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EXISTENCE AND CONCENTRATION OF SOLUTIONS FOR SUBLINEAR FOURTH-ORDER ELLIPTIC EQUATIONS

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Abstract. This article concerns the fourth-order elliptic equations

$$\Delta^2 u - \Delta u + \lambda V(x)u = f(x,u), \quad x \in \mathbb{R}^N,$$
$$u \in H^2(\mathbb{R}^N),$$

where $\lambda > 0$ is a parameter, $V \in C(\mathbb{R}^N)$ and $V^{-1}(0)$ has nonempty interior. Under some mild assumptions, we establish the existence of nontrivial solutions. Moreover, the concentration of solutions is also explored on the set $V^{-1}(0)$ as $\lambda \to \infty$.

1. Introduction and statement of main results

This article concerns the fourth-order elliptic equation

$$\begin{gathered} \Delta^2 u - \Delta u + \lambda V(x)u = f(x,u), \quad x \in \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \end{gathered} \tag{1.1}$$

where $\Delta^2 := \Delta(\Delta)$ is the biharmonic operator and $\lambda > 0$ is a parameter.

Problem (1.1) arises in the study of travelling waves in suspension bridge and the study of the static deflection of an elastic plate in a fluid, see [7, 8, 12]. There are many results for fourth-order elliptic equations, but most of them are focused on bounded domains, see [1, 2, 3, 6, 13, 21, 22, 24, 29, 33, 34] and the references therein. Recently, the case of the whole space $\mathbb{R}^N$ was also considered in some works, see [9, 18, 19, 25, 26, 27, 28, 30, 31, 32]. For the whole space $\mathbb{R}^N$ case, the main difficulty of this problem is the lack of compactness for Sobolev embedding theorem. To overcome this difficulty, some authors assumed that the potential V satisfies certain coercive condition, see [20, 26, 28, 30]. Later, the authors in [9, 27] considered the potential well case with a parameter. With the aid of parameter, they proved that the energy functional possess the property of locally compact. Moreover, the authors of these literatures proved the existence of infinitely many high energy solutions for superlinear case. For somewhat related sublinear case and the existence of infinitely many small negative-energy solutions, see [26, 31]. For singularly perturbed problem with superlinear nonlinearities and concentration

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phenomenon of semi-classical solutions, we refer readers to [10, 14, 15] and the references therein.

Motivated by the above articles, we continue to consider problem (1.1) with steep well potential and study the existence of nontrivial solution and concentration results (as $\lambda \to \infty$) under some mild assumptions different from those studied previously. To reduce our statements, we make the following assumptions for potential V:

- (V1) $V(x) \in C(\mathbb{R}^N)$ and $V(x) \geq 0$ on $\mathbb{R}^N$;
- (V2) There exists a constant $b > 0$ such that the set $V_b := \{x \in \mathbb{R}^N | V(x) < b\}$ is nonempty and has finite measure;
- (V3) $\Omega = int V^{-1}(0)$ is nonempty and has smooth boundary with $\bar{\Omega} = V^{-1}(0)$.

This kind of hypotheses was first introduced by Bartsch and Wang [4] (see also [5]) in the study of a nonlinear Schrödinger equation and the potential $\lambda V(x)$ with V satisfying (V1)–(V3) is referred as the steep well potential. It is worth mentioning that the above papers always assumed the potential V is positive ($V > 0$). Compared with the case $V > 0$, our assumptions on V are rather weak, and perhaps more important. Generally speaking, there may exist some behaviours and phenomenons for the solutions of problem (1.1) under condition (V3), such as the concentration phenomenon of solutions. We are also interested in the case that the nonlinearity $f(x, u)$ is sublinear and indefinite. To our knowledge, few works concerning on this case up to now. Based on the above facts, the main purpose of this paper is to prove the existence of nontrivial solutions and to investigate the concentration phenomenon of solutions on the set $V^{-1}(0)$ as $\lambda \to \infty$. To state our results, we need the following assumptions:

