

## CONDITIONS FOR HAVING A DIFFEOMORPHISM BETWEEN TWO BANACH SPACES

MAREK GALEWSKI, ELŻBIETA GALEWSKA, EWA SCHMEIDEL

ABSTRACT. We provide sufficient conditions for a mapping acting between two Banach spaces to be a diffeomorphism. Standard arguments allow us to obtain a local diffeomorphism. It is proved to be global using mountain pass geometry.

### 1. INTRODUCTION

Given two Banach spaces  $X$  and  $B$ , a continuously Fréchet - differentiable map  $f : X \rightarrow B$  is called a diffeomorphism if it is a bijection and its inverse  $f^{-1} : B \rightarrow X$  is continuously Fréchet - differentiable as well. A continuous linear mapping  $\Lambda : X \rightarrow B$ ,  $\Lambda \in L(X, B)$ , is a Fréchet - derivative of  $f$  at  $x \in X$  provided that for all  $h \in X$  it holds that

$$f(x+h) - f(x) = \Lambda h + o(\|h\|) \quad (1.1)$$

and where  $\lim_{\|h\| \rightarrow 0} \frac{\|o(\|h\|)\|}{\|h\|} = 0$ ;  $\Lambda$  is then typically denoted as  $f'(x)$  while its action on  $h$  as  $f'(x)h$ . Mapping  $f$  is continuously Fréchet - differentiable if  $f' : X \rightarrow L(X, B)$  is continuous in respective topologies. Obviously if a mapping  $f$  is a diffeomorphism, it is automatically a homeomorphism, while the vice versa is not correct as seen by example of a function  $f(x) = x^3$ . Recalling the Inverse Function Theorem a continuously Fréchet - differentiable mapping  $f : X \rightarrow B$  such that for any  $x \in X$  the derivative is surjective, i.e.  $f'(x)X = H$  and invertible, i.e. there exists a constant  $\alpha_x > 0$  such that

$$\|f'(x)h\| \geq \alpha_x \|h\|$$

defines a local diffeomorphism. This means that for each point  $x$  in  $X$ , there exists an open set  $U$  containing  $x$ , such that  $f(U)$  is open in  $B$  and  $f|_U : U \rightarrow f(U)$  is a diffeomorphism. If  $f$  is a diffeomorphism it obviously defines a local diffeomorphism. Thus the main problem to be overcome is to make a local diffeomorphism a global one. Or in other words: what assumptions should be imposed on the spaces involved and the mapping  $f$  to have global diffeomorphism from the local one. This task can be investigated within the critical point theory, or more precisely with mountain geometry.

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2000 *Mathematics Subject Classification.* 57R50, 58E05.

*Key words and phrases.* Global diffeomorphism; local diffeomorphism; mountain pass lemma.

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Submitted February 19, 2014. Published April 11, 2014.

Such research has apparently been started by Katriel [2]. His result can be summarized as follows, see also [5, Theorem 5.4].

**Theorem 1.1.** *Let  $X, B$  be finite dimensional Euclidean spaces. Assume that  $f : X \rightarrow B$  is a  $C^1$ -mapping such that*

- (a1)  $f'(x)$  is invertible for any  $x \in X$ ;
- (a2)  $\|f(x)\| \rightarrow \infty$  as  $\|x\| \rightarrow \infty$ ,

*then  $f$  is a diffeomorphism.*

Recently, Idczak, Skowron and Walczak [1] using the Mountain Pass Lemma and ideas contained in the proof of Theorem 1.1 (see [5] for some nice version) proved the result concerning diffeomorphism between a Banach and a Hilbert space. They further applied this abstract tool to the initial value problem for some integro-differential system in order to get differentiability of the solution operator. It seems that differentiable dependence on parameters for boundary value problems can be investigated by this method. The result from [1] reads as follows.

