t) + \int_0^t Z(\tau) d\tau \\ &\lesssim 1 + \int_0^t \|b_1\|_{L^p}^{\frac{2p}{p-3}} X d\tau + e^{c(p)\int_0^T \|u_{2,3}\|_{L^\infty}^2 d\lambda}\Big(1 + \Big(\int_0^T \|u_{2,3}\|_{L^\infty}^2 \|b_1\|_{L^p}^2 d\lambda\Big)^{\frac{p}{p-3}}\Big) \\ &\quad + \int_0^t \|u_3\|_{L^\infty}^{8/3} \|\nabla u\|_{L^2}^2 d\tau + \int_0^t \|b_1\|_{L^p}^{\frac{8p}{3p-10}} X d\tau \end{aligned}$$

$$
\begin{aligned}
&\quad + e^{c(p)\int_0^T \|u_{2,3}\|_{L^\infty}^2 d\lambda}\Big(1 + \Big(\int_0^T \|u_{2,3}\|_{L^\infty}^2 \|b_1\|_{L^p}^2 d\lambda\Big)^{\frac{4p}{3p-10}}\Big) \\
&\lesssim 1 + \int_0^t \Big(\|b_1\|_{L^p}^{\frac{8p}{3p-10}} + \|u_3\|_{L^\infty}^{8/3}\Big) X d\tau \\
&\quad + e^{c(p)T^{1/4}(\int_0^T \|u_{2,3}\|_{L^\infty}^{8/3} d\lambda)^{3/4}}\Big(1 + \Big(\Big(\int_0^T \|u_{2,3}\|_{L^\infty}^{8/3} d\lambda\Big)^{3/4} \\
&\quad \times \Big(\int_0^T \|b_1\|_{L^p}^8 d\lambda\Big)^{1/4}\Big)^{\frac{4p}{3p-10}}\Big) \\
&\lesssim 1 + \int_0^t \Big(\|b_1\|_{L^p}^{\frac{8p}{3p-10}} + \|u_3\|_{L^\infty}^{8/3}\Big) X d\tau + c(p, N_1)\Big(1 + \Big(\int_0^T \|b_1\|_{L^p}^8 d\lambda\Big)^{\frac{p}{3p-10}}\Big).
\end{aligned}
$$

Thus, the proof of Theorem 1.3 is complete because by Hölder's inequalities as in the proof of Theorem 1.1, we have

$$
\int_0^T \|b_1(\tau)\|_{L^p}^{\frac{8p}{3p-10}} + \|u_3(\tau)\|_{L^\infty}^{8/3} + \|b_1(\tau)\|_{L^p}^8 d\lambda < \infty.
$$

7. Proof of Theorem 1.4

The proof is similar to that of Theorem 1.2; we sketch it for completeness. For fixed $T > 0$, firstly, from (3.3) with $j_1 = 2, j_2 = 3, j_3 = 1$, we obtain by Hölder's inequality

$$
\sup_{t\in[0,T]} \|b_{2,3}(t)\|_{L^{10/3}}^2 \le N_2^2 e^{\frac{14}{3}T^{1/4}N_1^{3/4}} + \big(\frac{7}{3}\big) e^{\frac{14}{3}T^{1/4}N_1^{3/4}} N_2^2 T^{1/4} N_1^{3/4}
$$

so that by (4.13),

$$
\begin{aligned}
\sup_{t\in[0,T]} \|b(t)\|_{L^{10/3}}^2 &\le \sup_{t\in[0,T]} 2^{8/5}\Big(\|b_1\|_{L^{10/3}}^2 + \|b_{2,3}\|_{L^{10/3}}^2\Big) \\
&\le 2^{8/5}\Big(N_2^2 + N_2^2 e^{\frac{14}{3}T^{1/4}N_1^{3/4}} + \big(\frac{7}{3}\big) e^{\frac{14}{3}T^{1/4}N_1^{3/4}} N_2^2 T^{1/4} N_1^{3/4}\Big)
\end{aligned} \tag{7.1}
$$

similarly to (5.1). Now as in the proof of Theorem 1.2, we choose $t_1 \in [0, T]$ to be specified subsequently and use the previous estimate of

$$
\sup_{t\in[0,t_1]} X(t) + \int_0^{t_1} Z(\tau) d\tau \lesssim 1 + \sum_{i=1}^4 IIII_i \tag{7.2}
$$

from (5.2). By (2.2), we can choose $t_1 \in [0, T]$ so that

$$
\begin{aligned}
&c_1\Big(2^{8/5}\Big(N_2^2 + N_2^2 e^{\frac{14}{3}T^{1/4}N_1^{3/4}} + \big(\frac{7}{3}\big) e^{\frac{14}{3}T^{1/4}N_1^{3/4}} N_2^2 T^{1/4} N_1^{3/4}\Big)\Big)^{5/2} \\
&\times \Big(\int_0^{t_1} X(\tau) d\tau\Big)^{1/4} \\
&\le \frac{1}{8}.
\end{aligned} \tag{7.3}
$$

