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EXISTENCE AND UNIQUENESS FOR SUPERLINEAR SECOND-ORDER DIFFERENTIAL EQUATIONS ON THE HALF-LINE

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Abstract. We prove the existence and uniqueness, and study the global behavior of a positive continuous solution to the superlinear second-order differential equation

$$\frac{1}{A(t)}(A(t)u'(t))' = u(t)g(t,u(t)), \quad t \in (0,\infty),$$
$$u(0) = a, \quad \lim_{t\to\infty} \frac{u(t)}{\rho(t)} = b,$$

where a, b are nonnegative constants such that $a+b > 0$, A is a continuous function on $[0,\infty)$, positive and continuously differentiable on $(0,\infty)$ such that $1/A$ is integrable on $[0,1]$ and $\int_0^\infty 1/A(t)\,dt = \infty$. Here $\rho(t) = \int_0^t 1/A(s)\,ds$, for $t \geq 0$ and $g(t,s)$ is a nonnegative continuous function satisfying suitable integrability condition. Our Approach is based on estimates of the Green's function and a perturbation argument. Finally two illustrative examples are given.

1. Introduction

We are concerned with the existence, uniqueness and global behavior of a positive continuous solution to the second-order differential equation

$$\begin{gathered} \frac{1}{A(t)}(A(t)u'(t))' = u(t)g(t,u(t)), \quad t \in (0,\infty), \\ u(0) = a, \quad \lim_{t\to\infty} \frac{u(t)}{\rho(t)} = b, \end{gathered} \tag{1.1}$$

where a, b are nonnegative constants such that $a+b > 0$, A is a continuous function on $[0,\infty)$, positive and continuously differentiable on $(0,\infty)$ such that $\frac{1}{A}$ is integrable on $[0,1]$ and $\int_0^\infty 1/A(t)\,dt = \infty$.

Here $\rho(t) = \int_0^t 1/A(s)\,ds$, for $t \geq 0$. The nonnegative nonlinearity g is required to satisfy an appropriate condition related to the class $\mathcal{K}$, defined next.

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Definition 1.1. A Borel measurable function q in $(0,\infty)$ belongs to the class $\mathcal{K}$ if

$$\|q\| := \int_0^\infty A(r)\rho(r)|q(r)|dr < \infty. \tag{1.2}$$

The motivation for the present work stems from both practical and theoretical aspects. In fact, boundary value problems on the half-line arise quite naturally in the study of radially symmetric solutions of nonlinear elliptic equations, see for instance [4, 11], and various physical phenomena [9, 10], such as unsteady flow of gas through a semi-infinite, porous media and the theory of drain flows.

Note that boundary value problems for second-order differential equations have been considering widely and there are many results on the existence of solutions, see for example [1, 2, 5, 7, 8, 14].

Zhao [17] considered the second-order differential equation

$$\frac{1}{A(t)}(A(t)u'(t))' + h(t,u(t)) = 0, \quad t \in (0,\infty), \tag{1.3}$$

subject to the boundary conditions

$$u(0) = a, \quad \lim_{t\to\infty} \frac{u(t)}{\rho(t)} = b, \tag{1.4}$$

where $A(t) \equiv 1$, $a = 0$ and h is a measurable function on $(0,\infty)\times(0,\infty)$, dominated by a convex positive function. Then he showed that there exists $\mu > 0$ such that for each $b \in (0,\mu]$, there exists a positive continuous solution u of (1.3)–(1.4). This result has been generalized by Mâagli and Masmoudi [12]. On the other hand, Yan [15] studied equation (1.3) subject to the boundary condition $u(0) = a \geq 0$, $\lim_{t\to\infty} A(t)u'(t) = b \geq 0$, where A is a continuous function satisfying some appropriate conditions and

$$q_0(t)k(s) \leq h(t,s) \leq q_0(t)\widetilde{k}(s),$$

where q_0, k and $\widetilde{k}$ are nonnegative continuous functions on $(0,\infty)$ such that

$$\int_0^\infty A(r)|q_0(r)|dr < \infty, \quad \lim_{s\to 0^+} \frac{k(s)}{s} = \infty$$

and $\widetilde{k}$ satisfying some growth condition. By using fixed-point index theory, the existence of at least one nonnegative nonzero solution is established.

