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GLOBAL WELL-POSEDNESS FOR NONLINEAR SCHRÖDINGER EQUATIONS WITH ENERGY-CRITICAL DAMPING

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Abstract. We consider the Cauchy problem for the nonlinear Schrödinger equations with energy-critical damping. We prove the existence of global in-time solutions for general initial data in the energy space. Our results extend some results from [1, 2].

1. Introduction

In this article we study the Cauchy problem for the nonlinear Schrödinger (NLS) equation with energy-critical damping,

$$\begin{gathered} iu_t + \frac{1}{2}\Delta u = V(x)u + \lambda|u|^{2\sigma}u - ia|u|^{\alpha}u, \quad (t,x) \in [0,\infty) \times \mathbb{R}^N, \\ u|_{t=0} = u_0, \quad u_0 \in \Sigma, \end{gathered} \tag{1.1}$$

where $N \geq 3$, $\lambda \in \mathbb{R}$, $a > 0$, $0 < \sigma \leq \frac{2}{N-2}$, $\alpha = \frac{4}{N-2}$ and Σ denotes the energy space associated to the harmonic potential; i.e.,

$$\Sigma = \{u \in H^1(\mathbb{R}^N),\ xu \in L^2(\mathbb{R}^N)\},$$

equipped with the norm

$$\|u\|_\Sigma := \|u\|_{L^2} + \|\nabla u\|_{L^2} + \|xu\|_{L^2}.$$

The external potential V is supposed to be an anisotropic quadratic confinement, i.e.,

$$V(x) = \frac{1}{2}\sum_{j=1}^{N} \omega_j^2 x_j^2, \quad \omega_j \in \mathbb{R}. \tag{1.2}$$

Equation (1.1) appears in different physical contexts. For example, considering the three-body interaction in collapsing Bose-Einstein condensates (BECs), within the realm of Gross-Pitaevskii theory, the emittance of particles from the condensate is described by the dissipative model involving a quintic nonlinear damping term [14]; in nonlinear optics, equation (1.1) with $V = 0$ describes the propagation of a laser pulse within an optical fiber under the influence of additional multi-photon absorption processes, see, e.g., [5, 12].

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For $a = 0$, equation (1.1) simplifies to the classical NLS. It arises in various areas of physics, such as nonlinear optics and nonlinear plasmas; for a broader introduction, see [9, 19]. It also has received a great deal of attention from mathematicians, for instance, see [6, 9, 19, 20] and the references therein.

For $a > 0$, the last term in (1.1) is dissipative, see [1, 2]. Therefore, the energy of (1.1) is no longer conserved, in contrast to the usual case of Hamiltonian NLS. When $\sigma = \alpha$, and $0 < \sigma \le 1/N$, the asymptotic behavior in time of the small solution to (1.1) has been studied in [16, 18]. Numerical studies of (1.1) can be found in [3, 4, 13, 17]; in particular, the nonlinear-damping continuation of singular solutions for (1.1) with critical and supercritical nonlinearities has been considered in [13]. When $V \equiv 0$, under some assumptions, Feng, Zhao and Sun [11] have showed that as $a \to 0$ the solution of (1.1) converges to that of (1.1) with $a = 0$. In [10] the particular case of a mass critical nonlinearity $\sigma = 2/N$ and $V = 0$ has been studied. In there, global in-time existence of solutions is established if $\alpha > 4/N$ and it is claimed that finite time blow-up in the log-log regime occurs if $\alpha < 4/N$. The global well-posedness for a cubic NLS equation perturbed by higher-order nonlinear damping has been studied in [2], where, in particular, the energy-critical case of a quintic dissipation in three-dimensional space has been treated. Recently, Antonelli, Carles and Sparber [1] have done a more systematic study for NLS type equations with general energy-subcritical damping. However, equation (1.1) with an energy-critical damping or nonlinearity do not seem to have been discussed except $N = 3$ and $\sigma = 1$. The aim of this paper is to establish the global well-posedness for (1.1) with an energy-subcritical or critical nonlinearity and an energy-critical damping. To solve this problem, we mainly use the idea of [2]. This is shown in the following theorem.