- (F1) $f \in C(\mathbb{R}^N, \mathbb{R})$ and there exist constants $1 < \gamma_1 < \gamma_2 < \cdots < \gamma_m < 2$ and functions $\xi_i(x) \in L^{\frac{2}{2-\gamma_i}}(\mathbb{R}^N, \mathbb{R}^+)$ such that
$$|f(x,u)| \leq \sum_{i=1}^{m} \gamma_i \xi_i(x) |u|^{\gamma_i - 1}, \quad \forall (x,u) \in \mathbb{R}^N \times \mathbb{R}.$$
- (F2) There exist three constants $\eta, \delta > 0, \gamma_0 \in (1,2)$ such that
$$|F(x,u)| \geq \eta |u|^{\gamma_0} \quad \text{for all } x \in \Omega \text{ and all } |u| \leq \delta,$$
where $F(x,u) = \int_0^u f(x,s) ds$.

On the existence of solutions we have the following result.

Theorem 1.1. *Assume that the conditions* (V1)–(V3), (F1), (F2) *hold. Then there exists* $\Lambda_0 > 0$ *such that for every* $\lambda > \Lambda_0$*, problem* (1.1) *has at least a solution* u_λ.

On the concentration of solutions we have the following result.

Theorem 1.2. *Let* u_λ *be a solution of problem* (1.1) *obtained in Theorem 1.1, then* $u_\lambda \to u_0$ *in* $H^2(\mathbb{R}^N)$ *as* $\lambda \to \infty$*, where* $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ *is a nontrivial solution of the equation*
$$\begin{gathered} \Delta^2 u - \Delta u = f(x,u), \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{1.2}$$

The rest of this article is organized as follows. In Section 2, we establish the variational framework associated with problem (1.1), and we also give the proof of Theorem 1.1. In Section 3, we study the concentration of solutions and prove Theorem 1.2.

2. Variational setting and proof of Theorem 1.1

By $\|\cdot\|_q$ we denote the usual L^q-norm for $1 \le q \le \infty$, c_i, C, C_i stand for different positive constants. Let

$$X = \big\{u \in H^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} \big(|\Delta u|^2 + |\nabla u|^2 + V(x)u^2\big)\, dx < +\infty\big\},$$

be equipped with the inner product

$$(u,v) = \int_{\mathbb{R}^N} \big(\Delta u \Delta v + \nabla u \cdot \nabla v + V(x)uv\big)\, dx, \quad u,v \in X,$$

and the norm

$$\|u\| = \Big(\int_{\mathbb{R}^N} (|\Delta u|^2 + |\nabla u|^2 + V(x)u^2)\, dx\Big)^{1/2}, \quad u \in X.$$

For $\lambda > 0$, we also need the following inner product

$$(u,v)_\lambda = \int_{\mathbb{R}^N} (\Delta u \Delta v + \nabla u \cdot \nabla v + \lambda V(x)uv)\, dx, \quad u,v \in X,$$

and the corresponding norm $\|u\|_\lambda^2 = (u,u)_\lambda$. It is clear that $\|u\| \le \|u\|_\lambda$, for $\lambda \ge 1$.

Set $E_\lambda = (X, \|u\|_\lambda)$, then E_λ is a Hilbert space. By using (V1)-(V2) and the Sobolev inequality, we can demonstrate that there exist positive constants λ_0, c_0 (independent of λ) such that

$$\|u\|_{H^2(\mathbb{R}^N)} \le c_0 \|u\|_\lambda, \quad \text{for all } u \in E_\lambda,\ \lambda \ge \lambda_0.$$

