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LOCAL WELL-POSEDNESS AND BLOW-UP OF SOLUTIONS FOR WAVE EQUATIONS ON SHALLOW WATER WITH PERIODIC DEPTH

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Abstract. In this article, we consider a nonlinear evolution equation for surface waves in shallow water over periodic uneven bottom. The local well-posedness in Sobolev space $H^s(\mathbb{S})$ with $s > 3/2$ is established by applying Kato's theory. Then a blow up criterion is determined in $H^s(\mathbb{S})$, $s > 3/2$. Finally, some blow-up results are given for a simplified model.

1. Introduction

This article concerns an evolution equation which models the propagation of surface waves in shallow water over uneven bottom [17]:

$$\begin{aligned}&(1-\mu m\partial_x^2)u_t + cu_x + kc_xu + \sum_{j\in J}\varepsilon^j f_j u^j u_x + \mu g u_{xxx}\\ &= \varepsilon\mu[h_1 u u_{xxx} + \partial_x(h_2u)u_{xx} + u_x\partial_x^2(h_2u)],\end{aligned} \tag{1.1}$$

where $u(t,x)$ is the free surface elevation, $m \in \mathbb{R}^+$, $k \in \mathbb{R}$, J is a finite subset of $\mathbb{Z}^+$ and $c = \sqrt{1-\beta b^{(\alpha)}}$ ($b^{(\alpha)}(x) = b(\alpha x)$ is the bottom function), $f_j = f_j(c)$, $g = g(c)$, $h_1 = h_1(c)$ and $h_2 = h_2(c)$ are smooth functions of c. In order to give a detailed interpretation of the above equation, we introduce the following quantities: a is the order of amplitude of the waves; λ is the wave-length of the waves; b_0 is the order of amplitude of the variation of the bottom topography; λ_0 is the wavelength of the bottom variations; h_0 is the reference depth. Then the four dimensionless parameters in (1.1) are:

$$\varepsilon = \frac{a}{h_0}, \quad \mu = \frac{h_0^2}{\lambda^2}, \quad \alpha = \frac{\lambda}{\lambda_0}, \quad \beta = \frac{b_0}{h_0}.$$

Since μ is small, we assume that $|\mu m| < 1$.

Note that (1.1) is related to Constantin-Lannes equations [9], Camassa-Holm (CH) equation [2] and Degasperis-Procesi (DP) equation [10].

(I) From [17], choosing

$$m = \frac{1}{12}, \quad k = \frac{1}{2}, \quad J = \{1, 2, 3\},$$

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$$\begin{gathered} f_1(c)=\frac{3}{2c},\quad f_2(c)=-\frac{3}{8c^3},\quad f_3(c)=\frac{3}{16c^5},\\ g(c)=-\frac{1}{12}c^5+\frac{1}{12}c^5+\frac{1}{12}c,\quad h_1(c)=-\frac{1}{6}c^3-\frac{1}{8c},\\ h_2(c)=-\frac{5}{48}c^3-\frac{3}{16c},\quad \alpha=\varepsilon,\quad \beta=\mu^{3/2}, \end{gathered}$$

and neglecting the $O(\mu^2)$ terms, Equation (1.1) reads

$$\begin{aligned} &u_t+cu_x+\frac{1}{2}c_xu+\frac{3}{2}\varepsilon uu_x-\frac{3}{8}\varepsilon^2u^2u_x+\frac{3}{16}\varepsilon^3u^3u_x+\frac{\mu}{12}(u_{xxx}-u_{xxt})\\ &=-\frac{7}{24}\varepsilon\mu(uu_{xxx}+2u_xu_{xx}). \end{aligned}\tag{1.2}$$

If we take $b=0$ (i.e., we consider a flat bottom) in (1.2), then one recovers the Constantin-Lannes equations:

$$\begin{aligned} &u_t+u_x+\frac{3}{2}\varepsilon uu_x-\frac{3}{8}\varepsilon^2u^2u_x+\frac{3}{16}\varepsilon^3u^3u_x+\frac{\mu}{12}(u_{xxx}-u_{xxt})\\ &=-\frac{7}{24}\varepsilon\mu(uu_{xxx}+2u_xu_{xx}). \end{aligned}\tag{1.3}$$

(II) From [17], choosing $c=1$ (i.e., $b=0$):

$$\begin{gathered} m=-B,\quad k=\frac{3}{2},\quad J=\{1\},\quad f_1(c)=\frac{3}{2},\\ g(c)=A,\quad h_1(c)=E,\quad h_2(c)=F, \end{gathered}$$

where A, B, E, F, are constants, one gets the class of equations:

$$u_t+u_x+\frac{3}{2}\varepsilon uu_x+\mu(Au_{xxx}+Bu_{xxt})=\varepsilon\mu(Euu_{xxx}+Fu_xu_{xx}).\tag{1.4}$$

Furthermore, as in [9], (i) if we take:

$$A\neq B,\quad B=-2E,\quad F=2E,\quad U(x,t)=\frac{1}{a}u(\frac{x}{\gamma}+\frac{\nu}{\delta}t,\frac{t}{\delta}),$$

with $\hat{k}\neq 0$, $a=\frac{2}{\varepsilon\hat{k}}(1-\nu)$, $\gamma^2=-\frac{1}{B\mu}$, $\nu=\frac{A}{B}$, and $\delta=\frac{\gamma}{\hat{k}}(1-\nu)$, then we recover the CH equation

$$U_t+\hat{k}U_x+3UU_x-U_{txx}=2U_xU_{xx}+UU_{xxx}.$$

(ii) If we take:

$$A\neq B,\quad B=-\frac{3}{8}E,\quad F=3E,\quad U(x,t)=\frac{1}{a}u(\frac{x}{\gamma}+\frac{\nu}{\delta}t,\frac{t}{\delta}),$$

with $\hat{k}\neq 0$, $a=\frac{8}{3\varepsilon\hat{k}}(1-\nu)$, $\gamma^2=-\frac{1}{B\mu}$, $\nu=\frac{A}{B}$, and $\delta=\frac{\gamma}{\hat{k}}(1-\nu)$, then we recover the DP equation

$$U_t+\hat{k}U_x+4UU_x-U_{txx}=3U_xU_{xx}+UU_{xxx}.$$

As using the governing equations for water waves to study the property of waves has proved intractable, many approximate model equations have been proposed, which are based on linear theory and therefore inadequate to explain potential nonlinear behaviours like wave breaking (meaning solutions that remain bounded while its slope becomes unbounded in finite time) or solitary waves. Hence many competing nonlinear models have been suggested to manage these phenomena. One of the most prominent examples is the CH equation, which has been studied extensively in the last twenty years because of its many remarkable properties: infinity of conservation laws and complete integrability [2, 14], existence of peaked solitons and multi-peakons [1, 2], well-posedness and breaking waves [4, 6, 7, 8], and so on.

The relevance of the CH equation as a model for the propagation of shallow water waves was discussed by Johnson [18]. Later, Constantin and Lannes derived the evolution equation (1.3) for the free surface which approximates the governing equation to the same order as the CH equation, and they also proved that the Cauchy problem on the line associated to (1.3), is locally well-posed [9]. Employing a semigroup approach due to Kato [19], Duruk showed that this result also holds true for a larger class of initial data [11], as well as for the corresponding spatially periodic Cauchy problem [12]. Shortly afterwards, Mi and Mu [23] discussed the local well-posedness of (1.3) in Besov spaces $B^s_{p,r}$, p, $r \in [1, +\infty]$, $s > \max\{\frac{3}{2}, 1+\frac{1}{p}\}$ by using Littlewood-Paley decomposition and transport equation theory, along with a study about analytic solutions and persistence properties of strong solutions. Besides, the equation (1.3) captures the non-linear phenomenon of wave breaking [9, 12]. This model equation also possesses solitary travelling wave solutions decaying at infinity [16] and their orbital stability has been studied in [13].