**Theorem 1.2.** *Let  $X$  be a real Banach space,  $H$  - a real Hilbert space. If  $f : X \rightarrow H$  is a  $C^1$ -mapping such that*

- (b1) *for any  $y \in H$  the functional  $\varphi : X \rightarrow \mathbb{R}$  given by*

$$\varphi(x) = \frac{1}{2} \|f(x) - y\|^2$$

*satisfies Palais-Smale condition;*

- (b2) *for any  $x \in X$ ,  $f'(x)X = H$  and there exists a constant  $\alpha_x > 0$  such that*

$$\|f'(x)h\| \geq \alpha_x \|h\| \tag{1.2}$$

*then  $f$  is a diffeomorphism.*

The question aroused *whether the Hilbert space  $H$  in the formulation of the above theorem could be replaced by a Banach space.* This question is of some importance since one would expect diffeomorphism to act between two Hilbert spaces or two Banach spaces rather than between a Hilbert and a Banach space. The applications given in [1] work when both  $X$  and  $H$  are Hilbert spaces.

The aim of this note is to provide an affirmative answer to this question. We also simplify a bit the proof of Theorem 1.2 by using a weak version of the MPL Lemma due to Figueredo and Solimini, see [3], [4] which we recall below.

Functional  $J : X \rightarrow \mathbb{R}$  satisfies the Palais-Smale condition if every sequence  $(u_n)$  such that  $\{J(u_n)\}$  is bounded and  $J'(u_n) \rightarrow 0$ , has a convergent subsequence. We note that in a finite dimensional setting condition (a2) implies that the Palais-Smale condition holds for  $x \rightarrow \|f(x)\|$ . The version of the Mountain Pass Lemma (MPL Lemma) which we use is as follows.

**Lemma 1.3** ([3]). *Let  $X$  be a Banach space and  $J \in C^1(X, \mathbb{R})$  satisfies the Palais-Smale condition. Assume that*

$$\inf_{\|x\|=r} J(x) \geq \max\{J(0), J(e)\}, \tag{1.3}$$

*where  $0 < r < \|e\|$  and  $e \in X$ . Then  $J$  has a non-zero critical point  $x_0$ .*

**Remark 1.4.** From the proof of Lemma 1.3 it is seen that if  $\inf_{\|x\|=r} J(x) > \max\{J(0), J(e)\}$ , then also  $x \neq e$ .

## 2. MAIN RESULT

Our main result concerns extension of Theorem 1.2 to the case of  $H$  being a Banach space. We retain the assumption providing local diffeomorphism and modify assumption (b1) to get the global diffeomorphism. This is realized by replacing  $\|\cdot\|^2$  with some functional  $\eta$  for which functional  $x \rightarrow \eta(f(x) - y)$  satisfies the Palais-Smale condition for all  $y$ . One can think of  $\eta$  as  $\eta(x) = \int_0^1 |x(t)|^p dt$  for  $x \in L^p(0, 1)$ ,  $p > 1$ . Our main result reads as follows.

**Theorem 2.1.** *Let  $X, B$  be real Banach spaces. Assume that  $f : X \rightarrow B$  is a  $C^1$ -mapping,  $\eta : B \rightarrow \mathbb{R}_+$  is a  $C^1$  functional and that the following conditions hold*

- (c1)  $(\eta(x) = 0 \iff x = 0)$  and  $(\eta'(x) = 0 \iff x = 0)$ ;  
(c2) for any  $y \in B$  the functional  $\varphi : X \rightarrow \mathbb{R}$  given by

$$\varphi(x) = \eta(f(x) - y)$$

*satisfies Palais-Smale condition;*

- (c3) for any  $x \in X$  the Fréchet derivative is surjective, i.e.  $f'(x)X = B$ , and there exists a constant  $\alpha_x > 0$  such that for all  $h \in X$

$$\|f'(x)h\| \geq \alpha_x \|h\|;$$

- (c4) there exist positive constants  $\alpha, c, M$  such that

$$\eta(x) \geq c\|x\|^\alpha \text{ for } \|x\| \leq M$$

then  $f$  is a diffeomorphism.

*Proof.* We follow the ideas used in the proof of Main Theorem in [1] with necessary modifications. In view of the remarks made in the Introduction condition (c3) implies that  $f$  is a local diffeomorphism. Thus it is sufficient to show that  $f$  is onto and one to one.