Theorem 1.1. *Let $N \ge 3$, $a > 0$, $\alpha = \frac{4}{N-2}$ and $u_0 \in \Sigma$. Assume that V satisfy* (1.2) *and suppose further that*

(1) *either $\lambda \ge 0$ and $0 < \sigma \le \frac{2}{N-2}$,*
(2) *or $\lambda < 0$ and $0 < \sigma < \frac{2}{N-2}$.*

Then, the Cauchy problem (1.1) *has a unique global solution $u \in C([0,\infty), \Sigma)$.*

Remark. In the case of energy-critical, it is well-known (see, e.g. [9]) that the usual a-priori estimates on the H^1-norm is not sufficient to conclude global existence. The reason is that the local existence time of solutions does not only depend on the H^1-norm of u, but also on its profile. This is an essential difference with [1]. Enlightened mainly by the work in [2, 20, 21, 22], we will prove this theorem by combining a-priori estimates and a bootstrap argument.

We finally state the following estimate for the time-decay of solutions. The proof is the same as that of [1, Proposition 4.2], so we omit it.

Corollary 1.2. *Let $N \ge 3$, $a > 0$, $\omega_j \ne 0$ $(j = 1, \dots, N)$ and $u_0 \in \Sigma$. In either of the cases mentioned in Theorem 1.1, the solution to* (1.1) *satisfies $u \in L^\infty([0,\infty), \Sigma)$ and there exists $C > 0$ such that*

$$\|u(t)\|_{L^2}^2 \le Ct^{-\frac{N-2}{N+2}}, \quad \forall t \ge 1.$$

This article is organized as follows: in Section 2, we collect some lemmas such as Strichartz's estimates, and a-priori estimates for the solutions of (1.1). In section 3, we show Theorem 1.1.

Notation. In this article, we use the following notation. $C > 0$ will stand for a constant that may be different from line to line when it does not cause any confusion. Since we exclusively deal with $\mathbb{R}^N$, we often use the abbreviation $L^r = L^r(\mathbb{R}^N)$. Given any interval $I \subset \mathbb{R}$, the norms of mixed spaces $L^q(I, L^r(\mathbb{R}^N))$ are denoted by $\|\cdot\|_{L^q(I,L^r)}$. We denote by $U(t) := e^{itH}$, the Schrödinger group generated by $H = -\frac{1}{2}\Delta + V$. We recall that a pair of exponents (q, r) is Schrödinger-admissible if $\frac{2}{q} = N(\frac{1}{2} - \frac{1}{r})$ and $2 \le r \le \frac{2N}{N-2}$, ($2 \le r \le \infty$ if $N = 1$; $2 \le r < \infty$ if $N = 2$). Then, for any space-time slab $I \times \mathbb{R}^N$, we can define the Strichartz norm

$$\|u\|_{S(I)} = \sup_{(q,r)} \|u\|_{L^q(I,L^r)},$$

where the supremum is taken over all admissible pairs of exponents (q, r).

2. Some lemmas

We first recall the following Strichartz's estimates.

Lemma 2.1 ([2, 7, 8, 15]). *Let (q, r), (q_1, r_1) and (q_2, r_2) be admissible pairs. Assume that I is some finite time interval. Then it follows*

$$\|U(\cdot)\varphi\|_{L^q(I,L^r)} \le C(r, N)|I|^{1/q}\|\varphi\|_{L^2},$$

and

$$\Big\| \int_{I\cap\{s\le t\}} U(t-s)F(s)ds\Big\|_{L^{q_1}(I,L^{r_1})} \le C(r_1, r_2, N)|I|^{1/q_1}\|F\|_{L^{q_2'}(I,L^{r_2'})}.$$

Next, we show that (1.1) is locally well-posed for any $u_0 \in \Sigma$ and we also establish a blow-up alternative.