In fact, by using conditions (V1)-(V2) and the Sobolev inequality, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} (|\Delta u|^2 + |\nabla u|^2 + u^2)\, dx \\
&= \int_{\mathbb{R}^N} (|\Delta u|^2 + |\nabla u|^2)\, dx + \int_{V_b} u^2\, dx + \int_{\mathbb{R}^N \setminus V_b} u^2\, dx \\
&\le \int_{\mathbb{R}^N} (|\Delta u|^2 + |\nabla u|^2)\, dx + (\operatorname{meas}(V_b))^{\frac{2^*-2}{2^*}} \Big(\int_{\mathbb{R}^N} |u|^{2^*}\, dx\Big)^{2/2^*} + \int_{\mathbb{R}^N \setminus V_b} u^2\, dx \\
&\le \int_{\mathbb{R}^N} (|\Delta u|^2 + |\nabla u|^2)\, dx + (\operatorname{meas}(V_b))^{\frac{2^*-2}{2^*}} \Big(\int_{\mathbb{R}^N} |u|^{2^*}\, dx\Big)^{2/2^*} \\
&\quad + \frac{1}{\lambda b} \int_{\mathbb{R}^N \setminus V_b} \lambda V u^2\, dx \\
&\le \int_{\mathbb{R}^N} (|\Delta u|^2 + |\nabla u|^2)\, dx + S^{-1} (\operatorname{meas}(V_b))^{\frac{2^*-2}{2^*}} \int_{\mathbb{R}^N} |\nabla u|^2\, dx + \frac{1}{\lambda b} \int_{\mathbb{R}^N} \lambda V u^2\, dx \\
&\le \max\Big\{1, 1 + S^{-1} (\operatorname{meas}(V_b))^{\frac{2^*-2}{2^*}}, \frac{1}{\lambda b}\Big\} \int_{\mathbb{R}^N} (|\Delta u|^2 + |\nabla u|^2 + \lambda V u^2)\, dx \\
&:= c_0 \int_{\mathbb{R}^N} (|\Delta u|^2 + |\nabla u|^2 + \lambda V u^2)\, dx, \\
&\quad \text{for } \lambda \ge \lambda_0 := \frac{1}{b(1 + S^{-1} (\operatorname{meas}(V_b))^{\frac{2^*-2}{2^*}})}.
\end{aligned}$$

Here we use the fact that $H^2(\mathbb{R}^N) \subset H^1(\mathbb{R}^N)$. Furthermore, the embedding $E_\lambda \hookrightarrow L^p(\mathbb{R}^N)$ is continuous for $p \in [2, 2_*]$, and $E_\lambda \hookrightarrow L^p_{\rm loc}(\mathbb{R}^N)$ is compact for $p \in [2, 2_*)$, i.e., there are constants $c_p > 0$ such that

$$\|u\|_p \le c_p\|u\|_{H^2(\mathbb{R}^N)} \le c_p c_0 \|u\|_\lambda, \quad \text{for all } u \in E_\lambda,\ \lambda \ge \lambda_0,\ 2 \le p \le 2_*, \tag{2.1}$$

where $2_* = +\infty$ if $N \le 4$, and $2_* = \frac{2N}{N-4}$ if $N > 4$.

Let

$$\Phi_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\Delta u|^2 + |\nabla u|^2 + \lambda V(x)u^2\right) dx - \int_{\mathbb{R}^N} F(x,u)\,dx. \tag{2.2}$$

By a standard argument, it is easy to verify that $\Phi_\lambda \in C^1(E_\lambda, \mathbb{R})$ and

$$\langle \Phi_\lambda'(u), v\rangle = \int_{\mathbb{R}^N} [\Delta u \Delta v + \nabla u \cdot \nabla v + \lambda V(x)uv]\,dx - \int_{\mathbb{R}^N} f(x,u)v\,dx, \tag{2.3}$$

for all $u, v \in E_\lambda$. Then we can infer that $u \in E_\lambda$ is a critical point of Φ_λ if and only if it is a weak solution of problem (1.1). Next, we give a useful lemma.

