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WEAK ASYMPTOTIC SOLUTION FOR A NON-STRICTLY HYPERBOLIC SYSTEM OF CONSERVATION LAWS

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Abstract. In this article, we construct the weak asymptotic solution developed by Panov and Shelkovich for piecewise known solutions to a prolonged system of conservation laws. This is done by introducing four singular waves along a discontinuity curve, which in turn implies the existence of weak asymptotic solutions for the Riemann type initial data. By piecing together the Riemann problems, we construct weak asymptotic solution for general type initial data.

1. Introduction

Systems of conservation laws arise in many physical contexts are not strictly hyperbolic. For such systems classical theories of Glimm [2] and Lax [8] do not apply. Because of the appearance of product of distributions, it is difficult to define the notion of solutions for these problems. One way to avoid this is to work with the generalized space of Colmbeau. For details see [9] and [1].

A system of this kind was introduced by Joseph and Vasudeva Murthy[5], namely,

$$(u_j)_t + \sum_{i=1}^{j} (\frac{u_i u_{j-i+1}}{2})_x = 0, \quad j = 1, 2, \dots, n. \tag{1.1}$$

For $n = 1$, system (1.1) is Burger's equation, which is well studied by Hopf [3]. For $n = 2$ case is an one dimensional model for the large scale structure formation of universe, see, [12]. Using vanshing viscosity approach it is observed by Joseph [4] that the second component contain δ measure concentrated along the line of discontinuity. The case $n = 3$ is studied in [6]. Solution is constructed in the Colombeau setting. If $u_1 = u, u_2 = v, u_3 = w$, (1.1) becomes

$$u_t + (\frac{u^2}{2})_x = 0, \quad v_t + (uv)_x = 0, \quad w_t + (\frac{v^2}{2} + uw)_x = 0. \tag{1.2}$$

A similar system,

$$u_t + (u^2)_x = 0, \quad v_t + (2uv)_x = 0, \quad w_t + 2(v^2 + uw)_x = 0, \tag{1.3}$$

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is studied by Panov and Shelkovich [11]. In [11] a concept of weak asymptotic solution is introduced and a solution is constructed under this consideration and generalized integral formulation is introduced for piecewise continuous data. Note that the system (1.3) can be obtained from (1.2) using the transformation $(u, v, w) \to (2u, v, \frac{w}{2})$. The case $n = 4$ is studied by joseph and Sahoo [7]. In [7], using vanshing viscosity approach a solution is constructed for Riemann type initial data and based on this a weak integral formulation is given.

In this paper we use the weak asymptotic method introduced by Panov and Shelkovich [11] to study the case $n = 4$. Putting $u_1 = u$, $u_2 = v$, $u_3 = w$, $u_4 = z$ and followed by a linear transformation, the system (1.1) leads to the system

$$\begin{gathered} u_t + (u^2)_x = 0, \quad v_t + (2uv)_x = 0 \\ w_t + 2(v^2 + uw)_x = 0, \quad z_t + 2((3vw + uz)_x) = 0. \end{gathered} \tag{1.4}$$

The aim of this paper is to study the above system (1.4) with initial conditions

$$u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad w(x, 0) = w_0(x), \quad z(x, 0) = z_0(x). \tag{1.5}$$

The content of the paper is as follows. We construct weak asymptotic solution by connecting two known solutions from the left and right. As a special case we derive weak asymptotic solution for the Riemann type initial data. Then we construct a weak asymptotic solution when the initial data for u is a monotonic increasing function and initial data for v, w and z are locally integrable functions.

2. Weak asymptotic solution for Riemann type initial data

In this section we connect two classical solutions by introducing a discontinuity curve in asymptote level. First of all we recall the definition of *weak asymptotic solution* as introduced in [11, 10].

Definition 2.1. Let us define

$$\begin{gathered} L_1(u) = u_t + (u^2)_x, \quad L_2(u, v) = v_t + (2uv)_x \\ L_3(u, v, w) = w_t + 2(v^2 + uw)_x, \quad L_4(u, v, w, z) = z_t + 2((3vw + uz)_x). \end{gathered}$$

$(u^\epsilon, v^\epsilon, w^\epsilon, z^\epsilon)$ is said to be weak asymptotic solution to problem (1.4) with initial data (1.5) if

$$\begin{gathered} \int L_1[u(x, t, \epsilon)]\psi(x)dx = o(1), \\ \int L_2[u(x, t, \epsilon), v(x, t, \epsilon)]\psi(x)dx = o(1), \\ \int L_3[u(x, t, \epsilon), v(x, t, \epsilon), w(x, t, \epsilon)]\psi(x)dx = o(1), \\ \int L_4[u(x, t, \epsilon), v(x, t, \epsilon), w(x, t, \epsilon), z(x, t, \epsilon)]\psi(x)dx = o(1), \end{gathered} \tag{2.1}$$

and initial conditions satisfy

$$\begin{aligned}
&\int \Big(u(x,0,\epsilon)-u_0(x)\Big)\psi(x)dx=o(1),\\
&\int \Big(v(x,0,\epsilon)-v_0(x)\Big)\psi(x)dx=o(1),\\
&\int \Big(w(x,0,\epsilon)-w_0(x)\Big)\psi(x)dx=o(1),\\
&\int \Big(z(x,0,\epsilon)-z_0(x)\Big)\psi(x)dx=o(1),
\end{aligned}\tag{2.2}$$

for all $\psi \in D(R)$.