Proposition 2.2 (Local solution). *Let $N \ge 3$, $0 < \sigma \le \frac{2}{N-2}$, $\alpha = \frac{4}{N-2}$, $\lambda, a \in \mathbb{R}$ and V satisfy (1.2). For every $u_0 \in \Sigma$, there exist $T > 0$ and a unique strong solution u defined on $[0, T]$. Let $[0, T^*)$ be the maximal time interval on which u is well-defined, then, the following properties hold:*

(i) *$u, \nabla u, xu \in S([0, T])$ for $0 < T < T^*$.*
(ii) *If $T^* < \infty$, then $\|u\|_{S([0,T^*))} = +\infty$.*

Proof. The proof of this proposition is standard and based on contraction mapping arguments. Thus, we only present the main steps of the classical argument, which can be found for instance in [9]. Firstly, for some $T > 0$, we define

$$X_T = L^\infty((0, T); L^2) \cap L^q((0, T); L^r) \cap L^\gamma((0, T); L^\rho)$$

where $r = 2\sigma + 2$,

$$q = \frac{4\sigma + 4}{N\sigma}, \quad \gamma = \frac{2N}{N-2}, \quad \rho = \frac{2N^2}{N^2 - 2N + 4}.$$

Since $U(\cdot)\nabla u_0 \in X_T$ by Strichartz's estimates, we have $\|U(\cdot)\nabla u_0\|_{X_T} \to 0$ as $T \to 0$.

Next, we claim that there exists $\eta > 0$ such that if $u_0 \in \Sigma$ satisfies

$$\|U(\cdot)\nabla u_0\|_{X_T} \le \eta \tag{2.1}$$

for some $T > 0$, then there exists a unique solution $u \in S([0, T])$ of (1.1). Notice that (2.1) is satisfied for T small enough.

Indeed, fix $\eta > 0$, to be chosen later. Duhamel's formulation for (1.1) reads

$$u(t) = U(t)u_0 - i\lambda \int_0^t U(t-s)(|u|^{2\sigma}u)(s)ds - a\int_0^t U(t-s)(|u|^{\frac{4}{N-2}}u)(s)ds. \quad (2.2)$$

Denote the right hand side by $\Phi(u)(t)$. By Lemma 2.1 and Hölder's inequality, we have

$$\begin{aligned} \|\Phi(u)\|_{X_T} &\le C\|u_0\|_{L^2} + C\||u|^{2\sigma}u\|_{L^{q'}((0,T);L^{r'})} + C\||u|^{\frac{4}{N-2}}u\|_{L^{\gamma'}((0,T);L^{\rho'})} \\ &\le C\|u_0\|_{L^2} + CT^{2\sigma/\theta}\|u\|^{2\sigma}_{L^\infty((0,T);H^1)}\|u\|_{L^q((0,T);L^r)} \\ &\quad + C\|u\|_{L^\gamma((0,T);L^\rho)}\|\nabla u\|^{\frac{4}{N-2}}_{L^\gamma((0,T);L^\rho)}, \end{aligned} \quad (2.3)$$

where $\theta = \frac{2\sigma(2\sigma+2)}{2-(N-2)\sigma}$. Next, to estimate ∇u and xu, we notice that

$$[\partial_j, H] = \partial_j V(x), \quad [x_j, H] = \partial_j, \quad j = 1, \dots, N.$$

where $[A, B] = AB - BA$ denotes the usual commutator. Therefore,

$$\begin{aligned} \nabla\Phi(u)(t) = U(t)\nabla u_0 &- i\lambda\int_0^t U(t-s)\nabla(|u|^{2\sigma}u)(s)ds \\ &- a\int_0^t U(t-s)\nabla(|u|^{\frac{4}{N-2}}u)(s)ds \\ &- i\lambda\int_0^t U(t-s)\Phi(u)(s)\nabla V ds. \end{aligned} \quad (2.4)$$

Now we estimate the second term of the right-hand side as above. Since ∇V is sublinear by assumption,

$$\begin{aligned} \|\nabla\Phi(u)\|_{X_T} &\le C\|U(\cdot)\nabla u_0\|_{X_T} + CT^{2\sigma/\theta}\|u\|^{2\sigma}_{L^\infty((0,T);H^1)}\|\nabla u\|_{L^q((0,T);L^r)} \\ &\quad + C\|\nabla u\|^{\frac{N+2}{N-2}}_{L^\gamma((0,T);L^\rho)} + CT\|x\Phi(u)\|_{L^\infty((0,T);L^2)} \\ &\quad + CT\|\Phi(u)\|_{L^\infty((0,T);L^2)}. \end{aligned} \quad (2.5)$$