Recently, in [3], we have studied problem (1.3)-(1.4) where A is a continuous function satisfying some appropriate conditions, $h(t,s) = q(t)s^\sigma$, with $\sigma < 1$, $a = b = 0$ and $q(t)$ is a nonnegative continuous function that is required to satisfy some assumptions related to Karamata regular variation theory. Using monotonicity methods, we established the existence, uniqueness and the global asymptotic behavior of a positive continuous solution; see also [6].

Here, we shall use estimates of the Green's function and a perturbation argument to address existence, uniqueness and global behavior of a positive continuous solution to problem (1.1).

Throughout this paper, and without loss of generality, we assume that $\rho(1) = 1$. We let $a, b \geq 0$ such that $a + b > 0$ and we denote by $\omega(t) := a + b\rho(t)$, $t \geq 0$, the unique solution of the problem

$$\begin{gathered}\frac{1}{A(t)}(A(t)u'(t))'=0, \quad t\in(0,\infty),\\ u(0)=a, \quad \lim_{t\to\infty}\frac{u(t)}{\rho(t)}=b.\end{gathered}\tag{1.5}$$

We denote by $G(t,s)$ the Green's function of the operator $u\mapsto -\frac{1}{A}(Au')'$ on $(0,\infty)$ with the Dirichlet conditions $u(0)=0$ and $\lim_{t\to\infty}\frac{u(t)}{\rho(t)}=0$, which is given by

$$G(t,s)=A(s)\min(\rho(t),\rho(s)).\tag{1.6}$$

The outline of the article is as follows. In Section 2, we give some sharp estimates on the Green's function $G(t,s)$, including the following 3G-inequality: for each $t,s,r\in(0,\infty)$,

$$\frac{G(t,r)G(r,s)}{G(t,s)}\le A(r)\rho(r).$$

In particular, we derive from this 3G-inequality that for each $q\in\mathcal{K}$, we have

$$\alpha_q:=\sup_{t,s\in(0,\infty)}\int_0^\infty\frac{G(t,r)G(r,s)}{G(t,s)}|q(r)|dr=\|q\|<\infty.\tag{1.7}$$

In Section 3, for a given nonnegative function q in $\mathcal{K}\cap C((0,\infty))$, we prove that the Green's function $G_q(t,s)$ of the operator $u\mapsto -\frac{1}{A}(Au')'+qu$ on $(0,\infty)$ with the Dirichlet conditions $u(0)=0$ and $\lim_{t\to\infty}\frac{u(t)}{\rho(t)}=0$ is given by

$$G_q(t,s)=A(s)\rho(t)\rho(s)\varphi(t)\varphi(s)\int_{\max(t,s)}^\infty\frac{dr}{A(r)\rho^2(r)\varphi^2(r)},$$

where φ is the unique positive solution in $C([0,\infty))\cap C^2((0,\infty))$ of the equation

$$\frac{1}{A(t)\rho^2(t)}(A(t)\rho^2(t)u'(t))'-q(t)u(t)=0$$

$\lim_{t\to0}(A\rho^2\varphi')(t)=0$ and $\varphi(0)=1$.

In particular, we deduce the comparison result,

$$e^{-2\|q\|}G(t,s)\le G_q(t,s)\le G(t,s), \quad \text{for } t,s\ge0.$$

Moreover, we establish the resolvent equality

$$Vf=V_qf+V_q(qVf)=V_qf+V(qV_qf), \quad \text{for } f\in\mathcal{B}^+((0,\infty)),$$

where $\mathcal{B}^+((0,\infty))$ is the set of nonnegative Borel measurable functions in $(0,\infty)$ and the kernels V and V_q are defined on $\mathcal{B}^+((0,\infty))$ by

$$Vf(t):=\int_0^\infty G(t,s)f(s)ds, \quad V_qf(t):=\int_0^\infty G_q(t,s)f(s)ds, \quad t\ge0.$$

To state our existence results, we use the following assumptions:

- (H1) g is a nonnegative continuous function in $(0,\infty)\times[0,\infty)$.
- (H2) There exists a nonnegative function $q\in\mathcal{K}\cap C((0,\infty))$ such that for each $t\in(0,\infty)$, the map $s\to s(q(t)-g(t,s\omega(t)))$ is nondecreasing on $[0,1]$.
- (H3) For each $t\in(0,\infty)$, the function $s\to sg(t,s)$ is nondecreasing on $[0,\infty)$.