To study weak asymptotic analysis first we need the following Lemma as in [11], regarding the superpositions of the singular waves $\delta, \delta', \delta''$ and δ'''.

Lemma 2.2. *Let $\{w_i\}_{i\in I}$ be an indexed set of Friedrich mollifiers satisfying*

$$w_i(x)=w_i(-x), \quad \int w_i=1.$$

Define

$$H_i(x,\epsilon)=w_{0i}(\frac{x}{\epsilon})=\int_{-\infty}^{\frac{x}{\epsilon}} w_i(y)dy, \quad \delta_i(x,\epsilon)=\frac{1}{\epsilon}w_i(\frac{x}{\epsilon}), \quad \delta_i^k(x,\epsilon)=\frac{1}{\epsilon^{k+1}}w_i^k(\frac{x}{\epsilon}).$$

The above assumptions implies the following asymptotic expansions, in the sense of distributions,

$$\begin{gathered}
(H_i(x,\epsilon))^r=H(x)+O_{D'}(\epsilon), \quad (H_i(x,\epsilon)(H_j(x,\epsilon))=H(x)+O_{D'}(\epsilon)\\
(H_i(x,\epsilon))^r\delta_j(x,\epsilon)=\delta(x)\int w_{0i}^r(y)w_j(y)dy+O_{D'}(\epsilon)\\
(\delta_i(x,\epsilon))^2=\frac{1}{\epsilon}\delta(x)\int w_i^2(y)dy+O_{D'}(\epsilon)\\
H_i(x,\epsilon)\delta_j'(x,\epsilon)=-\frac{1}{\epsilon}\delta(x)\int w_i(y)w_j(y)dy+\delta'(x)\int w_{0i}(y)w_j(y)dy+O_{D'}(\epsilon)\\
H_i(x,\epsilon)\epsilon^2\delta_j'''(x,\epsilon)=\frac{1}{\epsilon}\delta(x)\int w_i'(y)\delta_j'(y)dy+O_{D'}(\epsilon),\\
\delta_i(x,\epsilon).\delta_j(x,\epsilon)=\frac{1}{\epsilon}\delta(x)\int w_i(y)w_j(y)dy+O_{D'}(\epsilon)\\
\delta_i(x,\epsilon)\delta_j'(x,\epsilon)=\frac{1}{\epsilon}\delta'(x)\int yw_i(y)w_j'(y)dy+O_{D'}(\epsilon),\\
H_i(x,\epsilon)\delta_j''(x,\epsilon)=\frac{1}{\epsilon}\delta(x)\int w_{0i}(y)w_j(y)dy+\frac{1}{2}\delta''(x)\int y^2w_{0i}(y)w_j(y)dy+O_{D'}(\epsilon)\\
\delta_i(x,\epsilon))\epsilon^2\delta_j'''(x,\epsilon)=\frac{1}{\epsilon}\delta'(x)\int yw_i(y)w_j'''(y)dy+O_{D'}(\epsilon)\\
H_i(x,\epsilon)\epsilon^2\delta_j''''(x,\epsilon)=\frac{1}{\epsilon}\delta'(x)\int yw_{0i}(y)w_j''''(y)dy+O_{D'}(\epsilon)
\end{gathered}$$

where $\langle O_{D'}(\epsilon),\psi(x)\rangle \to 0$ for every test function ψ.

Proof. Let $\psi \in D(\mathbb{R})$ be any test function. The first six relations can be found in [11]; so wee prove from the seventh onward.

Now we prove seventh asymptotic expansion. Using change of variable formula ($x = \epsilon y$), employing third order Taylor expansion $\psi(\epsilon y) = \psi(0) + \epsilon y\psi'(0) + \frac{1}{2}\epsilon^2 y^2\psi''(0) + \epsilon^3 y^3 O(1)$, and the fact that $\int y w_i(y)w_j(y)dy = 0$, we have

$$\begin{aligned}\langle \delta_i(x,\epsilon)\delta_j(x,\epsilon), \psi(x)\rangle &= \int \frac{1}{\epsilon} w_i(\frac{x}{\epsilon})\frac{1}{\epsilon} w_j(\frac{x}{\epsilon})\psi(x)dx \\ &= \frac{1}{\epsilon}\int w_i(y)w_j(y)\psi(\epsilon y)dy \\ &= \frac{1}{\epsilon}\psi(0)\int w_i(y)w_j(y)dy + \psi'(0)\int y w_i(y)w_j(y)dy + O(\epsilon) \\ &= \frac{1}{\epsilon}\delta(x)\int w_i(y)w_j(y)dy + O(\epsilon).\end{aligned}$$