Firstly we show that  $f$  is onto. Let us fix any point  $y \in B$ . Observe that  $\varphi$  is a composition of two  $C^1$  mappings, thus  $\varphi \in C^1(X, \mathbb{R})$ . Moreover,  $\varphi$  is bounded from below and satisfies the Palais-Smale condition. Thus from the Ekeland's Variational Principle it follows that there exists argument of a minimum which we denote by  $\bar{x}$ , see [3, Theorem 4.7]. We see by the chain rule for Fréchet derivatives and by Fermat's Principle that

$$\varphi'(\bar{x}) = \eta'(f(\bar{x}) - y) \circ f'(\bar{x}) = 0.$$

Since by (c3) mapping  $f'(\bar{x})$  is invertible we see that  $\eta'(f(\bar{x}) - y) = 0$ . Now by (c1) it follows that

$$f(\bar{x}) - y = 0.$$

Thus  $f$  is surjective.

Now we argue by contradiction that  $f$  is one to one. Suppose there are  $x_1$  and  $x_2$ ,  $x_1 \neq x_2$ ,  $x_1, x_2 \in X$ , such that  $f(x_1) = f(x_2) = a \in B$ . We will apply Lemma 1.3. Thus we put  $e = x_1 - x_2$  and define mapping  $g : X \rightarrow B$  by the following formula

$$g(x) = f(x + x_2) - a.$$

Observe that  $g(0) = g(e) = 0$ . We define functional  $\psi : X \rightarrow \mathbb{R}$  by

$$\psi(x) = \eta(g(x)).$$

By (c2) functional  $\psi$  satisfies the Palais-Smale condition. Next we see that  $\psi(e) = \psi(0) = 0$ . Using (1.1) and (1.2) we see that there is a number  $\rho > 0$  such that

$$\frac{1}{2}\alpha_{\bar{x}}\|x\| \leq \|g(x)\| \text{ for } x \in \overline{B(0, \rho)}. \quad (2.1)$$

Indeed, since  $\lim_{\|h\| \rightarrow 0} \frac{o(\|h\|)}{\|h\|} = 0$  we see that for  $\|h\|$  sufficiently small, say  $\|h\| \leq \delta$ , it holds that  $o(\|h\|) \leq \frac{1}{2}\alpha_{x_2}\|h\|$  and

$$g(0+h) - g(0) = g'(0)h + o(\|h\|).$$

By the definition of  $g$  and by (c3) we see that for  $\|h\| \leq \delta$ ,

$$\|g(h)\| + \frac{1}{2}\alpha_{x_2}\|h\| \geq \|g(h) - o(\|h\|)\| = \|f'(x_2)h\| \geq \alpha_{x_2}\|h\|.$$

We can always assume that  $\delta < \rho < \min\{\|e\|, M\}$ . Thus (2.1) holds. Take any  $0 < r < \rho$ . Recall that by (c4) we obtain since (2.1) holds

$$\psi(x) = \eta(g(x)) \geq c\|g(x)\|^\alpha \geq c\left(\frac{1}{2}\alpha_{x_2}\right)^\alpha \|x\|^\alpha.$$

Thus

$$\inf_{\|x\|=r} \psi(x) \geq c\left(\frac{1}{2}\alpha_{x_2}\right)^\alpha \|r\|^\alpha > 0 = \psi(e) = \psi(0)$$

We see that (1.3) is satisfied for  $J = \psi$ . Thus by Lemma 1.3 and by Remark 1.4 we note that  $\psi$  has a critical point  $v \neq 0$ ,  $v \neq e$  and such that

$$\psi'(v) = \eta'(f(v+x_2) - a) \circ f'(v+x_2) = 0.$$

Since  $f'(v+x_2)$  is invertible, we see that  $\eta'(f(v+x_2) - a) = 0$ . So by the assumption (c1) we calculate  $f(v+x_2) - a = 0$ . This means that either  $v = 0$  or  $v = e$ . Thus we obtain a contradiction which shows that  $f$  is a one to one operator.  $\square$

We supply our result with a few of remarks.

**Remark 2.2.** We see that from Theorem 2.1 by putting  $\eta(x) = \frac{1}{2}\|x\|^2$  we obtain easily Theorem 1.2. In that case  $c = 1$ ,  $M > 0$  is arbitrary,  $\alpha = 2$ . It seems there is no difference as concerns the finite and infinite dimensional context.

**Remark 2.3.** Since the deformation lemma is also true with Cerami condition, we can assume that  $\varphi$  satisfies the Cerami condition instead of the Palais-Smale condition. However, in the possible applications, in which the A-R condition could not be assumed, it seems that checking the Palais-Smale condition would be an easier task. We refer to [6, 7] for some other variational methods.