Following the ideas presented in [9], Samer Israwi derived equation (1.1), a model describing water waves over uneven bottoms [17]. Local well-posedness result of the initial value problem associated to (1.1) was first proved by Samer Israwi for initial data $u_0 \in H^s(\mathbb{R})$ with $s > 5/2$ [17]. In this article, we obtain the local well-posedness for the Cauchy problem corresponding to (1.1) for a class of initial data with less regular data $u_0 \in H^s(\mathbb{S})$, $s > 3/2$. The key point to get this desirable result is to transform (1.1) into the type of transport equation (3.4), which enables us to use Kato's theory. Furthermore, the blow-up criterion for periodic solutions of (1.1) is also presented in our paper. As for (1.2), a simplification of (1.1), we present the blow-up criterion in $H^s(\mathbb{S})$ with $s > 3/2$, an improvement compared with the parallel result in [17]. Besides, we give a sufficient condition (4.10) which ensures the occurrence of wave-breaking.

This article is organized as follows. In Section 2, we state the theory Kato proposed. In Section 3, we establish local well-posedness for periodic solutions of the Cauchy problem corresponding to (1.1). In Section 4, we investigate the wave-breaking phenomena of (1.1) and (1.2).

Notation. In this article, $a \lesssim b$ means that there is a uniform constant C that may be different on different lines, such that $a \leq Cb$. All of different positive constants might be denoted by the uniform constant C and C_κ denotes a constant related to κ.

2. Kato's theory

In this section, we state Kato's theorem in the form suitable for our purpose. We begin by fixing some notation. Let A denotes an operator, we denote by $D(A)$ the domain of the operator A. $[A, B]$ denotes the commutator of two linear operators A and B. $\|\cdot\|_X$ denotes the norm of the Banach space X.

Consider the abstract quasilinear equation

$$\begin{gathered} \frac{dv}{dt} + A(t,v)v = f(t,v), \quad t \geq 0, \\ v(0) = v_0. \end{gathered} \tag{2.1}$$

Let X and Y be Hilbert spaces, such that Y is continuously and densely embedded in X, and let $Q : Y \to X$ be a topological isomorphism. Let $L(Y, X)$ denotes the space of all bounded linear operators from Y to X ($L(X)$, if $X = Y$).

Assume the following:

(i) For each $t \geq 0$, $A(t, y) \in L(Y, X)$ for $y \in X$ with

$$\|(A(t,y) - A(t,z))w\|_X \leq \mu_1 \|y - z\|_X \|w\|_Y, \quad t \geq 0,\ y, z, w \in Y,$$

and $A(t, y) \in G(X, 1, \beta)$ (i.e., $A(t, y)$ is quasi-m-accretive), uniformly on bounded sets in Y.

(ii) $QA(t, y)Q^{-1} = A(t, y) + B(t, y)$, where $B(t, y) \in L(X)$ is bounded for each $t \geq 0$, uniformly on bounded sets in Y. Moreover,

$$\|(B(t,y) - B(t,z))w\|_X \leq \mu_2 \|y - z\|_Y \|w\|_X, \quad t \geq 0,\ y, z \in Y,\ w \in X.$$

(iii) For each $y \in Y$, $t \mapsto f(t, y)$ is continuous on $[0, +\infty)$. For each $t \geq 0$, $f(t, y) : Y \to Y$ and extends also to a map from X into X. f is uniformly bounded on bounded sets in Y, and

$$\begin{gathered}
\|(f(t,y) - f(t,z))\|_Y \leq \mu_3 \|y - z\|_Y, \quad t \geq 0,\ y, z \in Y, \\
\|(f(t,y) - f(t,z))\|_X \leq \mu_4 \|y - z\|_X, \quad t \geq 0,\ y, z, \in X.
\end{gathered}$$

Here μ_1, μ_2, μ_3, and μ_4 are constants depending only on $\max\{\|y\|_Y, \|z\|_Y\}$.

Theorem 2.1 ([19]). *Assume* (i)–(iii) *hold. Given $v_0 \in Y$, there is a maximal $T > 0$ depending only on $\|v_0\|_Y$ and a unique solution v to* (2.1)*, such that*

$$v = v(.; v_0) \in C([0, T); Y) \cap C^1([0, T); X).$$

Also, the map $v_0 \mapsto v(.; v_0)$ is continuous from Y to $C([0, T); Y) \cap C^1([0, T); X)$.

3. Local well-posedness

In this section, we will establish the local existence for periodic solutions to the Cauchy problem of (1.1) in $H^s(\mathbb{S})$ with $s > 3/2$ with $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ (the circle of unit length) by applying Kato's semigroup theorem. In sequence, $\|\cdot\|_s$ and $(\cdot, \cdot)_s$ denote the norm and the inner product of $H^s(\mathbb{S})$ respectively, and $b \in H^\infty(\mathbb{S})$.

First, we rewrite (1.1) in the form

$$\begin{aligned}
0 &= (1 - \mu m \partial_x^2) u_t - \frac{1}{m}(1 - \mu m \partial_x^2)(g u_x) + \frac{\varepsilon}{m}(1 - \mu m \partial_x^2)(h_1 u u_x) + k c_x u \\
&\quad + (\frac{g}{m} + c - \mu g_{xx}) u_x - 2\mu g_x u_{xx} + (\varepsilon \mu \partial_x^2 h_1 - \frac{\varepsilon}{m} h_1 - \varepsilon \mu \partial_x^2 h_2) u u_x \\
&\quad + \sum_{j \in J} \varepsilon^j f_j u^j u_x + \varepsilon \mu (2\partial_x h_1 - 2\partial_x h_2) u_x^2 + \varepsilon \mu (2\partial_x h_1 - \partial_x h_2) u u_{xx} \\
&\quad + \varepsilon \mu (3h_1 - 2h_2) u_x u_{xx}.
\end{aligned} \tag{3.1}$$

Then this equation is equivalent to

$$u_t + (-\frac{1}{m} g \partial_x + \frac{\varepsilon}{m} h_1 u \partial_x) u = F(u), \tag{3.2}$$

where

$$
\begin{aligned}
F(u) &= -(1-\mu m\partial_x^2)^{-1}[kc_x u + (\frac{g}{m} + c - \mu g_{xx})u_x - 2\mu g_x u_{xx} \\
&\quad + (\varepsilon\mu\partial_x^2 h_1 - \frac{\varepsilon}{m}h_1 - \varepsilon\mu\partial_x^2 h_2)uu_x + \sum_{j\in J}\varepsilon^j f_j u^j u_x \\
&\quad + \varepsilon\mu(2\partial_x h_1 - 2\partial_x h_2)u_x^2 + \varepsilon\mu(2\partial_x h_1 - \partial_x h_2)uu_{xx} + \varepsilon\mu(3h_1 - 2h_2)u_x u_{xx}] \\
&:= -(1-\mu m\partial_x^2)^{-1} f(u).
\end{aligned}
\tag{3.3}
$$

Now we present a local well-posedness result for the system

$$
\begin{gathered}
u_t + (-\frac{1}{m}g\partial_x + \frac{\varepsilon}{m}h_1 u\partial_x)u = F(u), \quad t>0,\ x\in\mathbb{R}, \\
u(0,x) = u_0(x), \quad x\in\mathbb{R}, \\
u(t,x+1) = u(t,x), \quad t>0,\ x\in\mathbb{R}, \\
b(x+1) = b(x), \quad x\in\mathbb{R}.
\end{gathered}
\tag{3.4}
$$

Theorem 3.1. *Given $u_0 \in H^s(\mathbb{S})$, $s>3/2$, there exists a maximal $T = T(u_0) > 0$ and a unique solution $u(t,x)$ to (3.4), such that*

$$
u = u(.;u_0) \in C([0,T);H^s(\mathbb{S})) \cap C^1\left([0,T);H^{s-1}(\mathbb{S})\right).
$$

Moreover, the solution depends continuously on the initial data; i.e., the mapping

$$
u_0 \mapsto u(.;u_0) : H^s(\mathbb{S}) \to C([0,T);H^s(\mathbb{S})) \cap C^1\left([0,T);H^{s-1}(\mathbb{S})\right)
$$

is continuous.