Now we prove eighth asymptotic expansion. Using change of variable formula ($x = \epsilon y$), employing third order Taylor expansion, $\psi(\epsilon y) = \psi(0) + \epsilon y\psi'(0) + \frac{1}{2}\epsilon^2 y^2\psi''(0) + \epsilon^3 y^3 O(1)$, and the fact that $\int y w_i(y)w_j(y)dy = 0$, we have

$$\begin{aligned}\langle \delta_i(x,\epsilon)\delta_j'(x,\epsilon), \psi(x)\rangle &= \frac{1}{\epsilon^2}\int w_i(y)w_j'(y)\psi(\epsilon y)dy \\ &= \frac{1}{\epsilon^2}\psi(0)\int w_i(y)w_j'(y)dy + \frac{1}{\epsilon}\psi'(0)\int y w_i(y)w_j'(y)dy \\ &\quad + \frac{1}{2}\psi''(0)\int y^2 w_i(y)w_j'(y)dy + O(\epsilon) \\ &= \frac{1}{\epsilon}\delta'(x)\int y w_i(y)w_j'(y)dy + O(\epsilon).\end{aligned}$$

In the above calculation we also used the identity

$$\int w_i(y)w_j'(y)dy = \int y^2 w_i(y)w_j'(y)dy = 0$$

.

Following an analysis similar as above, we prove the remaining identities. Details are as follows:

$$\begin{aligned}&\langle H_i(x,\epsilon)\delta_j''(x,\epsilon), \psi(x)\rangle \\ &= \int w_{0i}(y)\frac{1}{\epsilon^2} w_j''(y)\psi(\epsilon y)dy \\ &= \int w_{0i}(y)\frac{1}{\epsilon^2} w_j''(y)(\psi(0) + \epsilon y\psi'(0) + \frac{\epsilon^2 y^2}{2}\psi''(0))dy + O(\epsilon) \\ &= \frac{1}{\epsilon}\delta'(x)\int y w_{0i}(y)w_j''(y)dy + \frac{1}{2}\delta''(x)\int y^2 w_{0i}(y)w_j''(y)dy + O(\epsilon),\end{aligned}$$

$$\begin{aligned}&\langle \delta_i(x,\epsilon))\epsilon^2 w_j'''(x,\epsilon), \psi(x)\rangle \\ &= \frac{1}{\epsilon^2}\int w_i(y)w_j'''(y)\psi(\epsilon y)dy \\ &= \frac{1}{\epsilon^2}\int w_i(y)w_j'''(y)(\psi(0) + \epsilon y\psi'(0) + \frac{\epsilon^2 y^2}{2}\psi''(0))dy + O(\epsilon) \\ &= \frac{1}{\epsilon^2}\delta(x)\int w_i(y)w_j'''(y)dy + \frac{1}{\epsilon}\delta'(x)\int y w_i(y)w_j'''(y)dy\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\delta''(x)\int y^2 w_i(y)w_j'''(y)dy+O(\epsilon)\\
&=\frac{1}{\epsilon}\delta'(x)\int y w_i(y)w_j'''(y)dy+O(\epsilon)
\end{aligned}
$$

$$
\begin{aligned}
&\langle H_i(x,\epsilon)\epsilon^2\delta_j''''(x,\epsilon),\psi(x)\rangle\\
&=\int w_{0i}(y)\frac{1}{\epsilon^2}w_j''''(y)\psi(\epsilon y)(\psi(0)+\epsilon y\psi'(0)+\frac{\epsilon^2 y^2}{2}\psi''(0))dy+O(\epsilon)\\
&=\frac{1}{\epsilon}\delta'(x)\int y w_{0i}(y)w_j''''(y)dy+O(\epsilon)
\end{aligned}
$$

□

It is observed in [7], that the vanishing viscosity limit for the component z admits combinations of δ,δ',δ'' waves. So we choose ansatz as the combination of the above singular waves along the discontinuity curve. But this is not enough as it is clear in the construction of w, see [11]. In [11], a correction term is added in the component w to construct weak asymptotic solution. As the solution for the component is more complicated, extra care has to be taken to accomplish this.This is done by choosing the correction term carefully in the component z.

Theorem 2.3. *The following ansatz*

$$
\begin{aligned}
u(x,t,\epsilon)&=u_2(x,t)+[u]H_u(-x+\phi(t),\epsilon),\\
v(x,t,\epsilon)&=v_2(x,t)+[v]H_v(-x+\phi(t),\epsilon)+e(t)\delta_e(-x+\phi(t),\epsilon),\\
w(x,t,\epsilon)&=w_2(x,t)+[w]H_w(-x+\phi(t),\epsilon)+g(t)\delta_g(-x+\phi(t),\epsilon)\\
&\quad+h(t)\delta_h'(-x+\phi(t),\epsilon)+R_w(-x+\phi(t),\epsilon),\\
z(x,t,\epsilon)&=z_2(x,t)+[z]H_z(-x+\phi(t),\epsilon)+l(t)\delta_l(-x+\phi(t),\epsilon)\\
&\quad+m(t)\delta_m'(-x+\phi(t),\epsilon)+n(t)\delta_n''(-x+\phi(t),\epsilon)\\
&\quad+R_z(-x+\phi(t),\epsilon),
\end{aligned}
\tag{2.3}
$$