Similarly, we have

$$\begin{aligned} \|x\Phi(u)\|_{X_T} &\le C\|xu_0\|_{L^2} + CT^{2\sigma/\theta}\|u\|^{2\sigma}_{L^\infty((0,T);H^1)}\|xu\|_{L^q((0,T);L^r)} \\ &\quad + C\|xu\|_{L^\gamma((0,T);L^\rho)}\|\nabla u\|^{\frac{4}{N-2}}_{L^\gamma((0,T);L^\rho)} \\ &\quad + CT\|\nabla\Phi(u)\|_{L^\infty((0,T);L^2)}. \end{aligned} \quad (2.6)$$

It is thus easy to see that Φ maps the set

$$\begin{aligned} \mathcal{B} = \Big\{u; &\|\nabla u\|_{L^\gamma((0,T);L^\rho)} \le 2\eta,\ \|\nabla u\|_{L^\infty((0,T);L^2)\cap L^q((0,T);L^r)} \le 2C\|xu_0\|_{L^2}, \\ &\|xu\|_{X_T} \le 2C\|xu_0\|_{L^2}, \|u\|_{X_T} \le 2C\|u_0\|_{L^2}\Big\} \end{aligned}$$

to itself and is a contraction in the X_T norm, provided η and T are chosen sufficiently small. The contraction mapping theorem then implies the existence of a unique solution to (1.1) on $[0, T]$. Finally, by some standard arguments, (i) and (ii) follow. □

Remark. For more general potentials, as suggested in the proof, Proposition 2.2 remains valid if we assume more generally that $V(x)$ is smooth, and at most quadratic, i.e., $\partial^\alpha V \in L^\infty(\mathbb{R}^N)$ for all $|\alpha| \ge 2$.

In the following, we shall derive several a-priori estimates on the solutions of (1.1). By the analogous arguments to those of [1, Lemma 2.7] and [2, Lemma 3.1], we obtain the following lemma.

Lemma 2.3. *Let $u(t) \in \Sigma$ be a solution of (1.1) defined on the maximal interval $[0, T^*)$, and $V(x)$ satisfy (1.2). Then it follows*

$$\|u(t)\|_{L^2} \le \|u_0\|_{L^2}, \quad \forall\, t \in [0, T^*), \tag{2.7}$$

$$\int_0^{T^*} \int_{\mathbb{R}^N} |u(t,x)|^{\frac{2N}{N-2}} dxdt \le C(\|u_0\|_{L^2}). \tag{2.8}$$

The a-priori estimates in Lemma 2.1 are not sufficient to conclude global well-posedness for (1.1). We consequently follow the idea in [1] and [2] and consider the modified energy functional

$$\begin{aligned} E(t) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u(t,x)|^2 dx + \int_{\mathbb{R}^N} V(x) |u(t,x)|^2 dx \\ + \frac{\lambda}{\sigma+1} \int_{\mathbb{R}^N} |u(t,x)|^{2\sigma+2} dx + \kappa \int_{\mathbb{R}^N} |u(t,x)|^{\frac{2N}{N-2}} dx. \end{aligned} \tag{2.9}$$

Lemma 2.4. *Let $u(t) \in \Sigma$ be a solution of (1.1) defined on the maximal interval $[0, T^*)$, and $V(x)$ satisfy (1.2). Moreover, let $0 < \kappa < \frac{a(N-2)^2}{2N}$, and assume that*

1. *either $\lambda \ge 0$ and $0 < \sigma \le \frac{2}{N-2}$,*
2. *or $\lambda < 0$ and $0 < \sigma < \frac{2}{N-2}$.*

Then

$$E(t) \le E(0) + C(\|u_0\|_{L^2}), \quad \forall\, t \in [0, T^*), \tag{2.10}$$

$$\int_0^{T^*} \int_{\mathbb{R}^N} |u(x,t)|^{\frac{2(N+2)}{N-2}} dxdt \le C(E(0), \|u_0\|_{L^2}). \tag{2.11}$$