To prove our results, we apply Theorem 2.1 with $Y = H^s(\mathbb{S})$, $X = H^{s-1}(\mathbb{S})$, $s>3/2$, $Q = \Lambda = (1-\partial_x^2)^{1/2}$. Obviously, Q is an isomorphism of Y onto X. First of all, we need the following lemmas ensuring the validity of the assumptions (i)–(iii). For convenience, we may neglect the constant coefficients of the terms appearing in the evolution equation.

Lemma 3.2. *Let $A(u) = (g - h_1 u)\partial_x$ with $u\in H^s(\mathbb{S})$ and $s>3/2$. Then for each $t\ge 0$, $A(u)\in L(H^s(\mathbb{S}), H^{s-1}(\mathbb{S}))$ for $u\in H^s(\mathbb{S})$. Moreover,*

$$
\|(A(y)-A(z))w\|_{s-1} \le \mu_1\|y-z\|_{s-1}\|w\|_s, \quad t\ge 0,\ y,z,w\in H^s(\mathbb{S}).
$$

Proof. Let $y,z,w\in H^s(\mathbb{S})$, $s>3/2$. We have

$$
\begin{aligned}
\|(A(y)-A(z))w\|_{s-1} &\le \|h_1(y-z)w_x\|_{s-1} \\
&\le \|h_1(y-z)\|_{s-1}\|w_x\|_{s-1} \\
&\le \|h_1\|_{s-1}\|(y-z)\|_{s-1}\|w\|_s \\
&\le \mu_1\|(y-z)\|_{s-1}\|w\|_s,
\end{aligned}
$$

where $\mu_1 = \|h_1\|_{s-1}$. □

Next, we prove that $A(u)\in G(H^{s-1}(\mathbb{S}),1,\beta)$. First, we need the following lemmas.

Lemma 3.3 ([19]). *Let k, l be real numbers, such that $-k<l\le k$. Then*

$$
\|fg\|_l \le C\|f\|_k\|g\|_l, \quad \text{if } k>\frac{1}{2},
$$

where C is a positive constant depending on k, l.

Lemma 3.4 ([20]). *Let $f \in H^r, r > 3/2$. Then*
$$\|\Lambda^{-k}[\Lambda^{k+l+1}, M_f]\Lambda^{-l}\|_{L(L^2(\mathbb{S}))} \le C\|f\|_r, \quad |k|, |l| \le r-1,$$
where M_f is the operator of multiplication by f and C is a constant depending only on k, l.

Lemma 3.5 ([24]). *Let Z and X be two Banach spaces, such that X be continuously and densely embedded in Z. Let $-A$ be the infinitesimal generator of the C_0-semigroup $T(t)$ on Z and let Q be an isomorphism from X onto Z. Then X is $-A$-admissible [i.e., $T(t)X \subset X$; for all $t \ge 0$, and the restriction of $T(t)$ to X is a C_0-semigroup on X] if and only if $-A_1 = -QAQ^{-1}$ is the infinitesimal generator of the C_0-semigroup $T_1(t) = QT(t)Q^{-1}$ on Z. Moreover, if X is $-A$-admissible, then the part of $-A$ in X is the infinitesimal generator of the restriction of $T(t)$ to X.*

Lemma 3.6. *The operator $A(u) = (g - h_1 u)\partial_x$, with $u \in H^s(\mathbb{S})$, $s > 3/2$, belongs to $G(H^{s-1}(\mathbb{S}), 1, \beta)$.*

Proof. Because $H^{s-1}(\mathbb{S})$ is a Hilbert space, $A(u)$ belongs to $G(H^{s-1}(\mathbb{S}), 1, \beta)$ if and only if there is a real number, such that

(1) $(A(u)y, y)_{s-1} \ge -\beta\|y\|_{s-1}^2, y \in H^{s-1}(\mathbb{S})$,
(2) $-A(u)$ is the infinitesimal generator of a C_0-semigroup on $H^{s-1}(\mathbb{S})$.

First, let us prove (1). Since $u \in H^s(\mathbb{S}), s > 3/2$, it follows that u and u_x belong to $L^\infty(\mathbb{S})$, and $\|u\|_{L^\infty(\mathbb{S})}, \|u_x\|_{L^\infty(\mathbb{S})} \le \|u\|_s$. Note that
$$\begin{aligned}
\Lambda^{s-1}((g - h_1 u)\partial_x y) &= [\Lambda^{s-1}, g - h_1 u]\partial_x y + (g - h_1 u)\Lambda^{s-1}\partial_x y \\
&= [\Lambda^{s-1}, g - h_1 u]\partial_x y + (g - h_1 u)\partial_x \Lambda^{s-1} y.
\end{aligned}$$
Then we have
$$\begin{aligned}
(A(u)y, y)_{s-1} &= ([\Lambda^{s-1}, g - h_1 u]\partial_x \Lambda^{1-s}\Lambda^{s-1} y, \Lambda^{s-1} y)_0 \\
&\quad - \frac{1}{2}(\partial_x(g - h_1 u)\Lambda^{s-1} y, \Lambda^{s-1} y)_0 \\
&\le \|[\Lambda^{s-1}, g - h_1 u]\Lambda^{-(s-2)}\|_{L(L^2(\mathbb{S}))}\|\Lambda^{s-1} y\|_{L^2(\mathbb{S})}^2 \\
&\quad + \|g_x - \partial_x h_1 u + h_1 u_x\|_{L(L^\infty(\mathbb{S}))}\|\Lambda^{s-1} y\|_{L^2(\mathbb{S})}^2 \\
&\le C\|g - h_1 u\|_s \|y\|_{s-1}^2 + C\|u\|_s \|y\|_{s-1}^2 \\
&\le C\|u\|_s \|y\|_{s-1}^2,
\end{aligned}$$
where we have applied Lemma 3.4 with $k = 0, l = s - 2$. Let $\beta = C\|u\|_s$. Then
$$(A(u)y, y)_{s-1} \ge -\beta\|y\|_{s-1}^2.$$

Next, we prove (2). Let $Q = \Lambda^{s-1}$. Note that Q is an isomorphism of $H^{s-1}(\mathbb{S})$ onto $L^2(\mathbb{S})$ and that $H^{s-1}(\mathbb{S})$ is continuously and densely embedded in $L^2(\mathbb{S})$ as $s > 3/2$. Define
$$A_1(u) := QA(u)Q^{-1} = \Lambda^{s-1}A(u)\Lambda^{1-s}, \quad B_1(u) = A_1(u) - A(u).$$
Let $y \in L^2(\mathbb{S})$ and $u \in H^s(\mathbb{S}), s > 3/2$. Then we have
$$\begin{aligned}
\|B_1(u)y\|_0 &= \|[\Lambda^{s-1}, A]\Lambda^{1-s} y\|_0 \\
&\le \|[\Lambda^{s-1}, g - h_1 u]\Lambda^{2-s}\|_{L(L^2(\mathbb{S}))}\|\Lambda^{-1}\partial_x y\|_0 \\
&\le C\|u\|_s\|y\|_0 \le C\|y\|_0.
\end{aligned}$$

The above inequality implies $B_1(u) \in L(L^2(\mathbb{S}))$.

Note that $A_1(u) = A(u) + B_1(u)$. By a perturbation theorem for semigroups [24, Sec. 5.2 Theorem 2.3], we obtain $A_1 \in G(L^2(\mathbb{S}), 1, \beta')$ provided $A \in G(L^2(\mathbb{S}), 1, \beta')$. Applying Lemma 3.5 with $X = H^{s-1}(\mathbb{S})$, $Z = L^2(\mathbb{S})$ and $Q = \Lambda^{s-1}$, we conclude that $H^{s-1}(\mathbb{S})$ is $-A(u)$-admissible. Therefore, $-A(u)$ is the infinitesimal generator of a C_0-semigroup on $H^{s-1}(\mathbb{S})$. This will complete the proof of Lemma 3.6. $\square$

To complete the proof of Lemma 3.6, it remains to prove $A \in G(L^2(\mathbb{S}), 1, \beta')$.