where

$$
\begin{gathered}
R_w(x,t,\epsilon)=\epsilon^2P(t)\delta_P'''(-x+\phi(t),\epsilon),\\
R_z(x,t,\epsilon)=\epsilon^2(Q(t)\delta_R'''(-x+\phi(t),\epsilon)+R(t)\delta_R''''(-x+\phi(t),\epsilon).
\end{gathered}
$$

is weak asymptotic solution to the problem (1.4) *if the following relations hold:*

$$
\begin{gathered}
L_1[u_1]=0,\quad L_1[u_2]=0,\\
L_2[u_1,v_1]=0,\quad L_2[u_2,v_2]=0,\\
L_3[u_1,v_1,w_1]=0,\quad L_3[u_2,v_2,w_2]=0,\\
\dot\phi(t)=(u_1+u_2)\big|_{x=\phi(t)},\quad \dot e(t)=[u](v_1+v_2)\big|_{x=\phi(t)}\\
\dot g(t)=(2[v](v_1+v_2)+[u](w_1+w_2)\big|_{x=\phi(t)},\quad \frac{d}{dt}(h(t)[u(\phi(t),t)])=\frac{d}{dt}e^2(t),\\
\int w_{0u}(y)w_j(y)dy=\int y^2 w_{0v}(y)w_e(y)dy=\frac{1}{2},\quad j=e,g,h,
\end{gathered}
$$

$$\int w_u(y)w_h(y)dy = \int w_e^2(y)dy, \quad P(t) = \frac{A}{u_1(\phi(t),t)}, \quad \text{where } A \text{ is a constant},$$
$$L_4[u_1,v_1,w_1,z_1] = 0, \quad L_4[u_2,v_2,w_2,z_2] = 0,$$
$$\dot{l}(t) = -[z]\dot{\phi}(t) + 2[3vw + uz],$$
$$\int w_{0u}(y)w_l(y)dy = \frac{1}{2}\int y^2 w_{0u}(y)w_n(y)dy = \int w_{0u}(y)w_m(y)dy = \frac{1}{2},$$
$$\begin{aligned}\dot{m}(t) &= 2[3\{(v_2+[v]\int w_{0v}(y)w_g(y)dy)g(t) + (w_2+[w]\int w_{0w}(y)w_e(y)dy)e(t)\}\\ &\quad + 3\{(v_{2x}+[v_x]\int w_{0v}(y)w_h(y)dy)h(t)\}\\ &\quad + (u_{2x}+\frac{[u_x]}{2})m(t) + (u_{1xx}+\frac{[u_{xx}]}{2})n(t)],\end{aligned}$$
$$\dot{n}(t) = 2[3\{(v_2+[v]\int w_{0v}(y)w_h(y)dy)h(t)\} - (2u_{2x}+[u_x])n(t)],$$
$$\begin{aligned}R(t) &= \frac{1}{[u]\int w_{0u}(y)w_R''''(y)dy}\Big[3e(t)h(t)\int yw_e(y)w_h'(y)dy\\ &\quad + 3e(t)p(t)\int yw_e(y)w_P'''(y)dy\Big]\end{aligned}$$
$$\begin{aligned}Q(t) &= \frac{1}{[u]\int w_u'(y)w_Q'(y)dy}\Big[3e(t)g(t)\int w_e(y)w_g(y)dy\\ &\quad - 3[v]h(t)\int w_v(y)w_h(y)dy - [u]m(t)\int w_u(y)w_m(y)dy\\ &\quad + \frac{[u]n(t)}{2} + [u_x]R(t)\int w_{0u}(y)w_R''''(y)dy\Big]\end{aligned}$$

Proof. If the first thirteen relations above hold, then the expression for u, v and w in (2.3) is a weak asymptotic solution, is shown in [11]. So, we only prove that the expression for the component z in equation (2.3) is a weak asymptotic solution.

Multiplying the ansatz given for v and w in the equation (2.3) and using lemma 2.2, we obtain

$$\begin{aligned}&v(x,t,\epsilon)w(x,t,\epsilon)\\ &= v_2w_2 + [vw]H(-x+\phi(t)) + \Big\{(v_2+[v]\int w_{0v}(y)w_g(y)dy)g(t)\\ &\quad + (w_2+[w]\int w_{0w}(y)w_e(y)dy)e(t)\Big\}\delta(-x+\phi(t))\\ &\quad + (v_2+[v]\int w_{0v}(y)w_h(y)dy)h(t)\delta'(-x+\phi(t))\\ &\quad + (e(t)g(t)\int w_e(y)w_g(y)dy - [v]h(t)\int w_v(y)w_h(y)dy)\frac{1}{\epsilon}\delta(-x+\phi(t))\\ &\quad + (e(t)h(t)\int yw_e(y)w_h'(y)dy + e(t)p(t)\int yw_e(y)w_P'''(y)dy)\frac{1}{\epsilon}\delta'(-x+\phi(t))\\ &\quad + O_{D'}(\epsilon).\end{aligned}$$