Proof. This is done along the lines of [1, Proposition 3.1]. By their, we obtain

$$\begin{aligned} \frac{d}{dt} E(t) = &-\Big(a - \kappa\Big(\frac{4}{(N-2)^2} + \frac{2}{N-2}\Big)\Big) \int_{\mathbb{R}^N} |u|^{\frac{4}{N-2}} |\nabla u|^2 dx \\ &- 2a \frac{2}{N-2} \int_{\mathbb{R}^N} |u|^{\frac{4}{N-2}} |\nabla |u||^2 dx \\ &- \kappa\Big(\frac{4}{(N-2)^2} + \frac{2}{N-2}\Big) \int_{\mathbb{R}^N} |u|^{\frac{4}{N-2}} |Re(\bar{\phi}\nabla u) - \mathrm{Im}(\bar{\phi}\nabla u)|^2 dx \\ &- 2a \int_{\mathbb{R}^N} V(x) |u|^{\frac{2N}{N-2}} dx - 2a\lambda \int_{\mathbb{R}^N} |u|^{\frac{4}{N-2}+2\sigma} dx \\ &- 2a\kappa \frac{N}{N-2} \int_{\mathbb{R}^N} |u|^{\frac{8}{N-2}+2} dx, \end{aligned}$$

where

$$\phi(t,x) := \begin{cases} |u(t,x)|^{-1} u(t,x) & \text{if } u(t,x) \ne 0, \\ 0 & \text{if } u(t,x) = 0. \end{cases}$$

Therefore, if $\lambda \ge 0$, (2.10) follows by $\frac{d}{dt}E(t) \le 0$. If $\lambda < 0$, (2.10) follows by the Young inequality with ε. (2.11) follows by (2.10) and (2.8). $\square$

With Lemma 2.4 in hand, we can obtain the uniform bound on the Σ-norm of $u(t)$. The proof is analogue to that of Corollary 3.4 in [2], so we omit it.

Corollary 2.5. *Let $u(t) \in \Sigma$ be a solution of* (1.1) *defined on the maximal interval* $[0, T^*)$. *Then*

$$\|u(t)\|_\Sigma \le C(\|u_0\|_\Sigma), \quad \forall t \in [0, T^*).$$

3. Proof of main results

Proof of Theorem 1.1. Let I be some finite time interval, in the following, we set

$$W(I) = L^{\frac{2(N+2)}{N-2}}(I, L^{\frac{2(N+2)}{N-2}}), \quad V(I) = L^{\frac{2(N+2)}{N}}(I, L^{\frac{2(N+2)}{N}}).$$

We divide the proof into two steps: (i) $\frac{2}{N} < \sigma \le \frac{2}{N-2}$ and (ii) $0 < \sigma \le \frac{2}{N}$.

Step 1. We first treat the case (i) $\frac{2}{N} < \sigma \le \frac{2}{N-2}$. By applying Strichartz's estimates to (2.2) and Hölder's inequality, we can estimate as follows:

$$\begin{aligned}
&\|u\|_{L^q(I,L^r)} \\
&\le C|I|^{1/q}\Big(\|u_0\|_{L^2} + \||u|^{2\sigma}u\|_{L^{\frac{2(N+2)}{N+4}}(I,L^{\frac{2(N+2)}{N+4}})} \\
&\quad + \||u|^{\frac{4}{N-2}}u\|_{L^{\frac{2(N+2)}{N+4}}(I,L^{\frac{2(N+2)}{N+4}})}\Big) \\
&\le C|I|^{1/q}\Big(\|u_0\|_{L^2} + \|u\|^{2\sigma}_{L^{\sigma(N+2)}(I,L^{\sigma(N+2)})}\|u\|_{V(I)} + \|u\|^{\frac{4}{N-2}}_{W(I)}\|u\|_{V(I)}\Big) \\
&\le C|I|^{1/q}\Big(\|u_0\|_{L^2} + \|u\|^{N\sigma-2}_{W(I)}\|u\|^{3-\sigma(N-2)}_{V(I)} + \|u\|^{\frac{4}{N-2}}_{W(I)}\|u\|_{V(I)}\Big),
\end{aligned} \tag{3.1}$$

where C is independent of I.