Lemma 3.7. *The operator $A(u) = (g - h_1 u)\partial_x$, with $u \in H^s(\mathbb{S})$, $s > 3/2$, belongs to $G(L^2(\mathbb{S}), 1, \beta')$.*

Proof. Because $L^2(\mathbb{S})$ is a Hilbert space, $A(u) \in G(L^2(\mathbb{S}), 1, \beta')$ [21] if and only if there is a real number β', such that

(1) $(A(u)y, y)_0 \geq -\beta' \|y\|_0^2, y \in L^2(\mathbb{S})$,
(2) the range of $A + \lambda$ is all of X, for some (or all) $\lambda > \beta'$.

First, let us prove (1),

$$\begin{aligned}
(A(u)y, y)_0 &= ((g - h_1 u)\partial_x y, y)_0 \\
&= -\frac{1}{2}(\partial_x (g - h_1 u)y, y)_0 \\
&\leq \frac{1}{2}\|u_x\|_{L^\infty(\mathbb{S})}\|y\|_0^2 \leq C\|u\|_s\|y\|_0^2.
\end{aligned}$$

Setting $\beta' = C\|u\|_s$, we have $(A(u)y, y)_0 \geq -\beta'\|y\|_0^2$.

Next, we prove (2). Because $A(u)$ is a closed operator and satisfies (1), it follows that $(\lambda I + A)$ has closed range in $L^2(\mathbb{S})$ for all $\lambda > \beta'$. Thus, it suffices to show $(\lambda I + A)$ has dense range in $L^2(\mathbb{S})$ for all $\lambda > \beta'$.

Given $u \in H^s(\mathbb{S})$, $s > 3/2$, $y \in L^2(\mathbb{S})$, we obtain

$$\partial_x (g - h_1 u)y = (g_x - \partial_x h_1 u - h_1 u_x)y \in L^2(\mathbb{S}), \quad y \in L^2(\mathbb{S}).$$

Then

$$\begin{aligned}
D(A) &= \{y \in L^2(\mathbb{S}), (g - h_1 u)\partial_x y \in L^2(\mathbb{S})\} \\
&= \{z \in L^2(\mathbb{S}), -\partial_x((g - h_1 u)z) \in L^2(\mathbb{S})\} \\
&= D(A^*).
\end{aligned}$$

Assume that the range of $(\lambda I + A)$ is not all of $L^2(\mathbb{S})$. Then there exists $z \in L^2(\mathbb{S})$, $z \neq 0$, such that $((\lambda I + A)y, z)_0 = 0$ for all $y \in D(A)$. Since $H^1(\mathbb{S}) \subset D(A)$, we have that $D(A) = D(A^*)$ is dense in $L^2(\mathbb{S})$. This means that there exists a sequence $z_k \in D(A^*)$ which converges to an element $z \in L^2(\mathbb{S})$. Recalling that $D(A^*)$ is closed, we find that $z \in D(A^*)$ and $\lambda z + A^* z = 0$ in $L^2(\mathbb{S})$. Note that $D(A) = D(A^*)$. Multiplying by z and then integrating by parts, we obtain

$$0 = ((\lambda I + A^*)z, z)_0 = (\lambda z, z)_0 + (z, Az)_0 \geq (\lambda - \beta')\|z_0\|_0^2, \lambda > \beta'.$$

Thus, we obtain $z = 0$. This contradicts the previous assumption $z \neq 0$ and completes the proof. $\square$

Lemma 3.8. $B(u) = \Lambda A(u)\Lambda^{-1} - A = [\Lambda, A(u)]\Lambda^{-1} \in L(H^{s-1}(\mathbb{S}))$, *for* $u \in H^s(\mathbb{S})$. *Moreover,*

$$\|(B(y) - B(z))w\|_{s-1} \leq \mu_2\|y - z\|_s\|w\|_{s-1}, \; y, z \in H^s(\mathbb{S}), w \in H^{s-1}(\mathbb{S}).$$

Proof.
$$B(u) = \Lambda A(u)\Lambda^{-1} - A(u) = \Lambda A(u)\Lambda^{-1} - A(u)\Lambda\Lambda^{-1} = [\Lambda, A(u)]\Lambda^{-1},$$
and for $y, z \in H^s(\mathbb{S})$, $w \in H^{s-1}(\mathbb{S})$, we have
$$\begin{aligned}
\|(B(y) - B(z))w\|_{s-1} &= \|\Lambda^{s-1}[\Lambda, (A(y) - A(z)]\Lambda^{-1}w\|_0 \\
&\le \|\Lambda^{s-1}[\Lambda, h_1(y-z)]\Lambda^{-1}\partial_x w\|_0 \\
&\le \|\Lambda^{s-1}[\Lambda, h_1(y-z)]\Lambda^{1-s}\|_{L(L^2(\mathbb{S}))}\|\Lambda^{s-2}\partial_x w\|_0 \\
&\le \|h_1(y-z)\|_s\|w\|_{s-1} \\
&\le \mu_2\|(y-z)\|_s\|w\|_{s-1}.
\end{aligned}$$
Taking $z = 0$ in the above inequality, we obtain $B(u) \in L(H^{s-1}(\mathbb{S}))$, $t \ge 0$, $u \in H^s(\mathbb{S})$. This completes the proof. □

Lemma 3.9. *The function $F(u)$ defined by (3.3) is uniformly bounded on bounded sets in $H^s(\mathbb{S})$, and for all $s > 3/2$, it satisfies*

(1) $\|F(y) - F(z)\|_s \le \mu_3\|y - z\|_s$,
(2) $\|F(y) - F(z)\|_{s-1} \le \mu_4\|y - z\|_{s-1}$.

Proof. Observe that $F(u) = -(1 - \mu m\partial_x^2)^{-1}f(u)$ and
$$\begin{aligned}
\|(1 - \mu m\partial_x^2)^{-1}f(u)\|_s &= \Big(\sum_{k=-\infty}^{+\infty}(1+|k|^2)^s|\mathcal{F}((1-\mu m\partial_x^2)^{-1}f)(k)|^2\Big)^{1/2} \\
&= \Big(\sum_{k=-\infty}^{+\infty}(1+|k|^2)^s|\mathcal{F}[\mathcal{F}^{-1}((1+\mu m|k|^2)^{-1}\mathcal{F}f)(k)]|^2\Big)^{1/2} \\
&= \Big(\sum_{k=-\infty}^{+\infty}(1+|k|^2)^s|(1+\mu m|k|^2)^{-1}\hat{f}(k)|^2\Big)^{1/2} \\
&= \Big(\sum_{k=-\infty}^{+\infty}(1+|k|^2)^s(1+\mu m|k|^2)^{-2}|\hat{f}(k)|^2\Big)^{1/2} \\
&= \Big(\sum_{k=-\infty}^{+\infty}(1+|k|^2)^s(\mu m)^{-2}(\frac{1}{\mu m}+|k|^2)^{-2}|\hat{f}(k)|^2\Big)^{1/2} \\
&\le C\Big(\sum_{k=-\infty}^{+\infty}(1+|k|^2)^s(1+|k|^2)^{-2}|\hat{f}(k)|^2\Big)^{1/2} \\
&= C\|f(u)\|_{s-2},
\end{aligned}$$
where we used that $|\mu m| < 1$. Thus,
$$\begin{aligned}
&\|F(y) - F(z)\|_{s-1} \\
&\le \|f(y) - f(z)\|_{s-3} \\
&\le \|kc_x(y-z)\|_{s-3} + \|(\frac{g}{m} + c - \mu g_{xx})(y_x - z_x)\|_{s-3} \\
&\quad + \|2\mu g_x(y_{xx} - z_{xx})\|_{s-3} + \|(\varepsilon\mu\partial_x^2 h_1 - \frac{\varepsilon}{m}h_1 - \varepsilon\mu\partial_x^2 h_2)(yy_x - zz_x)\|_{s-3} \\
&\quad + \|\sum_{j\in J}\varepsilon^j f_j(y^j y_x - z^j z_x)\|_{s-3} + \|\varepsilon\mu(2\partial_x h_1 - 2\partial_x h_2)(y_x^2 - z_x^2)\|_{s-3}
\end{aligned}$$

$$
\begin{aligned}
&+ \|\varepsilon\mu(2\partial_x h_1 - \partial_x h_2)(yy_{xx} - zz_{xx})\|_{s-3} \\
&+ \|\varepsilon\mu(3h_1 - 2h_2)(y_x y_{xx} - z_x z_{xx})\|_{s-3}.
\end{aligned}
$$