Lemma 2.1 ([16]). *Let E be a real Banach space and $\Phi \in C^1(E, \mathbb{R})$ satisfy the (PS)-condition. If Φ is bounded from below, then $c = \inf_E \Phi$ is a critical value of Φ.*

Lemma 2.2. *Suppose that* (V1)-(V3), (F1), (F2) *are satisfied. There exists $\Lambda_0 > 0$ such that for every $\lambda > \Lambda_0$, Φ_λ is bounded from below in E.*

Proof. From (2.1), (2.2), (F1) and the Hölder inequality, we have

$$\begin{aligned}
\Phi_\lambda(u) &= \frac{1}{2}\|u\|_\lambda^2 - \int_{\mathbb{R}^N} F(x,u)\,dx \\
&\ge \frac{1}{2}\|u\|_\lambda^2 - \sum_{i=1}^m \Big(\int_{\mathbb{R}^N} |\xi_i(x)|^{\frac{2}{2-\gamma_i}}\,dx\Big)^{(2-\gamma_i)/2} \Big(\int_{\mathbb{R}^N} |u|^2\,dx\Big)^{\gamma_i/2} \\
&\ge \frac{1}{2}\|u\|_\lambda^2 - \sum_{i=1}^m c_2^{\gamma_i} c_0^{\gamma_i} \|\xi_i\|_{\frac{2}{2-\gamma_i}} \|u\|_\lambda^{\gamma_i},
\end{aligned} \tag{2.4}$$

which implies that $\Phi_\lambda(u) \to +\infty$ as $\|u\|_\lambda \to +\infty$, since $1 < \gamma_1 < \gamma_2 < \dots < \gamma_m < 2$. Consequently, there exists $\Lambda_0 := \max\{1, \lambda_0\} > 0$ such that for every $\lambda > \Lambda_0$, Φ_λ is bounded from below. ☐

Lemma 2.3. *Suppose that* (V1)–(V3), (F1), (F2) *are satisfied. Then Φ_λ satisfies the (PS)-condition for each $\lambda > \Lambda_0$.*

Proof. Assume that $\{u_n\} \subset E_\lambda$ is a sequence such that $\Phi_\lambda(u_n)$ is bounded and $\Phi_\lambda'(u_n) \to 0$ as $n \to \infty$. By Lemma 2.2, it is clear that $\{u_n\}$ is bounded in E_λ. Thus, there exists a constant $C > 0$ such that for all $n \in \mathbb{N}$

$$\|u_n\|_p \le c_p c_0 \|u_n\|_\lambda \le C, \quad \text{for all } u \in E_\lambda,\ \lambda \ge \lambda_0,\ 2 \le p \le 2_*. \tag{2.5}$$

Passing to a subsequence if necessary, we may assume that $u_n \rightharpoonup u_0$ in E_λ. For any $\epsilon > 0$, since $\xi_i(x) \in L^{\frac{2}{2-\gamma_i}}(\mathbb{R}^N, \mathbb{R}^+)$, we can choose $R_\epsilon > 0$ such that

$$\Big(\int_{\mathbb{R}^N \setminus B_{R_\epsilon}} |\xi_i(x)|^{\frac{2}{2-\gamma_i}}\,dx\Big)^{(2-\gamma_i)/2} < \epsilon, \quad 1 \le i \le m. \tag{2.6}$$

By Sobolev's embedding theorem, $u_n \rightharpoonup u_0$ in E_λ implies

$$u_n \to u_0 \quad \text{in } L^2_{\rm loc}(\mathbb{R}^N),$$

and hence,

$$\lim_{n\to\infty}\int_{B_{R_\epsilon}} |u_n-u_0|^2\,dx=0. \tag{2.7}$$

By (2.7), there exists $N_0\in\mathbb{N}$ such that

$$\int_{B_{R_\epsilon}} |u_n-u_0|^2\,dx<\epsilon^2,\quad \text{for } n\geq N_0. \tag{2.8}$$