Using properties of the Green's function $G_q(t,s)$ and using a perturbation argument, we prove the following result.

Theorem 1.2. *Assume* (H1)–(H3). *Then* (1.1) *has a unique positive solution* $u \in C([0,\infty)) \cap C^2((0,\infty))$ *satisfying*

$$c\omega(t) \le u(t) \le \omega(t), \tag{1.8}$$

where c *is a constant in* $(0,1]$.

Corollary 1.3. *Let* f *be a nonnegative function in* $C^1([0,\infty))$ *such that the map* $s \to \theta(s) = sf(s)$ *is nondecreasing on* $[0,\infty)$. *Let* p *be a nonnegative continuous function on* $(0,\infty)$ *such that the function* $t \to q(t) := p(t)\max_{0\le\xi\le\omega(t)}\theta'(\xi)$ *belongs to the class* $\mathcal{K}$. *Then the problem*

$$\begin{gathered} \frac{1}{A(t)}(A(t)u'(t))' = p(t)u(t)f(u(t)), \quad t \in (0,\infty), \\ u(0) = a, \quad \lim_{t\to\infty}\frac{u(t)}{\rho(t)} = b, \end{gathered} \tag{1.9}$$

has a unique positive solution $u \in C([0,\infty)) \cap C^2((0,\infty))$ *satisfying*

$$e^{-2\|q\|}\omega(t) \le u(t) \le \omega(t), \quad t \ge 0. \tag{1.10}$$

Observe that in Theorem 1.2 we obtain a positive continuous solution u, to (1.1), which behavior is not affected by the perturbed term. That is, it behaves like the solution ω of the homogeneous problem (1.5).

As a typical example of nonlinearity satisfying (H1)–(H3), we have $g(t,s) = p(t)s^\sigma$, for $\sigma \ge 0$, where p is a positive continuous function on $(0,\infty)$ such that

$$\int_0^\infty A(r)\rho(r)(a+b\rho(r))^\sigma p(r)dr < \infty.$$

2. Estimates on the Green's function

In this section, we prove some estimates on the Green's function $G(t,s)$.

Proposition 2.1. *(i) For each* $t, s \in [0,\infty)$, *we have*

$$A(s)\min(1,\rho(s))\min(1,\rho(t)) \le G(t,s) \le A(s)\min(1,\rho(s))\max(1,\rho(t)). \tag{2.1}$$

(ii) For $f \in \mathcal{B}^+((0,\infty))$, *the function* $t \to Vf(t)$ *is continuous on* $[0,\infty)$ *if and only if the integral* $\int_0^\infty A(s)\min(1,\rho(s))f(s)ds$ *converges.*

Proof. (i) The inequalities in (2.1) follow from (1.6) and the fact that for $\alpha, \beta \ge 0$,

$$\min(1,\alpha)\min(1,\beta) \le \min(\alpha,\beta) \le \min(1,\alpha)\max(1,\beta).$$

(ii) Using (1.6), (2.1) and the dominated convergence theorem, we obtain the required assertion. □

Corollary 2.2. *Let* $f \in \mathcal{B}^+((0,\infty))$ *such that* $s \mapsto A(s)\min(1,\rho(s))f(s)$ *is continuous and integrable on* $(0,\infty)$. *Then* Vf *is the unique continuous solution of the boundary-value problem*

$$\begin{gathered} \frac{1}{A(t)}(A(t)u'(t))' = -f, \quad \text{in } (0,\infty), \\ u(0) = 0, \quad \lim_{t\to\infty}\frac{u(t)}{\rho(t)} = 0. \end{gathered} \tag{2.2}$$

We have the following 3G-inequality.