Similarly,

$$u(x,t,\epsilon)z(x,t,\epsilon)$$

$$
\begin{aligned}
&= u_2z_2 + [uz]H(-x+\phi(t)) \\
&\quad + [u_2 + [u]\int w_{0u}(y)w_l(y)dy]l(t)\delta(-x+\phi(t)) \\
&\quad + [u_2 + [u]\int w_{0u}(y)w_m(y)dy]m(t)\delta'(-x+\phi(t)) \\
&\quad + [u_2 + \frac{[u]}{2}\int y^2 w_{0u}(y)w_n(y)dy]n(t)\delta''(-x+\phi(t)) \\
&\quad + [-[u]m(t)\int w_u(y)w_m(y)dy + [u]n(t)\int w_{0u}(y)w_n(y)dy \\
&\quad + [u]Q(t)\int w'_u(y)w'_Q(y)dy]\frac{1}{\epsilon}\delta(-x+\phi(t)) \\
&\quad + [u]R(t)\int w_{0u}(y)w''''_R(y)dy\frac{1}{\epsilon}\delta'(-x+\phi(t)) + O_{D'}(\epsilon).
\end{aligned}
$$

Arranging the coefficient of δ and the derivatives, $\frac{1}{\epsilon}\delta$ and $\frac{1}{\epsilon}\delta'$ of $3v(x,t,\epsilon)w(x,t,\epsilon)+u(x,t,\epsilon)z(x,t,\epsilon)$, we obtain

$$
\begin{aligned}
&3v(x,t,\epsilon)w(x,t,\epsilon) + u(x,t,\epsilon)z(x,t,\epsilon) \\
&= (3v_2w_2 + u_2z_2) + [3vw + uz]H(-x+\phi(t)) \\
&\quad + [3\{(v_2 + [v]\int w_{0v}(y)w_g(y)dy)g(t) + (w_2 + [w]\int w_{0w}(y)w_e(y)dy)e(t)\} \\
&\quad + (u_2 + [u]\int w_{0u}(y)w_l(y)dy)l(t) + 3\{(v_{2x} + [v_x]\int w_{0v}(y)w_h(y)dy)h(t)\} \\
&\quad + (u_{2x} + [u_x]\int w_{0u}(y)w_m(y)dy)m(t) \\
&\quad + (u_{2xx} + \frac{[u_{xx}]}{2}\int y^2 w_{0u}(y)w_n(y)dy)n(t)]\big|_{x=\phi(t)}\delta(-x+\phi(t)) \\
&\quad + [3\{(v_2 + [v]\int w_{0v}(y)w_h(y)dy)h(t)\} + (u_2 + [u]\int w_{0u}(y)w_m(y)dy)m(t) \\
&\quad - 2(u_{2x} + \frac{[u_x]}{2}\int y^2 w_{0u}(y)w_n(y)dy)n(t)]\big|_{x=\phi(t)}\delta'(-x+\phi(t)) \\
&\quad + [u_2 + \frac{[u]}{2}\int y^2 w_{0u}(y)w_n(y)dy]n(t)\big|_{x=\phi(t)}\delta''(-x+\phi(t)) \\
&\quad + \Big[3e(t)g(t)\int w_e(y)w_g(y)dy - 3[v]h(t)\int w_v(y)w_h(y)dy \\
&\quad - [u]m(t)\int w_u(y)w_m(y)dy + [u]n(t)\int w_{0u}(y)w_n(y)dy \\
&\quad + [u]Q(t)\int w'_u(y)w'_Q(y)dy \\
&\quad + [u_x]R(t)\int w_{0u}(y)w''''_R(y)dy\Big]\big|_{x=\phi(t)}\frac{1}{\epsilon}\delta(-x+\phi(t)) \\
&\quad + \Big[3e(t)h(t)\int yw_e(y)w'_h(y)dy + 3e(t)p(t)\int yw_e(y)w'''_P(y)dy \\
&\quad + [u]R(t)\int w_{0u}(y)w''''_R(y)dy\Big]\big|_{x=\phi(t)}\frac{1}{\epsilon}\delta'(-x+\phi(t)) + O_{D'}(\epsilon).
\end{aligned}
$$