Hence, by (F1), (2.5), (2.8) and the Hölder inequality, for any $n\geq N_0$, we have

$$\begin{aligned}
&\int_{B_{R_\epsilon}} |f(x,u_n)-f(x,u_0)|\,|u_n-u_0|\,dx\\
&\leq \Big(\int_{B_{R_\epsilon}} |f(x,u_n)-f(x,u_0)|^2\,dx\Big)^{1/2}\Big(\int_{B_{R_\epsilon}} |u_n-u_0|^2\,dx\Big)^{1/2}\\
&\leq \Big(\int_{B_{R_\epsilon}} 2\left(|f(x,u_n)|^2+|f(x,u_0)|^2\right)\,dx\Big)^{1/2}\epsilon\\
&\leq 2\Big[\sum_{i=1}^m \gamma_i^2\Big(\int_{B_{R_\epsilon}} |\xi_i(x)|^2\left(|u_n|^{2(\gamma_i-1)}+|u_0(x)|^{2(\gamma_i-1)}\right)\,dx\Big)^{1/2}\Big]\epsilon\\
&\leq 2\Big[\sum_{i=1}^m \gamma_i^2\|\xi_i\|^2_{\frac{2}{2-\gamma_i}}\left(\|u_n\|_2^{2(\gamma_i-1)}+\|u_0\|_2^{2(\gamma_i-1)}\right)\Big]^{1/2}\epsilon\\
&\leq 2\Big[\sum_{i=1}^m \gamma_i^2\|\xi_i\|^2_{\frac{2}{2-\gamma_i}}\left(C^{2(\gamma_i-1)}+\|u_0\|_2^{2(\gamma_i-1)}\right)\Big]^{1/2}\epsilon.
\end{aligned} \tag{2.9}$$

On the other hand, by (2.5), (2.6), (2.8) and (F1), we have

$$\begin{aligned}
&\int_{\mathbb{R}^N\setminus B_{R_\epsilon}} |f(x,u_n)-f(x,u_0)|\,|u_n-u_0|\,dx\\
&\leq 2\sum_{i=1}^m\int_{\mathbb{R}^N\setminus B_{R_\epsilon}} \gamma_i|\xi_i(x)|\left(|u_n|^{\gamma_i}+|u_0|^{\gamma_i}\right)\,dx\\
&\leq 2\epsilon\sum_{i=1}^m c_2^{\gamma_i}c_0^{\gamma_i}\left(\|u_n\|_\lambda^{\gamma_i}+\|u_0\|_\lambda^{\gamma_i}\right)\\
&\leq 2\epsilon\sum_{i=1}^m c_2^{\gamma_i}c_0^{\gamma_i}\left(C^{\gamma_i}+\|u_0\|_\lambda^{\gamma_i}\right),\quad n\in\mathbb{N}.
\end{aligned} \tag{2.10}$$

Since ϵ is arbitrary, combining $(2.9) with (2.10)$, we have

$$\int_{\mathbb{R}^N} |f(x,u_n)-f(x,u_0)|\,|u_n-u_0|\,dx<\epsilon,\quad \text{as } n\to\infty. \tag{2.11}$$

It follows from (2.3) that

$$\begin{aligned}
&\langle\Phi_\lambda'(u_n)-\Phi_\lambda'(u_0),u_n-u_0\rangle\\
&=\|u_n-u_0\|_\lambda^2+\int_{\mathbb{R}^N} |f(x,u_n)-f(x,u_0)|\,|u_n-u_0|\,dx.
\end{aligned} \tag{2.12}$$

It is clear that $\langle\Phi_\lambda'(u_n)-\Phi_\lambda'(u_0),u_n-u_0\rangle\to 0$, thus, from (2.11) and (2.12), we get $u_n\to u_0$ in E_λ. Hence, Φ_λ satisfies (PS)-condition. □