Now, we estimate each of the items above.

$$
\begin{aligned}
\|kc_x(y-z)\|_{s-3} &\lesssim \|y-z\|_{s-3} \lesssim \|y-z\|_{s-1}, \\
\|(\frac{g}{m} + c - \mu g_{xx})(y_x - z_x)\|_{s-3} &\lesssim \|y-z\|_{s-2} \lesssim \|y-z\|_{s-1}, \\
\|2\mu g_x(y_{xx} - z_{xx})\|_{s-3} &\lesssim \|y-z\|_{s-1}, \\
\|(\varepsilon\mu\partial_x^2 h_1 - \frac{\varepsilon}{m} h_1 - \varepsilon\mu\partial_x^2 h_2)(yy_x - zz_x)\|_{s-3} &\lesssim \|y^2 - z^2\|_{s-2} \\
&\lesssim \|y+z\|_{s-1}\|y-z\|_{s-2} \\
&\lesssim (\|y\|_{s-1} + \|z\|_{s-1})\|y-z\|_{s-1}, \\
\|\sum_{j\in J} \varepsilon^j f_j(y^j y_x - z^j z_x)\|_{s-3} &\lesssim \|\sum_{j\in J}(y^{j+1} - z^{j+1})\|_{s-2} \\
&\lesssim \sum_{j\in J} \|y^{j+1} - z^{j+1}\|_{s-2} \\
&\lesssim \sum_{j\in J} \|y-z\|_{s-2}\|y^j + y^{j-1}z + \dots + z^j\|_{s-1} \\
&\le C_{(\|y\|_{s-1}, \|z\|_{s-1})}\|y-z\|_{s-1}, \\
\|\varepsilon\mu(2\partial_x h_1 - 2\partial_x h_2)(y_x^2 - z_x^2)\|_{s-3} &\lesssim \|\partial_x(y+z)\partial_x(y-z)\|_{s-3} \\
&\lesssim \|\partial_x(y+z)\|_{s-1}\|\partial_x(y-z)\|_{s-3} \\
&\lesssim (\|y\|_s + \|z\|_s)\|y-z\|_{s-1}, \\
\|\varepsilon\mu(3h_1 - 2h_2)(y_x y_{xx} - z_x z_{xx})\|_{s-3} &\lesssim \|y_x^2 - z_x^2\|_{s-2} \\
&\lesssim (\|y\|_s + \|z\|_s)\|y-z\|_{s-1}, \\
\|\varepsilon\mu(2\partial_x h_1 - \partial_x h_2)(yy_{xx} - zz_{xx})\|_{s-3} &\lesssim \|(yy_x)_x - y_x^2 - (zz_x)_x + z_x^2\|_{s-3} \\
&\lesssim \|yy_x - zz_x\|_{s-2} + \|y_x^2 - z_x^2\|_{s-3} \\
&\lesssim \|y^2 - z^2\|_{s-1} + \|y_x^2 - z_x^2\|_{s-3} \\
&\lesssim (\|y\|_{s-1} + \|z\|_{s-1} + \|y\|_s + \|z\|_s)\|y-z\|_{s-1},
\end{aligned}
$$

here we have used the imbedding property of Sobolev spaces $H^s(\mathbb{S})$ (i.e., if $s_1 \le s_2$, then $\|\cdot\|_{s_1} \le \|\cdot\|_{s_2}$), and Cauchy-Schwarz inequality. So, we obtain

$$
\|F(y) - F(z)\|_{s-1} \le \mu_4\|y-z\|_{s-1}.
$$

Similarly, we can obtain $\|F(y) - F(z)\|_s \le \mu_3\|y-z\|_s$. This completes the proof. □

Proof of Theorem 3.1. Combining Theorem 2.1 and Lemmas 3.2, 3.6, 3.8, 3.9, we have the proof of Theorem 3.1. □

Theorem 3.10. *The existence time $T > 0$ in Theorem 3.1 can be chosen independently of s in the following sense. If $u \in C([0,T); H^s(\mathbb{S})) \cap C^1([0,T); H^{s-1}(\mathbb{S}))$ is a solution of (3.4), and if $u_0 \in H^{s'}(\mathbb{S})$ for some $s' \ne s$, $s' > 3/2$, then $u \in C([0,T); H^{s'}(\mathbb{S})) \cap C^1([0,T); H^{s'-1}(\mathbb{S}))$ with the same T. In particular, if $u_0 \in H^\infty(\mathbb{S})$, then $u \in ([0,T); H^\infty(\mathbb{S}))$.*

Proof. If $s' < s$, the result follows from the uniqueness of the solution guaranteed by Theorem 3.1 and $H^s(\mathbb{S}) \subset H^{s'}(\mathbb{S})$. So it suffices to consider the case $s' > s$. We suppose that $s < s' \leq s+1$, otherwise if $s' > s+1$, we can obtain the result by iterated application of the argument below.

For $u \in C([0,T); H^s(\mathbb{S})) \cap C^1([0,T); H^{s-1}(\mathbb{S}))$ and $u_0 \in H^{s'}(\mathbb{S})$, set $y(t) = (1-\mu m\partial_x^2)u(t,x)$, and

$$\begin{gathered}
A(t)y = \partial_x((-\frac{1}{m}g + \frac{\varepsilon}{m}h_1 u)y), \\
B(t)y = [\frac{1}{m}g_x - \frac{\varepsilon}{m}(\partial_x(h_1 u) + \partial_x(h_2 u) + h_2 u_x)]y, \\
f(t) = -cu_x - kc_x u - \sum_{j\in J}\varepsilon^j f_j u^j u_x - \frac{1}{m}g u_x + \frac{\varepsilon}{m}h_1 u u_x \\
+ \varepsilon\mu\partial_x^2 h_2 u u_x + 2\varepsilon\mu\partial_x h_2 u_x^2 + \frac{\varepsilon}{m}\partial_x(h_2 u)u + \frac{\varepsilon}{m}h_2 u u_x.
\end{gathered}$$

From (3.4) we obtain the abstract evolution equation

$$\frac{dy}{dt} + A(t)y + B(t)y = f(t), \quad y(0) = u(0) - \mu m\partial_x^2 u(0).$$

Since $u \in C([0,T); H^s(\mathbb{S}))$ and $u_0 \in H^{s'}(\mathbb{S})$, it follows that $y \in C([0,T); H^{s-2}(\mathbb{S}))$ and $y(0) \in H^{s'-2}(\mathbb{S})$. It is our purpose to show $y \in C([0,T); H^{s'-2}(\mathbb{S}))$ for the same T, which implies that $u \in C([0,T); H^{s'}(\mathbb{S}))$, because $(1-\mu m\partial_x^2)$ is an isomorphism form $H^{s'}(\mathbb{S})$ to $H^{s'-2}(\mathbb{S})$ ($|\mu m| < 1$). This will complete the proof.

Following the argument in [20], it is easy to see that the family $A(t)$ generates a unique evolution operator $U(t,\tau)$ associated with the space $X = H^l(\mathbb{S})$ and $Y = H^k(\mathbb{S})$, where $-s \leq l \leq s-2$, $1-s \leq k \leq s-1$, and $k \geq l+1$. Accordingly, an evolution operator $U(t,\tau)$ for the family $A(t)$ exists and is unique. In particular, $U(t,\tau)$ maps $H^r(\mathbb{S})$ into itself for $-s \leq r \leq s-1$.