Firstly, by (7.1),

$$
IIII_1 \le c_1\Big(2^{8/5}\Big(N_2^2 + N_2^2 e^{\frac{14}{3}T^{1/4}N_1^{3/4}} + \big(\frac{7}{3}\big) e^{\frac{7}{3}T^{1/4}N_1^{3/4}} N_2^2 T^{1/4} N_1^{3/4}\Big)\Big)^{10}. \tag{7.4}
$$

We use the same estimates of (4.16) and (4.17) for $IIII_2$ and $IIII_3$ as before. Finally, from (2.2), (7.1) and (7.3) and Hölder's inequality,

$$\begin{aligned} IIII_4 &\le c_1\Big(\int_0^{t_1}\|b\|_{L^{10/3}}^{20}X(\tau)d\tau\Big)^{1/4}\Big(\int_0^{t_1}Z(\tau)d\tau\Big) \\ &\le c_1\Big(\sup_{t\in[0,T]}\|b(t)\|_{L^{10/3}}^2\Big)^{5/2}\Big(\int_0^{t_1}X(\tau)d\tau\Big)^{1/4}\Big(\int_0^{t_1}Z(\tau)d\tau\Big) \\ &\le c_1\Big(2^{8/5}\Big(N_2^2+N_2^2e^{\frac{14}{3}T^{1/4}N_1^{3/4}}+\big(\frac{7}{3}\big)e^{\frac{14}{3}T^{1/4}N_1^{3/4}}N_2^2T^{1/4}N_1^{3/4}\Big)\Big)^{5/2} \quad (7.5)\\ &\quad\times\Big(\int_0^{t_1}Z(\tau)d\tau\Big) \\ &\le \frac{1}{8}\int_0^{t_1}Z(\tau)d\tau. \end{aligned}$$

After absorbing dissipative and diffusive terms, due to (4.16), (4.17), (7.4) and (7.5) applied to (7.2), Gronwall's inequality gives the bound on

$$\sup_{t\in[0,t_1]}X(t)+\frac{1}{2}\int_0^{t_1}Z(\tau)d\tau.$$

Reiterating on $[t,2t_1]$, after finite times we obtain the bound on the whole interval $[0,T]$ completing the proof of Theorem 1.4.

8. Appendix

In this section we give details of the decomposition in the $\|\nabla_h u\|_{L^2}^2+\|\nabla_h b\|_{L^2}^2$-estimate, namely (4.2) (cf. [5, 12]). The following lemma is useful:

Lemma 8.1 ([17])**.** *Assume that $u\in H^2(\mathbb{R}^3)$ is smooth and $\nabla\cdot u=0$. Then*

$$\sum_{i,j=1}^2\int u_i\partial_iu_j\Delta_hu_j=\sum_{i,j=1}^2\frac{1}{2}\int\partial_iu_j\partial_iu_j\partial_3u_3-\int\partial_1u_1\partial_2u_2\partial_3u_3+\int\partial_1u_2\partial_2u_1\partial_3u_3$$

Firstly, for I_1 applying this Lemma and integrating by parts, we have

$$\sum_{i,j=1}^2\int u_i\partial_iu_j\Delta_hu_j\lesssim\int|u_3||\nabla_hu||\nabla\nabla_hu|.$$

Thus,

$$\begin{aligned} I_1 &= \sum_{i,j=1}^2\int u_i\partial_iu_j\Delta_hu_j+\sum_{j=1}^2\int u_3\partial_3u_j\Delta_hu_j+\sum_{i=1}^3 u_i\partial_iu_3\Delta_hu_3 \\ &\lesssim\int|u_3||\nabla u||\nabla\nabla_hu|. \end{aligned}$$

Next, we decompose I_2: integrating by parts

$$\begin{aligned} I_2 &= -\sum_{i,j,k=1}^2\int b_i\partial_ib_j\partial_{kk}^2u_j-\sum_{i,k=1}^2\int b_i\partial_ib_3\partial_{kk}^2u_3-\sum_{j=1}^3\sum_{k=1}^2\int b_3\partial_3b_j\partial_{kk}^2u_j \\ &= -\sum_{i,j,k=1}^2\int b_i\partial_ib_j\partial_{kk}^2u_j-\sum_{i,k=1}^2\int b_i\partial_ib_3\partial_{kk}^2u_3 \end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{3}\sum_{k=1}^{2}\int \partial_k b_3\partial_3 b_j\partial_k u_j + b_3\partial_{k3}^2 b_j\partial_k u_j\\
&= -\sum_{i,j,k=1}^{2}\int b_i\partial_i b_j\partial_{kk}^2 u_j - \sum_{i,k=1}^{2}\int b_i\partial_i b_3\partial_{kk}^2 u_3\\
&+\sum_{j=1}^{3}\sum_{k=1}^{2}\int -\partial_{3k}^2 b_3 b_j\partial_k u_j - \partial_k b_3 b_j\partial_{3k}^2 u_j + b_3\partial_{k3}^2 b_j\partial_k u_j\\
&\lesssim \int |b||\nabla_h b||\nabla\nabla_h u| + |b||\nabla_h u||\nabla\nabla_h b|.
\end{aligned}
$$