Proof of Theorem 1.1. From Lemmas 2.1, 2.2, 2.3, we know that $c_\lambda = \inf_{E_\lambda} \Phi_\lambda(u)$ is a critical value of Φ_λ; that is, there exists a critical point $u_\lambda \in E_\lambda$ such that $\Phi_\lambda(u_\lambda) = c_\lambda$. Next, similar to the argument in [17], we show that $u_\lambda \neq 0$. Let $u^* \in \big(H^2(\Omega) \cap H_0^1(\Omega)\big) \setminus \{0\}$ and $\|u^*\|_\infty \leq 1$, then by (F2) and (2.2), we have

$$\begin{aligned} \Phi_\lambda(tu^*) &= \frac{1}{2}\|tu^*\|_\lambda^2 - \int_{\mathbb{R}^N} F(x, tu^*)\,dx \\ &= \frac{t^2}{2}\|u^*\|_\lambda^2 - \int_\Omega F(x, tu^*)\,dx \\ &\leq \frac{t^2}{2}\|u^*\|_\lambda^2 - \eta t^{\gamma_0} \int_\Omega |u^*|^{\gamma_0}\,dx, \end{aligned} \tag{2.13}$$

where $0 < t < \delta$, δ be given in (F2). Since $1 < \gamma_0 < 2$, it follows from (2.13) that $\Phi_\lambda(tu^*) < 0$ for $t > 0$ small enough. Hence, $\Phi_\lambda(u_\lambda) = c_\lambda < 0$, therefore, u_λ is a nontrivial critical point of Φ_λ and so u_λ is a nontrivial solution of problem (1.1). The proof is complete. □

3. Concentration of solutions

In the following, we study the concentration of solutions for problem (1.1) as $\lambda \to \infty$. Define

$$\tilde{c} = \inf_{u \in H^2(\Omega) \cap H_0^1(\Omega)} \Phi_\lambda|_{H^2(\Omega)\cap H_0^1(\Omega)}(u),$$

where $\Phi_\lambda|_{H^2(\Omega)\cap H_0^1(\Omega)}$ is a restriction of Φ_λ on $H^2(\Omega) \cap H_0^1(\Omega)$; that is,

$$\Phi_\lambda|_{H^2(\Omega)\cap H_0^1(\Omega)}(u) = \frac{1}{2}\int_\Omega \left(|\Delta u|^2 + |\nabla u|^2\right)\,dx - \int_\Omega F(x, u)\,dx,$$

for $u \in H^2(\Omega) \cap H_0^1(\Omega)$. Similar to the proof of Theorem 1.1, it is easy to prove that $\tilde{c} < 0$ can be achieved. Since $\big(H^2(\Omega) \cap H_0^1(\Omega)\big) \subset E_\lambda$ for all $\lambda > 0$, we get

$$c_\lambda \leq \tilde{c} < 0, \quad \text{for all } \lambda > \Lambda_0.$$

Proof of Theorem 1.2. We follow the arguments in [5]. For any sequence $\lambda_n \to \infty$, let $u_n := u_{\lambda_n}$ be the critical points of Φ_{λ_n} obtained in Theorem 1.1. Thus

$$\Phi_{\lambda_n}(u_n) \leq \tilde{c} < 0 \tag{3.1}$$

and

$$\begin{aligned} \Phi_{\lambda_n}(u_n) &= \frac{1}{2}\|u_n\|_{\lambda_n}^2 - \int_{\mathbb{R}^N} F(x, u_n)\,dx \\ &\geq \frac{1}{2}\|u_n\|_{\lambda_n}^2 - \sum_{i=1}^m c_2^{\gamma_i} c_0^{\gamma_i} \|\xi_i\|_{\frac{2}{2-\gamma_i}} \|u_n\|_{\lambda_n}^{\gamma_i}, \end{aligned}$$

which implies

$$\|u_n\|_{\lambda_n} \leq c_1, \tag{3.2}$$

where the constant c_1 is independent of λ_n. Therefore, we may assume that $u_n \rightharpoonup u_0$ in E_λ and $u_n \to u_0$ in $L^p_{\rm loc}(\mathbb{R}^N)$ for $2 \leq p < 2_*$. From Fatou's lemma, we have