Choose $X = H^{s-3}(\mathbb{S})$ and $Y = H^{s-2}(\mathbb{S})$. Obviously,

$$y \in C([0,T); H^{s-2}(\mathbb{S})) \cap C^1([0,T); H^{s-3}(\mathbb{S})).$$

By the properties of the evolution operator $U(t,\tau)$, we obtain

$$\frac{d}{d\tau}(U(t,\tau)y(\tau)) = U(t,\tau)(-B(\tau)y(\tau) + f(\tau)).$$

Integrating with respect to $\tau \in [0,t]$ gives

$$y(t) = U(t,0)y(0) + \int_0^t U(t,\tau)(-B(\tau)y(\tau) + f(\tau))d\tau.$$

If $s < s' \leq s+1$, we have

$$\begin{gathered}
f(t) \in C\left([0,T); H^{s-1}(\mathbb{S})\right) \subset C\left([0,T); H^{s'-2}(\mathbb{S})\right), \\
B(t) = [\frac{1}{m}g_x - \frac{\varepsilon}{m}(\partial_x(h_1 u) + \partial_x(h_2 u) + h_2 u_x)](t) \in L(H^{s'-2}(\mathbb{S}))
\end{gathered}$$

is strongly continuous in $[0,t)$, and

$$H^{s-1}(\mathbb{S})H^{s'-2}(\mathbb{S}) \subset H^{s'-2}(\mathbb{S}).$$

Due to $-s < s-2 < s'-2 < s-1$, the family $\{U(t,\tau)\}$ is strongly continuous from $H^{s'-2}(\mathbb{S})$ into itself. Observe that $y(0) \in H^{s'-2}(\mathbb{S})$, (3) can be regarded as an

Volterra-type integral equation and can be solved for y by successive approximation. This completes the proof of the theorem. $\square$

4. Wave breaking

In this section, we address the question of the formation of singularities for solutions to (1.1) and we also give some blow up results for (1.2).

4.1. **Blow-up condition for** (1.1)**.** As in the case of flat bottoms, it is possible to give some information on the blow-up pattern for (1.1). First we rewrite (1.1) (i.e., the first equation in (3.4)) in an equivalent form that is better suited for our purpose:

$$u_t + (-\frac{1}{m}g + \frac{\varepsilon}{m}h_1 u)u_x = f(t,u) \tag{4.1}$$

with

$$\begin{aligned}
&f(t,u)\\
&= -(1-\mu m\partial_x^2)^{-1}\Big[kc_x u + (\frac{g}{m} + c + \mu g_{xx})u_x\\
&\quad - (\varepsilon\mu\partial_x^2 h_1 + \frac{\varepsilon}{m}h_1)uu_x + \sum_{j\in J}\varepsilon^j f_j u^j u_x - \frac{3}{2}\varepsilon\mu\partial_x h_1 u_x^2\Big]\\
&\quad - \partial_x(1-\mu m\partial_x^2)^{-1}[-2\mu g_x u_x + \varepsilon\mu(2\partial_x h_1 - \partial_x h_2)uu_x + \varepsilon\mu(\frac{3}{2}h_1 - h_2)u_x^2]\\
&:= -P * f_1(t,u) - \partial_x P * f_2(t,u),
\end{aligned} \tag{4.2}$$

where $*$ denotes the convolution and $P(x)$ stands for the Green's function of the operator $(1-\mu m\partial_x^2)$ in the periodic case. Before giving the result, we need the following lemmas.

Lemma 4.1 ([22])**.** *If $s > 0$, then*

$$\|[\Lambda^s, g]f\|_{L^2(\mathbb{S})} \le C(\|\partial_x g\|_{L^\infty(\mathbb{S})}\|\Lambda^{s-1}f\|_{L^2(\mathbb{S})} + \|\Lambda^s g\|_{L^2(\mathbb{S})}\|f\|_{L^\infty(\mathbb{S})}),$$

where C is a constant depending only on s.

Lemma 4.2 ([22])**.** *Assume that $s > 0$. Then $H^s(\mathbb{S}) \cap L^\infty(\mathbb{S})$ is an algebra. Moreover,*

$$\|fg\|_s \le C(\|f\|_{L^\infty(\mathbb{S})}\|g\|_s + \|f\|_s\|g\|_{L^\infty(\mathbb{S})}),$$

where C is a constant depending only on s.

Theorem 4.3. *Assume $b \in H^\infty(\mathbb{S})$ and let $u_0 \in H^s(\mathbb{S})$ with $s > 3/2$. If T is the existence time of the corresponding solution of initial data u_0, then the $H^s(\mathbb{S})$ -norm of $u(t,x)$ to* (1.1) *blows up on $[0,T)$ if and only if*

$$\limsup_{t\uparrow T}\{\|u(t,x)\|_{L^\infty(\mathbb{S})} + \|u_x(t,x)\|_{L^\infty(\mathbb{S})}\} = +\infty.$$

Proof. Let $u(t,x)$ be the solution of (1.1) with the initial data $u_0 \in H^s(\mathbb{S})$, $s > 3/2$, which is guaranteed by Theorem 3.1. If

$$\limsup_{t\uparrow T}\{\|u(t,x)\|_{L^\infty(\mathbb{S})} + \|u_x(t,x)\|_{L^\infty(\mathbb{S})}\} = +\infty,$$

by Sobolev's embedding theorem, we obtain that the solution $u(t,x)$ will blows up in finite time.

Next, we prove that if there exists $M > 0$ such that

$$\limsup_{t\uparrow T}\{\|u(t,x)\|_{L^\infty(\mathbb{S})} + \|u_x(t,x)\|_{L^\infty(\mathbb{S})}\} \le M,$$

then $\|u(t)\|_{H^s(\mathbb{S})}$ with $s > \frac{3}{2}$ remains bounded on $[0,T)$.

Applying the operator Λ^s to (4.1), multiplying the obtained equation by $\Lambda^s u$, and integrating with respect to x over [0,1], we obtain

$$\frac{d}{dt}(u,u)_s = -2((-\frac{1}{m}g\partial_x + \frac{\varepsilon}{m}h_1 u\partial_x)u,u)_s + 2(f(t,u),u)_s. \tag{4.3}$$

Similar to [15], using Lemma 4.1, we obtain

$$|(-\frac{1}{m}g\partial_x + \frac{\varepsilon}{m}h_1 u\partial_x)u,u)_s| \le C(\|u\|_{L^\infty(\mathbb{S})} + \|u_x\|_{L^\infty(\mathbb{S})})\|u\|_s^2 \le CM\|u\|_s^2. \tag{4.4}$$

On the other hand, we estimate the second term on the right-hand side of (4.3) as

$$\begin{aligned}
&(f(t,u),u)_s\\
&= (-P * f_1(t,u) - \partial_x P * f_2(t,u),u)_s\\
&\lesssim \|u\|_s(\|f_1(t,u)\|_{s-1} + \|f_2(t,u)\|_{s-1})\\
&\lesssim \|u\|_s(\|c_x u\|_{s-1} + \|(\frac{g}{m} + c + \mu g_{xx})u_x\|_{s-1} + \|(\varepsilon\mu\partial_x^2 h_1 + \frac{\varepsilon}{m}h_1)uu_x\|_{s-1}\\
&\quad + \|\sum_{j\in J}\varepsilon^j f_j u^j u_x\|_{s-1} + \|\frac{3}{2}\varepsilon\mu\partial_x h_1 u_x^2\|_{s-1} + \|2\mu g_x u_x\|_{s-1}\\
&\quad + \|\varepsilon\mu(2\partial_x h_1 - \partial_x h_2)uu_x\|_{s-1} + \|\varepsilon\mu(\frac{3}{2}h_1 - h_2)u_x^2\|_{s-1}).
\end{aligned} \tag{4.5}$$

Now we estimate the above items individually.