Next, we write

$$
\begin{aligned}
I_3 &= \sum_{i,j,k=1}^{2}\int u_i\partial_i b_j\partial_{kk}^2 b_j + \sum_{i,k=1}^{2}\int u_i\partial_i b_3\partial_{kk}^2 b_3 + \sum_{j=1}^{3}\sum_{k=1}^{2} u_3\partial_3 b_j\partial_{kk}^2 b_j\\
&= I_{31} + I_{32} + I_{33}.
\end{aligned}
$$

Integrating by parts,

$$
\begin{aligned}
I_{31} &= -\sum_{i,j,k=1}^{2}\int \partial_k u_i\partial_i b_j\partial_k b_j + u_i\partial_{ik}^2 b_j\partial_k b_j\\
&= \sum_{i,j,k=1}^{2}\int \partial_{kk}^2 u_i\partial_i b_j b_j + \partial_k u_i\partial_{ik}^2 b_j b_j + \frac{1}{2}\partial_i u_i\partial_k b_j\partial_k b_j\\
&= \sum_{i,j,k=1}^{2}\int \partial_{kk}^2 u_i\partial_i b_j b_j + \partial_k u_i\partial_{ik}^2 b_j b_j - \frac{1}{2}(\partial_{ik}^2 u_i\partial_k b_j b_j + \partial_i u_i\partial_{kk}^2 b_j b_j)\\
&\lesssim \int |b||\nabla_h b||\nabla\nabla_h u| + |b||\nabla_h u||\nabla\nabla_h b|,
\end{aligned}
$$

$$
\begin{aligned}
I_{32} &= -\sum_{i,k=1}^{2}\int \partial_k u_i\partial_i b_3\partial_k b_3 + \frac{1}{2}u_i\partial_i(\partial_k b_3)^2\\
&= \sum_{i,k=1}^{2}\int \partial_{kk}^2 u_i\partial_i b_3 b_3 + \partial_k u_i\partial_{ik}^2 b_3 b_3 + \frac{1}{2}\partial_i u_i\partial_k b_3\partial_k b_3\\
&= \sum_{i,k=1}^{2}\int \partial_{kk}^2 u_i\partial_i b_3 b_3 + \partial_k u_i\partial_{ik}^2 b_3 b_3 - \frac{1}{2}(\partial_{ik}^2 u_i\partial_k b_3 b_3 + \partial_i u_i\partial_{kk}^2 b_3 b_3)\\
&\lesssim \int |b||\nabla_h b||\nabla\nabla_h u| + |b||\nabla_h u||\nabla\nabla_h b|,
\end{aligned}
$$

$$
\begin{aligned}
I_{33} &= -\sum_{j=1}^{3}\sum_{k=1}^{2}\int \partial_k u_3\partial_3 b_j\partial_k b_j + \frac{1}{2}u_3\partial_3(\partial_k b_j)^2\\
&= \sum_{j=1}^{3}\sum_{k=1}^{2}\int \partial_{3k}^2 u_3 b_j\partial_k b_j + \partial_k u_3 b_j\partial_{3k}^2 b_j + \frac{1}{2}\partial_3 u_3\partial_k b_j\partial_k b_j
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{j=1}^{3}\sum_{k=1}^{2}\int \partial_{3k}^2 u_3 b_j \partial_k b_j + \partial_k u_3 b_j \partial_{3k}^2 b_j - \frac{1}{2}\sum_{i=1}^{2}\partial_i u_i \partial_k b_j \partial_k b_j \\
&= \sum_{j=1}^{3}\sum_{k=1}^{2}\int \partial_{3k}^2 u_3 b_j \partial_k b_j + \partial_k u_3 b_j \partial_{3k}^2 b_j + \frac{1}{2}\sum_{i=1}^{2}(\partial_{ik}^2 u_i \partial_k b_j b_j + \partial_i u_i \partial_{kk}^2 b_j b_j) \\
&\lesssim \int |b||\nabla_h b||\nabla\nabla_h u| + |b||\nabla_h u||\nabla\nabla_h b|.
\end{aligned}
$$

Therefore,

$$
I_3 \lesssim \int |b||\nabla_h b||\nabla\nabla_h u| + |b||\nabla_h u||\nabla\nabla_h b|.
$$

Finally,

$$
\begin{aligned}
I_4 &= -\sum_{i,j,k=1}^{2}\int b_i \partial_i u_j \partial_{kk}^2 b_j - \sum_{i,k=1}^{2} b_i \partial_i u_3 \partial_{kk}^2 b_3 - \sum_{j=1}^{3}\sum_{k=1}^{2} b_3 \partial_3 u_j \partial_{kk}^2 b_j \\
&\lesssim \int |b||\nabla_h b||\nabla\nabla_h u| + |b||\nabla u||\nabla\nabla_h b|.
\end{aligned}
$$

Hence,

$$
I_2 + I_3 + I_4 \lesssim |b||\nabla b||\nabla\nabla_h u| + |b||\nabla u||\nabla\nabla_h b|.
$$

This completes the decomposition claimed.

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