Proposition 2.3. *For each $t, s, r \in (0, \infty)$, we have*

$$\frac{G(t,r)G(r,s)}{G(t,s)} \leq A(r)\rho(r). \tag{2.3}$$

Proof. Using (1.6), for each $t, s, r \in (0, \infty)$, we deduce that

$$\frac{G(t,r)G(r,s)}{G(t,s)} = \frac{A(r)\min(\rho(t),\rho(r))\min(\rho(r),\rho(s))}{\min(\rho(t),\rho(s))}.$$

We claim that

$$\frac{\min(\rho(t),\rho(r))\min(\rho(r),\rho(s))}{\min(\rho(t),\rho(s))} \leq \rho(r).$$

Indeed, by symmetry, we may assume that $t \leq s$. Therefore, we obtain

$$\begin{aligned}\frac{\min(\rho(t),\rho(r))\min(\rho(r),\rho(s))}{\min(\rho(t),\rho(s))} &= \frac{\min(\rho(t),\rho(r))\min(\rho(r),\rho(s))}{\rho(t)} \\ &\leq \frac{\rho(t)\rho(r)}{\rho(t)} = \rho(r).\end{aligned}$$

This completes the proof. □

In the sequel, we denote

$$\alpha_q = \sup_{t,s\in(0,\infty)} \int_0^\infty \frac{G(t,r)G(r,s)}{G(t,s)}|q(r)|\, dr, \quad \|q\| = \int_0^\infty A(r)\rho(r)|q(r)|dr.$$

Proposition 2.4. *Let q be a nonnegative function in $\mathcal{K}$, then: (i) For $t \in [0, \infty)$, we have*

$$V(q)(t) \leq \alpha_q. \tag{2.4}$$

In particular,

$$\alpha_q = \|q\| < \infty. \tag{2.5}$$

(ii) For $t \in [0, \infty)$, we have

$$V(q\rho)(t) \leq \alpha_q \rho(t). \tag{2.6}$$

In particular for $t \in [0, \infty)$, we obtain

$$V(q\omega)(t) \leq \alpha_q \omega(t). \tag{2.7}$$

(iii) Let $f \in \mathcal{B}^+(0, \infty)$, then

$$V(qV(f))(t) \leq \alpha_q V(f)(t). \tag{2.8}$$

Proof. Let q be a nonnegative function in $\mathcal{K}$.

(i) Since for each $t, s \in (0, \infty)$, we have $\lim_{r\to 0} \frac{G(s,r)}{G(t,r)} = 1$, then by Fatou's lemma and (1.7), we deduce that

$$V(q)(t) = \int_0^\infty G(t,s)q(s)ds \leq \liminf_{r\to 0} \int_0^\infty G(t,s)\frac{G(s,r)}{G(t,r)}q(s)ds \leq \alpha_q.$$

This proves (2.4).

To prove (2.5), observe that $\|q\| = \|V(q)\|_\infty := \sup_{t>0}|V(q)(t)|$. So it follows from (2.4) that $\|q\| = \|V(q)\|_\infty \leq \alpha_q$. On the other hand, by using (2.3), for $t, s \in (0, \infty)$, we have

$$\int_0^\infty \frac{G(t,r)G(r,s)}{G(t,s)}q(r)dr \leq \int_0^\infty A(r)\rho(r)q(r)dr = \|q\|.$$

Hence $\alpha_q \leq \|q\| < \infty$. Therefore $\alpha_q = \|q\| < \infty$.

(ii) Since for each $t,s\in(0,\infty)$, we have $\lim_{r\to\infty}\frac{G(s,r)}{G(t,r)}=\frac{\rho(s)}{\rho(t)}$, then we deduce by Fatou's lemma and (1.7), that

$$\int_0^\infty \frac{G(t,s)}{\rho(t)}\rho(s)q(s)dr \le \liminf_{r\to\infty}\int_0^\infty G(t,s)\frac{G(s,r)}{G(t,r)}q(s)ds \le \alpha_q.$$

This proves (2.6). Inequality (2.7) follows from inequalities (2.4), (2.6) and the fact that $\omega(t)=a+b\rho(t)$.