By an analogous argument to that of (3.1), we obtain

$$\begin{aligned}
&\|\nabla u\|_{L^q(I,L^r)} + \|xu\|_{L^q(I,L^r)} \\
&\le C|I|^{1/q}\Big(\|\nabla u_0\|_{L^2} + \||u|^{2\sigma}\nabla u\|_{L^{\frac{2(N+2)}{N+4}}(I,L^{\frac{2(N+2)}{N+4}})} \\
&\quad + \||u|^{\frac{4}{N-2}}\nabla u\|_{L^{\frac{2(N+2)}{N+4}}(I,L^{\frac{2(N+2)}{N+4}})}\Big) \\
&\quad + C|I|^{1/q}\Big(\|xu_0\|_{L^2} + \||u|^{2\sigma}xu\|_{L^{\frac{2(N+2)}{N+4}}(I,L^{\frac{2(N+2)}{N+4}})} \\
&\quad + \||u|^{\frac{4}{N-2}}xu\|_{L^{\frac{2(N+2)}{N+4}}(I,L^{\frac{2(N+2)}{N+4}})}\Big) \\
&\le C|I|^{1/q}\Big(\|\nabla u_0\|_{L^2} + \|u\|^{N\sigma-2}_{W(I)}\|u\|^{2-\sigma(N-2)}_{V(I)}\|\nabla u\|_{V(I)} \\
&\quad + \|u\|^{\frac{4}{N-2}}_{W(I)}\|\nabla u\|_{V(I)}\Big) + C|I|^{1/q}\Big(\|xu_0\|_{L^2} \\
&\quad + \|u\|^{N\sigma-2}_{W(I)}\|u\|^{2-\sigma(N-2)}_{V(I)}\|xu\|_{V(I)} + \|u\|^{\frac{4}{N-2}}_{W(I)}\|xu\|_{V(I)}\Big).
\end{aligned} \tag{3.2}$$

Denoting the Strichartz norm in Σ by

$$\|u\|_{S_\Sigma(I)} := \|u\|_{S(I)} + \|\nabla u\|_{S(I)} + \|xu\|_{S(I)},$$

it follows from (3.1) and (3.2) that

$$\begin{aligned}
&\|u\|_{S_\Sigma(I)} \\
&\le C\sup_q |I|^{1/q}\Big(\|u_0\|_\Sigma + \|u\|^{N\sigma-2}_{W(I)}\|u\|^{3-\sigma(N-2)}_{S_\Sigma(I)} + \|u\|^{\frac{4}{N-2}}_{W(I)}\|u\|_{S_\Sigma(I)}\Big).
\end{aligned} \tag{3.3}$$

On the other hand, for every $T \in [0, T^*)$, we deduce from (2.11) that there exists $M > 0$ such that $\|u\|_{W([0,T])} \le M$, where M is independent of the length of I. Therefore, we can divide $[0,T]$ into subintervals $[0,T] = I_1 \cup \ldots \cup I_K$, where $I_k = [t_{k-1}, t_k]$ and such that in each I_k, we have

$$\|u\|_{W(I_k)} \le \varepsilon, \quad \text{for all } k = 1, \ldots, K,$$

for some $\varepsilon < 1$, which only depends on $\|u_0\|_\Sigma$.

Considering the first interval, $I_1 = [0, t_1]$, from (3.3) it follows that

$$\|u\|_{S_\Sigma(I_1)} \le C \sup_q |I_1|^{1/q} (\|u_0\|_\Sigma + \varepsilon^{N\sigma-2} \|u\|^{3-\sigma(N-2)}_{S_\Sigma(I_1)} + \varepsilon^{\frac{4}{N-2}} \|u\|_{S_\Sigma(I_1)}).$$

A standard continuity argument yields

$$\|u\|_{S_\Sigma(I_1)} \le C(\|u_0\|_\Sigma, |I_1|).$$

Similarly, we can show that

$$\|u\|_{S_\Sigma(I_k)} \le C(\|u_{t_{k-1}}\|_\Sigma, |I_k|), \quad k = 2, \ldots, K,$$

which, together with Corollary 2.5 implies

$$\|u\|_{S_\Sigma(I_k)} \le C(\|u_0\|_\Sigma, |I_k|), \quad k = 1, \ldots, K.$$

Summing up all the subintervals I_k, it follows that

$$\|u\|_{S_\Sigma([0,T])} \le C(\|u_0\|_\Sigma, M), \quad \text{for every } T < T^*$$

which implies $\|u\|_{S_\Sigma([0,T^*))} < \infty$. According to the blow-up alternative in Proposition 2.2, we conclude that the Cauchy problem (1.1) with $\frac{2}{N} < \sigma \le \frac{2}{N-2}$ is globally well-posedness.