### 3. CONCLUSION AND OTHER RESULTS

We would like to mention [8] for some other approach connected with the non-negative auxiliary scalar coercive function and the main assumption that for all positive  $r$ :  $\sup_{\|x\| \leq r} \|f'(x)^{-1}\| < +\infty$  and  $\|f(x)\| \rightarrow +\infty$  as  $\|x\| \rightarrow +\infty$ . The methods of the proof are quite different as well. One of the results of [8] most closely connected to ours and to those of [1] reads as follows

**Theorem 3.1.** *Let  $X, B$  be a real Banach spaces. Assume that  $f : X \rightarrow B$  is a  $C^1$ -mapping,  $\|f(x)\| \rightarrow +\infty$  as  $\|x\| \rightarrow +\infty$ , for all  $x \in X$   $f'(x) \in \text{Isom}(X, B)$  and for all  $x \in X$   $\sup_{\|x\| \leq r} \|f'(x)^{-1}\| < +\infty$  for all  $r > 0$ . Then  $f$  is a diffeomorphism.*

The main difference between our results and the existing one is that we do not require condition  $\sup_{\|x\| \leq r} \|f'(x)^{-1}\| < +\infty$  for all  $r > 0$ . We have boundedness of  $\|f'(x)^{-1}\|$  but in a pointwise manner. Still it seems that checking the condition  $\|f(x)\| \rightarrow +\infty$  as  $\|x\| \rightarrow +\infty$  might be difficult in a direct manner. Recall that  $\varphi(x) = \eta(f(x) - y)$  is bounded from below,  $C^1$  and satisfies the Palais-Smale condition and therefore it is coercive as well. However, coercivity alone does not provide the existence of exactly one minimizer. We would have to add strict convexity to the assumptions. Thus we can obtain easily the following result

**Theorem 3.2.** *Let  $X, B$  be a real Banach spaces. Assume that  $f : X \rightarrow B$  is a  $C^1$ -mapping,  $\eta : B \rightarrow \mathbb{R}_+$  is a  $C^1$  functional and that the following conditions hold*

(d1)  $(\eta(x) = 0 \iff x = 0)$  and  $(\eta'(x) = 0 \iff x = 0)$ .

(d2) for any  $y \in B$  the functional  $\varphi : X \rightarrow \mathbb{R}$  given by the formula

$$\varphi(x) = \eta(f(x) - y)$$

is coercive and strictly convex;

(d3) for any  $x \in X$  the Fréchet derivative is surjective, i.e.  $f'(x)X = B$ , and there exists a constant  $\alpha_x > 0$  such that for all  $h \in X$

$$\|f'(x)h\| \geq \alpha_x \|h\|$$

then  $f$  is a diffeomorphism.

*Proof.* Let us fix  $y \in B$ . Note that by (d2)  $\varphi$  has exactly one minimizer  $\bar{x}$ . Thus by Fermat's Principle we see that

$$\varphi'(\bar{x}) = \eta'(f(\bar{x}) - y) \circ f'(\bar{x}) = 0.$$

Since by (d3) mapping  $f'(\bar{x})$  is invertible we see that  $\eta'(f(\bar{x}) - y) = 0$ . Now by (d1) it follows that

$$f(\bar{x}) - y = 0.$$

Thus  $f$  is surjective and obviously one to one since  $\bar{x}$  is unique.  $\square$

We believe that checking that  $\varphi$  is strictly convex is still more demanding than proving that  $\varphi$  satisfies the Palais-Smale condition.

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MAREK GALEWSKI  
INSTITUTE OF MATHEMATICS, LODZ UNIVERSITY OF TECHNOLOGY, WÓLCZANSKA 215, 90-924  
LÓDZ, POLAND

*E-mail address:* `marek.galewski@p.lodz.pl`

ELŻBIETA GALEWSKA  
CENTRE OF MATHEMATICS AND PHYSICS, LODZ UNIVERSITY OF TECHNOLOGY, AL. POLITECHNIKI  
11, 90-924 LÓDZ, POLAND

*E-mail address:* `elzbieta.galewska@p.lodz.pl`

EWA SCHMEIDEL  
FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, UNIVERSITY OF BIAŁYSTOK, AKADEMICKA  
2, 15-267 BIAŁYSTOK, POLAND

*E-mail address:* `eschmeidel@math.uwb.edu.pl`