$$\begin{gathered}
\|c_x u\|_{s-1} \lesssim \|u\|_s,\\
\|(\frac{g}{m} + c + \mu g_{xx})u_x\|_{s-1} \lesssim \|u\|_s,\\
\|(\varepsilon\mu\partial_x^2 h_1 + \frac{\varepsilon}{m}h_1)uu_x\|_{s-1} \lesssim \|\partial_x(u^2)\|_{s-1} \lesssim \|u^2\|_s \lesssim \|u\|_{L^\infty(\mathbb{S})}\|u\|_s,\\
\|\sum_{j\in J}\varepsilon^j f_j u^j u_x\|_{s-1} \lesssim \sum_{j\in J}\|u^{j+1}\|_s \lesssim \|u\|_s\sum_{j\in J}\|u\|_{L^\infty(\mathbb{S})}^j \lesssim C_{\|u\|_{L^\infty(\mathbb{S})}}\|u\|_s,\\
\|\frac{3}{2}\varepsilon\mu\partial_x h_1 u_x^2\|_{s-1} \lesssim \|u_x^2\|_{s-1} \lesssim \|u_x\|_{L^\infty(\mathbb{S})}\|u_x\|_{s-1} \lesssim \|u_x\|_{L^\infty(\mathbb{S})}\|u\|_s,\\
\|2\mu g_x u_x\|_{s-1} \lesssim \|u\|_s,\\
\|\varepsilon\mu(2\partial_x h_1 - \partial_x h_2)uu_x\|_{s-1} \lesssim \|u\|_{L^\infty(\mathbb{S})}\|u\|_s,\\
\|\varepsilon\mu(\frac{3}{2}h_1 - h_2)u_x^2\|_{s-1} \lesssim \|u_x\|_{L^\infty(\mathbb{S})}\|u\|_s,
\end{gathered}$$

where we have used Lemma 4.2 and the imbedding property of Sobolev spaces $H^s(\mathbb{S})$. Inserting the above set of inequalities into (4.5), we obtain

$$(f(t,u),u)_s \le C_M\|u\|_s^2. \tag{4.6}$$

From (4.3), (4.4) and (4.6), we obtain

$$\frac{d}{dt}\|u\|_s^2 \le C_M\|u\|_s^2.$$

In view of Gronwall's inequality, we have

$$\|u\|_s^2 \le \|u_0\|_s^2 e^{C_M t}.$$

This means $\|u\|_s^2$ does not blow up in finite time under the assumption of the Theorem. This completes the proof. $\square$

4.2. **Blow-up results for (1.2).** In the following, we deduce that for solutions of the evolution equation

$$\begin{aligned} &u_t + cu_x + \frac{1}{2}c_x u + \frac{3}{2}\varepsilon u u_x - \frac{3}{8}\varepsilon^2 u^2 u_x + \frac{3}{16}\varepsilon^3 u^3 u_x + \frac{\mu}{12}(u_{xxx} - u_{xxt}) \\ &= -\frac{7}{24}\varepsilon\mu(u u_{xxx} + 2u_x u_{xx}), \end{aligned} \tag{4.7}$$

singularities can occur in finite time only in the form of wave breaking, more specifically surging breakers. In other words, there exists a breaking time for the solution which remains bounded while its slope becomes unbounded.

Proposition 4.4. *Let $b \in H^\infty(\mathbb{S})$. If for some initial data $u_0 \in H^s(\mathbb{S})$, $s > 3/2$, the maximal existence time $T > 0$ of the periodic solution to (1.2) is finite, then the solution $u(t,x) \in C([0,T), H^s(\mathbb{S})) \cap C^1([0,T), H^{s-1}(\mathbb{S}))$ satisfies:*

$$\sup_{t\in[0,T),x\in[0,1]} \{|u(t,x)|\} < \infty, \quad \lim_{t\uparrow T} \sup_{x\in[0,1]} \{u_x(t,x)\} = +\infty.$$

Proof. By Theorems 3.1 and 3.10 and a simple density argument, the bow-up conditions for (1.2) in [17] in the Sobolev space $H^s(\mathbb{S})$ with $s \geq 3$ are correct in $H^s(\mathbb{S})$ with $s > 3/2$. Thus, we obtain the above proposition. $\square$

Next we show that there exist solutions to (1.2) that blow up in finite time in the form of breaking waves. From Proposition 4.4, we know that to ensure the blow-up solutions exist, its key to guarantee the existence of at least one real valued point where the supremum of the slope approaches infinity. Therefore, we analyze the equation that describes the evolution of

$$S(t) := \sup_{x\in[0,1]} \{u_x(t,x)\}. \tag{4.8}$$

Before giving the result, we need to reformulate (1.2). Applying $(1 - \frac{\mu}{12}\partial_x^2)$ to (1.2), we obtain

$$\begin{aligned} &u_t + \tilde{P}_x * (cu) - \frac{1}{2}\tilde{P} * (c_x u) + \frac{3\varepsilon}{4}\tilde{P}_x * u^2 - \frac{\varepsilon^2}{8}\tilde{P}_x * u^3 + \frac{3\varepsilon^3}{64}\tilde{P}_x * u^4 \\ &+ \frac{\mu}{12}\partial_x^3 \tilde{P} * u + \frac{7\varepsilon\mu}{24}\tilde{P}_x * u_x^2 + \frac{7\varepsilon\mu}{24}\tilde{P} * (u u_{xxx}) = 0, \end{aligned}$$

where $\tilde{P}(x)$ is the Green function of the operator $(1 - \frac{\mu}{12}\partial_x^2)$ in the periodic case. Differentiating this equation with respect to x, we obtain

$$\begin{aligned} &u_{xt} + \partial_x^2 \tilde{P} * (cu) - \frac{1}{2}\tilde{P} * (c_x u)_x + \frac{3}{4}\varepsilon \partial_x^2 \tilde{P} * u^2 - \frac{\varepsilon^2}{8}\partial_x^2 \tilde{P} * u^3 + \frac{3\varepsilon^3}{64}\partial_x^2 \tilde{P} * u^4 \\ &+ \frac{\mu}{12}\partial_x^4 \tilde{P} * u + \frac{7\varepsilon\mu}{24}\partial_x^2 \tilde{P} * u_x^2 + \frac{7\varepsilon\mu}{24}\tilde{P}_x * (u u_{xxx}) = 0. \end{aligned}$$

Noticing the identity $u u_{xxx} = \partial_x^2(u u_x) - 3u_x u_{xx}$ and using the fact

$$\partial_x^2 \tilde{P} * f = \frac{12}{\mu}\tilde{P} * f - \frac{12}{\mu} f,$$

we deduce that

$$\begin{aligned} &u_{xt} - u_{xx} - \frac{7\varepsilon}{4}u_x^2 - \frac{7\varepsilon}{4}\tilde{P} * u_x^2 - \frac{7\varepsilon}{2}uu_{xx} \\ &- \frac{1}{2}\tilde{P} * (c_x u)_x + \frac{12}{\mu}\tilde{P} * g(u) - \frac{12}{\mu}g(u) = 0, \end{aligned} \tag{4.9}$$

where

$$g(u) = (1 + c)u + \frac{5\varepsilon}{2}u^2 - \frac{\varepsilon^2}{8}u^3 + \frac{3\varepsilon^3}{64}u^4.$$

Also, we denote

$$\|\tilde{P}(x)\|_{L^1[0,1]} := n_1, \quad \|\tilde{P}(x)\|_{L^2[0,1]} := n_2, \quad \|\tilde{P}(x)\|_{L^\infty[0,1]} := n_\infty.$$

Then we present a condition which guarantees the solutions must blow up in finite time.

Proposition 4.5. *If the initial wave profile $u_0 \in H^3(\mathbb{S})$ satisfies*

$$\begin{aligned} |\inf_{x\in[0,1]}\{\partial_x u_0(x)\}|^2 > \frac{12}{\varepsilon\mu}[(n_\infty + M)(\frac{17\varepsilon}{4}C_0 + \frac{\varepsilon^2}{8}\sqrt{M}C_0^{3/2} + \frac{3\varepsilon^3}{64}MC_0^2) \\ + (1 + C_1)(n_2 + \sqrt{M})\sqrt{C_0}] + \frac{(1 + M)}{2}n_\infty C_1\sqrt{C_0}, \end{aligned} \tag{4.10}$$

where

$$C_0 = \int_0^1 (u_0^2 + \frac{\mu}{12}u_{0x}^2)dx > 0, \quad C_1 = \|c\|_{W^{2,\infty}(\mathbb{S})}, \quad M = \max\{\frac{13}{\mu}, \frac{13}{12}\},$$

then wave breaking for the solutions of (1.2) *occurs in finite time, $T = O(1/\varepsilon)$.*