$$\int_{\mathbb{R}^N} V(x)|u_0|^2\,dx \leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} V(x)|u_n|^2\,dx \leq \liminf_{n\to\infty} \frac{\|u_n\|_{\lambda_n}^2}{\lambda_n} = 0,$$

which implies that $u_0 = 0$ a.e. in $\mathbb{R}^N \setminus V^{-1}(0)$ and $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ by (V3). Now for any $\varphi \in C_0^\infty(\Omega)$, since $\langle \Phi'_{\lambda_n}(u_n), \varphi\rangle = 0$, it is easy to verify that

$$\int_\Omega (\Delta u_0 \Delta\varphi + \nabla u_0 \cdot \nabla\varphi)\, dx - \int_\Omega f(x, u_0)\varphi\, dx = 0,$$

which implies that u_0 is a weak solution of problem (1.2) by the density of $C_0^\infty(\Omega)$ in $H^2(\Omega) \cap H_0^1(\Omega)$.

Next, we show that $u_n \to u_0$ in $L^p(\mathbb{R}^N)$ for $2 \le p < 2_*$. Otherwise, by Lions vanishing lemma [11, 22], there exist $\delta > 0, \rho > 0$ and $x_n \in \mathbb{R}^N$ such that

$$\int_{B_\rho(x_n)} |u_n - u_0|^2\, dx \ge \delta.$$

Since $u_n \to u_0$ in $L^2_{\rm loc}(\mathbb{R}^N)$, $|x_n| \to \infty$. Hence meas$\,(B_\rho(x_n) \cap V_b) \to 0$. By the Hölder inequality, we have

$$\int_{B_\rho(x_n)\cap V_b} |u_n - u_0|^2\, dx \le (\text{meas}\,(B_\rho(x_n) \cap V_b))^{\frac{2_*-2}{2_*}} \Big(\int_{\mathbb{R}^N} |u_n - u_0|^{2_*}\Big)^{2/2_*} \to 0.$$

Consequently,

$$\begin{aligned}
\|u_n\|^2_{\lambda_n} &\ge \lambda_n b \int_{B_\rho(x_n)\cap\{x\in\mathbb{R}^N : V(x)\ge b\}} |u_n|^2\, dx \\
&= \lambda_n b \int_{B_\rho(x_n)\cap\{x\in\mathbb{R}^N : V(x)\ge b\}} |u_n - u_0|^2\, dx \\
&= \lambda_n b\Big(\int_{B_\rho(x_n)} |u_n - u_0|^2\, dx - \int_{B_\rho(x_n)\cap V_b} |u_n - u_0|^2\, dx + o(1)\Big) \\
&\to \infty,
\end{aligned}$$

which contradicts (3.2). Next, we show that $u_n \to u_0$ in $H^2(\mathbb{R}^N)$. By virtue of $\langle \Phi'_{\lambda_n}(u_n), u_n\rangle = \langle \Phi'_{\lambda_n}(u_n), u_0\rangle = 0$ and the fact that $u_n \to u_0$ in $L^p(\mathbb{R}^N)$ for $2 \le p < 2_*$, we have

$$\lim_{n\to\infty} \|u_n\|^2_{\lambda_n} = \lim_{n\to\infty} (u_n, u_0)_{\lambda_n} = \lim_{n\to\infty} (u_n, u_0) = \|u_0\|^2;$$

therefore

$$\limsup_{n\to\infty} \|u_n\|^2 \le \|u_0\|^2.$$

On the other hand, the weakly lower semi-continuity of norm yields

$$\|u_0\|^2 \le \liminf_{n\to\infty} \|u_n\|^2.$$

Hence,

$$u_n \to u_0 \quad \text{in } H^2(\mathbb{R}^N).$$

From (3.1), we have

$$\frac{1}{2}\int_\Omega \big(|\Delta u_0|^2 + |\nabla u_0|^2\big)\, dx - \int_\Omega F(x, u_0)\, dx \le \tilde{c} < 0,$$

which implies that $u_0 \ne 0$. This completes the proof. □

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