Step 2. Next we treat case (ii) $0 < \sigma \le \frac{2}{N}$. We deduce from Strichartz's estimates and Hölder's inequality that

$$\begin{aligned}
&\|u\|_{L^q(I,L^r)} \\
&\le C|I|^{1/q} \Big(\|u_0\|_{L^2} + \||u|^{2\sigma} u\|_{L^{q_1'}(I,L^{r_1'})} + \||u|^{\frac{4}{N-2}} u\|_{L^{\frac{2(N+2)}{N+4}}(I,L^{\frac{2(N+2)}{N+4}})} \Big) \\
&\le C|I|^{1/q} \Big(\|u_0\|_{L^2} + \|u\|^{2\sigma}_{L^\beta(I,L^{r_1})} \|u\|_{L^{q_1}(I,L^{r_1})} + \|u\|^{\frac{4}{N-2}}_{W(I)} \|u\|_{V(I)} \Big) \\
&\le C|I|^{1/q} \Big(\|u_0\|_{L^2} + |I|^{1/\gamma} \|u\|^{1-\theta}_{L^\infty(I,L^2)} \|u\|^{\theta}_{W(I)} \|u\|_{L^{q_1}(I,L^{r_1})} + \|u\|^{\frac{4}{N-2}}_{W(I)} \|u\|_{V(I)} \Big),
\end{aligned} \tag{3.4}$$

where

$$\beta = \frac{2\sigma(2\sigma+2)}{2-(N-2)\sigma}, \quad \theta = \frac{\sigma(N+2)}{4(\sigma+1)} < 1, \quad \gamma = \frac{8\sigma(\sigma+1)}{4 - 2\sigma(N-2) - \sigma^2(N-2)} > 0,$$

$r_1 = 2\sigma + 2$, taking q_1 such that (q_1, r_1) is an admissible pair. By an analogous argument to that of (3.4), we have

$$
\begin{aligned}
&\|\nabla u\|_{L^q(I,L^r)} + \|xu\|_{L^q(I,L^r)} \\
&\le C|I|^{1/q}\Big(\|\nabla u_0\|_{L^2} + \||u|^{2\sigma}\nabla u\|_{L^{q_1'}(I,L^{r_1'})} + \||u|^{\frac{4}{N-2}}\nabla u\|_{L^{\frac{2(N+2)}{N+4}}(I,L^{\frac{2(N+2)}{N+4}})}\Big) \\
&\quad + C|I|^{1/q}\Big(\|xu_0\|_{L^2} + \||u|^{2\sigma}xu\|_{L^{q_1'}(I,L^{r_1'})} + \||u|^{\frac{4}{N-2}}xu\|_{L^{\frac{2(N+2)}{N+4}}(I,L^{\frac{2(N+2)}{N+4}})}\Big) \\
&\le C|I|^{1/q}\Big(\|\nabla u_0\|_{L^2} + |I|^{1/\gamma}\|u\|_{L^\infty(I,L^2)}^{1-\theta}\|u\|_{W(I)}^{\theta}\|\nabla u\|_{L^{q_1}(I,L^{r_1})} \\
&\quad + \|u\|_{W(I)}^{\frac{4}{N-2}}\|\nabla u\|_{V(I)}\Big) + C|I|^{1/q}\Big(\|xu_0\|_{L^2} \\
&\quad + |I|^{1/\gamma}\|u\|_{L^\infty(I,L^2)}^{1-\theta}\|u\|_{W(I)}^{\theta}\|xu\|_{L^{q_1}(I,L^{r_1})} + \|u\|_{W(I)}^{\frac{4}{N-2}}\|xu\|_{V(I)}\Big).
\end{aligned}
\tag{3.5}
$$

It follows from (3.4) and (3.5) that

$$
\|u\|_{S_\Sigma} \le C\sup_q |I|^{1/q}\Big(\|u_0\|_\Sigma + |I|^{1/\gamma}\|u\|_{W(I)}^{\theta}\|u\|_{S_\Sigma}^{2-\theta} + \|u\|_{W(I)}^{\frac{4}{N-2}}\|u\|_{S_\Sigma}\Big). \tag{3.6}
$$

Arguing as Step 1, we can conclude that the Cauchy problem (1.1) with $0 < \sigma \le 2/N$ is global well-posedness. This completes the proof. □

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