Proof. In view of [5, Lemma 2], for $u \in H^3(\mathbb{S})$,

$$\max_{x\in[0,1]} u^2(x) \le \max\{\frac{13}{\mu}, \frac{13}{12}\}C_0 = MC_0.$$

Furthermore, using Young's inequality, we obtain

$$\begin{aligned} \|\tilde{P} * (1 + c)u\|_{L^\infty[0,1]} &\le \|\tilde{P}\|_{L^2[0,1]}\|1 + c\|_{L^\infty[0,1]}\|u\|_{L^2[0,1]} \\ &\le (1 + C_1)n_2\sqrt{C_0}, \end{aligned} \tag{4.11}$$

$$\begin{gathered} \|\tilde{P} * u^2\|_{L^\infty[0,1]} \le \|\tilde{P}\|_{L^\infty[0,1]}\|u^2\|_{L^2[0,1]} \le \|\tilde{P}\|_{L^\infty[0,1]}\|u\|_{L^2[0,1]}^2 \le n_\infty C_0, \\ \|\tilde{P} * u^3\|_{L^\infty[0,1]} \le \|\tilde{P}\|_{L^\infty[0,1]}\|u\|_{L^\infty[0,1]}\|u\|_{L^2[0,1]}^2 \le n_\infty\sqrt{M}C_0^{3/2}, \\ \|\tilde{P} * u^4\|_{L^\infty[0,1]} \le \|\tilde{P}\|_{L^\infty[0,1]}\|u^2\|_{L^\infty[0,1]}\|u\|_{L^2[0,1]}^2 \le n_\infty MC_0^2, \end{gathered}$$

$$\begin{aligned} \|\tilde{P} * (c_x u)_x\|_{L^\infty[0,1]} &\le \|\tilde{P}\|_{L^\infty[0,1]}\|c_{xx}u + c_x u_x\|_{L^2[0,1]} \\ &\le n_\infty(\|c_x\|_{L^\infty[0,1]} + \|c_{xx}\|_{L^\infty[0,1]})(1 + \frac{12}{\mu})C_0^{1/2} \\ &\le n_\infty C_1(1 + M)C_0^{1/2}, \end{aligned} \tag{4.12}$$

$$\|\tilde{P} * u_x^2\|_{L^\infty[0,1]} \le \|\tilde{P}\|_{L^\infty[0,1]}\|u_x\|_{L^2[0,1]}^2 \le n_\infty\frac{12}{\mu}C_0.$$

Since (4.9) is an equality in the space of the continuous function, we can evaluate both sides at some fixed time t at a point $\xi(t) \in \mathbb{R}$, where

$$S(t) = u_x(t, \xi(t)),$$

with $S(t)$ defined by (4.8). Besides, $u_{xx}(t,\xi(t)) = 0$ due to u is C^2 in the spatial variable and the result on the evolution of extrema [7] imply an equivalent form of (4.9),

$$S'(t) - \frac{7\varepsilon}{4}S(t) = -\frac{12}{\mu}(\tilde{P} * g(u)) + \frac{12}{\mu}g(u) + \frac{7\varepsilon}{4}\tilde{P} * u_x^2 + \frac{1}{2}\tilde{P} * (c_x u)_x.$$

The previous estimates enable us to derive the differential inequality

$$\begin{aligned} S'(t) &\le \frac{7\varepsilon}{4}S(t) + \frac{12}{\mu}[(1+C_1)n_2\sqrt{C_0} + \frac{17\varepsilon}{4}n_\infty C_0 + \frac{\varepsilon^2}{8}n_\infty\sqrt{M}C_0^{3/2} \\ &\quad + \frac{3\varepsilon^3}{64}n_\infty MC_0^2 + (1+C_1)\sqrt{MC_0} + \frac{5\varepsilon}{2}MC_0 + \frac{\varepsilon^2}{8}(MC_0)^{3/2} \\ &\quad + \frac{3\varepsilon^3}{64}(MC_0)^2] + \frac{(1+M)}{2}n_\infty C_1 C_0^{1/2} \\ &\le \frac{7\varepsilon}{4}S(t) + \frac{12}{\mu}[(n_\infty + M)(\frac{17\varepsilon}{4}C_0 + \frac{\varepsilon^2}{8}\sqrt{M}C_0^{3/2} + \frac{3\varepsilon^3}{64}MC_0^2) \\ &\quad + (1+C_1)(n_2+\sqrt{M})\sqrt{C_0}] + \frac{(1+M)}{2}n_\infty C_1\sqrt{C_0} \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} S'(t) &\ge \frac{7\varepsilon}{4}S(t) - \frac{12}{\mu}[(n_\infty + M)(\frac{17\varepsilon}{4}C_0 + \frac{\varepsilon^2}{8}\sqrt{M}C_0^{3/2} + \frac{3\varepsilon^3}{64}MC_0^2) \\ &\quad + (1+C_1)(n_2+\sqrt{M})\sqrt{C_0}] - \frac{(1+M)}{2}n_\infty C_1\sqrt{C_0} \end{aligned} \tag{4.14}$$

for a.e. $t \in (0,T)$. Notice that $u_0 \not\equiv 0$ ensures $S(0) > 0$. By our assumption on the initial wave profile, at $t = 0$, the right hand of (4.14) is strictly positive. We infer that, up to the maximal existence time $T > 0$ of the solution, the function $S(t)$ must be strictly increasing and moreover

$$S'(t) \ge \frac{3}{4}\varepsilon S^2(t) \quad \text{for a.e. } t \in (0,T).$$

Dividing by $S^2(t) \ge S^2(0) > 0$, $t \in (0,T)$, and integrating, we have

$$\frac{1}{S(t)} \le \frac{1}{S(0)} - \frac{3}{4}\varepsilon t, \quad t \in (0,T).$$

As $S(t) > 0$, we have $\lim_{t\uparrow T} S(t) = \infty$, and

$$T \le \frac{4}{3\varepsilon S(0)}. \tag{4.15}$$

Furthermore, the inequality (4.13) combined with our assumption on $S(0)$ yield

$$S'(t) \le \frac{11}{4}\varepsilon S^2(t) \quad \text{for a.e. } t \in (0,T).$$

Since $\lim_{t\uparrow T} S(t) = \infty$, we obtain

$$T \ge \frac{4}{11\varepsilon S(0)}. \tag{4.16}$$

From the estimates (4.15) and (4.16), we deduce the finite maximal existence time $T > 0$ is of order $O(1/\varepsilon)$. □

Remark 4.6. Considering the case that the bottom to be flat, we have $c \equiv 1$ as a result of $b = 0$ and the definition of $c = \sqrt{1 - \beta b^{(\alpha)}}$. From estimates (4.11) and (4.12) in the proof of Proposition 4.5, we have that condition (4.10) to guarante the solutions must blow up in finite time reduces to

$$\begin{aligned} \big|\inf_{x\in[0,1]}\{\partial_x u_0(x)\}\big|^2 > \frac{12}{\varepsilon\mu}[(n_\infty + M)(\frac{17\varepsilon}{4}C_0 + \frac{\varepsilon^2}{8}\sqrt{M}C_0^{3/2} + \frac{3\varepsilon^3}{64}MC_0^2) \\ + 2(n_2 + \sqrt{M})\sqrt{C_0}], \end{aligned} \tag{4.17}$$

Assume that there exists a point subjecting to $b(\alpha x) = 0$, implying that $\|c\|_{L^\infty[0,1]} \geq 1$, then we obtain

$$\begin{aligned} |\inf_{x\in[0,1]}\{\partial_x u_0(x)\}|^2 > \frac{12}{\varepsilon\mu}[(n_\infty + M)(\frac{17\varepsilon}{4}C_0 + \frac{\varepsilon^2}{8}\sqrt{M}C_0^{3/2} + \frac{3\varepsilon^3}{64}MC_0^2) \\ + (1 + \|c\|_{L^\infty[0,1]})(n_2 + \sqrt{M})\sqrt{C_0}] \\ + \frac{(1+M)}{2}n_\infty(\|c_x\|_{L^\infty[0,1]} + \|c_{xx}\|_{L^\infty[0,1]})\sqrt{C_0}. \end{aligned} \tag{4.18}$$

Comparing (4.17) with (4.18), we find that it is more restrictive for the initial wave profile u_0 in the case of the variable bottom than the analogous condition in the case of the flat bottom, which means that the infimum of the slope for the initial value has to be steeper to ensure the existence of the blow-up solutions.

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