$$
\begin{aligned}
&(3v(x,t,\epsilon)w(x,t,\epsilon)+u(x,t,\epsilon)z(x,t,\epsilon))_x \\
&=(3v_2w_2+u_2z_2)_x+[(3vw+uz)_x]H(-x+\phi(t))-[3vw+uz]\delta(-x+\phi(t)) \\
&\quad -[3\{(v_2+[v]\int w_{0v}(y)w_g(y)dy)g(t)+(w_2+[w]\int w_{0w}(y)w_e(y)dy)e(t)\} \\
&\quad +(u_2+[u]\int w_{0u}(y)w_l(y)dy)l(t)+3\{(v_{2x}+[v_x]\int w_{0v}(y)w_h(y)dy)h(t)\} \\
&\quad +(u_{2x}+[u_x]\int w_{0u}(y)w_m(y)dy)m(t) \\
&\quad +(u_{2xx}+\frac{[u_{xx}]}{2}\int y^2w_{0u}(y)w_n(y)dy)n(t)]\big|_{x=\phi(t)}\delta'(-x+\phi(t)) \\
&\quad -[3\{(v_2+[v]\int w_{0v}(y)w_h(y)dy)h(t)\}+(u_2+[u]\int w_{0u}(y)w_m(y)dy)m(t) \\
&\quad -2(u_{2x}+\frac{[u_x]}{2}\int y^2w_{0u}(y)w_n(y)dy)n(t)]\big|_{x=\phi(t)}\delta''(-x+\phi(t)) \\
&\quad -[u_2+\frac{[u]}{2}\int y^2w_{0u}(y)w_n(y)dy]n(t)\big|_{x=\phi(t)}\delta'''(-x+\phi(t)) \\
&\quad -\Big[3e(t)g(t)\int w_e(y)w_g(y)dy-3[v]h(t)\int w_v(y)w_h(y)dy \\
&\quad -[u]m(t)\int w_u(y)w_m(y)dy \\
&\quad +[u]n(t)\int w_{0u}(y)w_n(y)dy+[u]Q(t)\int w_u'(y)w_Q'(y)dy \\
&\quad +[u_x]R(t)\int w_{0u}(y)w_R''''(y)dy\Big]\big|_{x=\phi(t)}\frac{1}{\epsilon}\delta'(-x+\phi(t)) \\
&\quad -\Big[3e(t)h(t)\int yw_e(y)w_h'(y)dy+3e(t)p(t)\int yw_e(y)w_P'''(y)dy \\
&\quad +[u]R(t)\int w_{0u}(y)w_R''''(y)dy\Big]\big|_{x=\phi(t)}\frac{1}{\epsilon}\delta''(-x+\phi(t))+O_{D'}(\epsilon).
\end{aligned} \tag{2.4}
$$

Differentiating z with respect to t,

$$
\begin{aligned}
&z_t(x,t,\epsilon) \\
&=z_{1t}+[z_t]H(-x+\phi(t))+\Big[[z]\dot\phi(t)+\dot l(t)\Big]\delta(-x+\phi(t)) \\
&\quad +\big[l(t)\dot\phi(t)+\dot m(t)\big]\delta'(-x+\phi(t))+\big[m(t)\dot\phi(t)+\dot n(t)\big]\delta''(-x+\phi(t)) \\
&\quad +n(t)\dot\phi(t)\delta'''(-x+\phi(t))+O_{D'}(\epsilon).
\end{aligned} \tag{2.5}
$$

Putting the value of $z_t(x,t,\epsilon)$ from the equations (2.5) and $(3v(x,t,\epsilon)w(x,t,\epsilon)+u(x,t,\epsilon)z(x,t,\epsilon))_x$ from the equations (2.4) in the fourth equation of (1.4), we obtain

$$
\begin{aligned}
&z_t+2((3vw+uz)_x) \\
&=z_{1t}+2(3v_2w_2+u_2z_2)_x+\Big[[z_t]+2[(3vw+uz)_x]\Big]H(-x+\phi(t)) \\
&\quad +\Big[[z]\dot\phi(t)+\dot l(t)-2[3vw+uz]\Big]\delta(-x+\phi(t))
\end{aligned}
$$

$$
\begin{aligned}
&+ \Big[l(t)\dot\phi(t) + \dot m(t) - 2[3\{(v_2 + [v]\int w_{0v}(y)w_g(y)dy)g(t)\\
&+ (w_2 + [w]\int w_{0w}(y)w_e(y)dy)e(t)\}\\
&+ (u_2 + [u]\int w_{0u}(y)w_l(y)dy)l(t) + 3\{(v_{2x} + [v_x]\int w_{0v}(y)w_h(y)dy)h(t)\}\\
&+ (u_{2x} + [u_x]\int w_{0u}(y)w_m(y)dy)m(t)\\
&+ (u_{2xx} + \frac{[u_{xx}]}{2}\int y^2 w_{0u}(y)w_n(y)dy)n(t)]\Big]\delta'(-x+\phi(t))\\
&+ \Big[m(t)\dot\phi(t) + \dot n(t) - 2[3\{(v_2 + [v]\int w_{0v}(y)w_h(y)dy)h(t)\}\\
&+ (u_2 + [u]\int w_{0u}(y)w_m(y)dy)m(t)\\
&- (2u_{2x} + [u_x]\int y^2 w_{0u}(y)w_n(y)dy)n(t)]\Big]\delta''(-x+\phi(t))\\
&+ \Big[n(t)\dot\phi(t) - [2u_2 + [u]\int y^2 w_{0u}(y)w_n(y)dy]n(t)\Big]\delta'''(-x+\phi(t))\\
&- 2\Big[3e(t)g(t)\int w_e(y)w_g(y)dy - 3[v]h(t)\int w_v(y)w_h(y)dy\\
&- [u]m(t)\int w_u(y)w_m(y)dy\\
&+ [u]n(t)\int w_{0u}(y)w_n(y)dy + [u]Q(t)\int w_u'(y)w_Q'(y)dy\\
&+ [u_x]R(t)\int w_{0u}(y)w_R''''(y)dy\Big]\big|_{x=\phi(t)}\frac{1}{\epsilon}\delta'(-x+\phi(t))\\
&- 2\Big[3e(t)h(t)\int y w_e(y)w_h'(y)dy + 3e(t)p(t)\int y w_e(y)w_P'''(y)dy\\
&+ [u]R(t)\int w_{0u}(y)w_R''''(y)dy\Big]\big|_{x=\phi(t)}\frac{1}{\epsilon}\delta''(-x+\phi(t)) + O_{D'}(\epsilon).
\end{aligned}
$$

So if the relations 14-21 holds then the coefficients of δ and their derivatives, $\frac{1}{\epsilon}\delta$ and $\frac{1}{\epsilon}\delta'$ vanishes. The proof is complete. ∎

For Riemann type data the above expression is simple, and it is described in the following corollary.

Corollary 2.4. *If u_i, v_i, w_i z_i for $i = 1, 2$ are constants then expression* (2.3) *is a weak asymptotic solution provided the following equalities hold.*

$$
\begin{gathered}
\dot\phi(t) = (u_1 + u_2)\big|_{x=\phi(t)}, \quad \dot e(t) = [u](v_1 + v_2)\big|_{x=\phi(t)},\\
\dot g(t) = (2[v](v_1 + v_2) + [u](w_1 + w_2)\big|_{x=\phi(t)}, quad\frac{d}{dt}(h(t)[u(\phi(t),t)]) = \frac{d}{dt}e^2(t)\\
\int w_{0u}(y)w_j(y)dy = \int y^2 w_{0v}(y)w_e(y)dy = \frac{1}{2}, \quad j = e, g, h,\\
\int w_u(y)w_h(y)dy = \int w_e^2(y)dy, \quad P(t) = \frac{A}{u_1(\phi(t),t)}, \quad \textit{where } A \textit{ is a constant},
\end{gathered}
$$

$$
\begin{gathered}
\dot{l}(t) = -[z]\dot{\phi}(t) + 2[3vw + uz], \\
\int w_{0u}(y) w_l(y) dy = \frac{1}{2}\int y^2 w_{0u}(y) w_n(y) dy = \int w_{0u}(y) w_m(y) dy = \frac{1}{2}, \\
\dot{m}(t) = 2[3\{(v_2 + [v]\int w_{0v}(y) w_g(y) dy) g(t) + (w_2 + [w]\int w_{0w}(y) w_e(y) dy) e(t)\}, \\
\dot{n}(t) = 2[3\{(v_2 + [v]\int w_{0v}(y) w_h(y) dy) h(t)\}, \\
\begin{aligned}
R(t) = \frac{1}{[u]\int w_{0u}(y) w_R''''(y) dy}&[3e(t)h(t)\int y w_e(y) w_h'(y) dy \\
&+ 3e(t)p(t)\int y w_e(y) w_P'''(y) dy],
\end{aligned}
\end{gathered}
$$

$$
\begin{aligned}
Q(t) = \frac{1}{[u]\int w_u'(y) w_Q'(y) dy}&\Big[3e(t)g(t)\int w_e(y) w_g(y) dy \\
&- 3[v]h(t)\int w_v(y) w_h(y) dy - [u]m(t)\int w_u(y) w_m(y) dy + \frac{[u]n(t)}{2}\Big]
\end{aligned} \tag{2.6}
$$

Piecing together the Riemann problems we construct a weak asymptotic solution for general type initial data under the assumption that u is a monotonic increasing function.

Theorem 2.5. *If u_0, v_0, w_0 and z_0 are locally integrable functions on $\mathbb{R}$, and u_0 is monotonic increasing, then there exists weak asymptotic solution (u, v, w, z) to the system* (1.4) *with initial data* (1.5).

Proof. Let ϕ be a test function on $\mathbb{R}$ having support in $[-K, K]$. Given $\epsilon > 0$, there exist piecewise constant functions $(u_{0\epsilon}, v_{0\epsilon}, w_{0\epsilon}, z_{0\epsilon})$ such that

$$
\begin{gathered}
\int_{[-K,K]} |u_0(x) - u_{0\epsilon}(x)| dx < \epsilon, \quad \int_{[-K,K]} |v_0(x) - v_{0\epsilon}(x)| dx < \epsilon, \\
\int_{[-K,K]} |w_0(x) - w_{0\epsilon}(x)| dx < \epsilon, \quad \int_{[-K,K]} |z_0(x) - z_{0\epsilon}(x)| dx < \epsilon.
\end{gathered}
$$

In addition to this we can take $u_{0\epsilon}$ monotonic increasing and all functions have same points of discontinuities. $(u_{0\epsilon}, v_{0\epsilon}, w_{0\epsilon}, z_{0\epsilon})$ in $[-K, K]$ can be represented as

$$
\begin{aligned}
u_{0\epsilon} &= \sum_{i=1}^{n} u_{0i}(H(x - a_{i-1}) - H(x - a_i)), \\
v_{0\epsilon} &= \sum_{i=1}^{n} v_{0i}(H(x - a_{i-1}) - H(x - a_i)), \\
w_{0\epsilon} &= \sum_{i=1}^{n} w_{0i}(H(x - a_{i-1}) - H(x - a_i)), \\
z_{0\epsilon} &= \sum_{i=1}^{n} z_{0i}(H(x - a_{i-1}) - H(x - a_i)).
\end{aligned}
$$

Since $u_{0\epsilon}$ is a monotonic increasing function, discontinuity curve arising in the solution of (u, v, w, z) do not intersect for any time. So the following functions are weak asymptotic solutions

$$\begin{aligned}
u(x,t,\eta) &= u_{01}H_u(-x+c_1t+a_1,\eta) + \sum_{i=2}^{n-1} u_{0i}\Big(H_u(x-c_{i-1}t-a_{i-1},\eta) \\
&\quad - H_u(x-c_it-a_i,\eta)\Big) + u_{0n}(H_u(x-c_{n-1}t-a_{n-1},\eta), \\
v(x,t,\eta) &= v_{01}H_v(-x+c_1t+a_1,\eta) + \sum_{i=2}^{n-1} v_{0i}\Big(H_v(x-c_{i-1}t-a_{i-1},\eta) \\
&\quad - H_v(x-c_it-a_i,\eta)\Big) + v_{0n}H_v(x-c_{n-1}t-a_{n-1},\eta) \\
&\quad + \sum_{i=1}^{n-1} e_i(t)\delta_e(-x+c_it,\eta), \\
w(x,t,\eta) &= w_{01}H_w(-x+c_1t+a_1,\eta) + \sum_{i=2}^{n-1} w_{0i}\Big(H_w(x-c_{i-1}t-a_{i-1},\eta) \\
&\quad - H_w(x-c_it-a_i,\eta)\Big) + w_{0n}H_w(x-c_{n-1}t-a_{n-1},\eta) \\
&\quad + \sum_{i=1}^{n-1} g_i(t)\delta_g(-x+c_it,\eta) + \sum_{i=1}^{n-1} h_i(t)\delta_h'(-x+c_it,\eta) \\
&\quad + \sum_{i=1}^{n-1} R_{wi}(-x+c_it,\eta), \\
z(x,t,\eta) &= z_{01}H_z(-x+c_1t+a_1,\eta) + \sum_{i=2}^{n-1} z_{0i}\Big(H_z(x-c_{i-1}t-a_{i-1},\eta) \\
&\quad - H_z(x-c_it-a_i,\eta)\Big) + z_{0n}(H_z(x-c_{n-1}t-a_{n-1},\eta)) \\
&\quad + \sum_{i=1}^{n-1} l_i(t)\delta_l(-x+c_it,\eta) + \sum_{i=1}^{n-1} m_i(t)\delta_m'(-x+c_it,\eta) \\
&\quad + \sum_{i=1}^{n-1} n_i(t)\delta_n''(-x+c_it,\eta) + \sum_{i=1}^{n-1} R_{zi}(-x+c_it,\eta),
\end{aligned}$$

where e_i, g_i, h_i, l_i, m_i, n_i, R_{wi} and R_{zi} satisfy (2.6) with u_1, u_2, v_1, v_2, w_1, w_2, z_1, z_2, e, g, h, l, m, n, R_w and R_z replaced by u_{i-1}, u_i, v_{i-1}, v_i, w_{i-1}, w_i, z_{i-1}, z_i, e_i, g_i, h_i, l_i, m_i, n_i, R_{wi} and R_{zi}. Given $\epsilon > 0$ choose $\eta(\epsilon)$ small enough such that the following estimates hold.

$$\Big|\int L_1[u(x,t,\eta(\epsilon))]\psi(x)dx\Big| < \eta(\epsilon), \quad \Big|\int L_2[u(x,t,\eta(\epsilon)),v(x,t,\eta(\epsilon))]\psi(x)\Big| < \epsilon,$$

$$\Big|\int L_3[u(x,t,\eta(\epsilon)),v(x,t,\eta(\epsilon)),w(x,t,\eta(\epsilon))]\psi(x)dx\Big| < \epsilon,$$

$$\Big|\int L_4[u(x,t,\eta(\epsilon)),v(x,t,\eta(\epsilon)),w(x,t,\eta(\epsilon)),z(x,t,\eta(\epsilon))]\psi(x)dx\Big| < \epsilon,$$

$$\Big|\int \Big(u(x,0,\eta(\epsilon)) - u_0(x)\Big)\psi(x)dx\Big| < 2\epsilon,$$

$$
\begin{aligned}
&|\int \Big(v(x,0,\eta(\epsilon)) - v_0(x)\Big)\psi(x)dx\Big| < 2\epsilon,\\
&|\int \Big(w(x,0,\eta(\epsilon)) - w_0(x)\Big)\psi(x)dx\Big| < 2\epsilon,\\
&|\int \Big(z(x,0,\eta(\epsilon)) - z_0(x)\Big)\psi(x)dx\Big| < 2\epsilon.
\end{aligned}
$$

Define

$$
\begin{aligned}
&(\bar{u}(x,t,\epsilon), \bar{v}(x,t,\epsilon), \bar{w}(x,t,\epsilon), \bar{z}(x,t,\epsilon))\\
&= (u(x,t,\eta(\epsilon), v(x,t,\eta(\epsilon), w(x,t,\eta(\epsilon), z(x,t,\eta(\epsilon)).
\end{aligned}
$$

Then $(\bar{u}, \bar{v}, \bar{w}, \bar{z})$ is a weak asymptotic solution of system (1.4)-(1.5). □

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