(iii) Using (1.7) and Fubini-Tonelli's theorem, we obtain

$$\begin{aligned} V(qV(f))(t) &= \int_0^\infty [\int_0^\infty G(t,r)G(r,s)q(r)dr]f(s)ds \\ &\le \int_0^\infty \alpha_q G(t,s)f(s)ds = \alpha_q V(f)(t). \end{aligned}$$

This completes the proof. □

3. Proofs of main results

In this section, we prove Theorem 1.2 and Corollary 1.3. First, for a given nonnegative function q in $\mathcal{K}\cap C((0,\infty))$, we aim at determining the Green's function $G_q(t,s)$ of the linear problem

$$\begin{gathered} \frac{1}{A(t)}(A(t)u'(t))' - q(t)u(t) = -f(t), \quad t\in(0,\infty), \\ u(0)=0, \quad \lim_{t\to\infty}\frac{u(t)}{\rho(t)}=0. \end{gathered} \tag{3.1}$$

Put $u(t):=\rho(t)v(t)$. It is easy to check that u is a solution of (3.1) if and only if v is a solution of the problem

$$\begin{gathered} \frac{1}{A(t)\rho^2(t)}(A(t)\rho^2(t)v'(t))' - q(t)v(t) = \frac{-f(t)}{\rho(t)}, \quad t\in(0,\infty), \\ \lim_{t\to 0}(A\rho^2v')(t)=0, \quad \lim_{t\to\infty}v(t)=0. \end{gathered} \tag{3.2}$$

Therefore, to obtain $G_q(t,s)$ it is sufficient to determine the Green's function $H_q(t,s)$ of the operator $u\mapsto \frac{-1}{A\rho^2}(A\rho^2v')'+qv$ on $(0,\infty)$ with the Dirichlet conditions $\lim_{t\to0}(A\rho^2v')(t)=0$, $\lim_{t\to\infty}v(t)=0$. To this end, we need the following results.

Proposition 3.1. *Let q be a nonnegative function in $\mathcal{K}\cap C((0,\infty))$, then the problem*

$$\begin{gathered} \frac{1}{A(t)\rho^2(t)}(A(t)\rho^2(t)u'(t))' - q(t)u(t) = 0, \quad t\in(0,\infty), \\ \lim_{t\to 0}(A\rho^2u')(t)=0, \quad u(0)=1, \end{gathered} \tag{3.3}$$

has a unique positive solution $\varphi\in C([0,\infty))\cap C^2((0,\infty))$. Moreover, φ is nondecreasing and for $t\ge0$ satisfies

$$1\le\varphi(t)\le\exp\Big(\int_0^t\frac{1}{A(s)\rho^2(s)}(\int_0^s A(r)\rho^2(r)q(r)dr)ds\Big)\le\exp(\|q\|). \tag{3.4}$$

In particular, $\varphi(\infty):=\lim_{t\to\infty}\varphi(t)$ exists and $1\le\varphi(\infty)\le\exp(\|q\|)$.

Proof. (see [16]). Let K be the operator defined on $C([0,\infty))$ by

$$Kf(t) := \int_0^t \frac{1}{A(s)\rho^2(s)}\Big(\int_0^s A(r)\rho^2(r)q(r)f(r)dr\Big)ds, \quad t\in[0,\infty).$$

We put $K^j = K^{j-1}\circ K$ for any integer $j\geq 2$. Then we claim that for each $t\geq 0$ and $m\in\mathbb{N}$, we have

$$0 \leq K^m\mathbf{1}(t) \leq \frac{(K\mathbf{1})^m(t)}{m!}. \tag{3.5}$$

Indeed, if $m=0$ or 1, (3.5) is valid. Now for a given $m\in\mathbb{N}$, suppose (3.5), then we have

$$\begin{aligned}
K^{m+1}\mathbf{1}(t) &= K(K^m\mathbf{1})(t)\\
&\leq \frac{1}{m!}K((K\mathbf{1})^m)(t)\\
&= \frac{1}{m!}\int_0^t \frac{1}{A(s)\rho^2(s)}\Big(\int_0^s A(r)\rho^2(r)q(r)(K\mathbf{1})^m(r)\,dr\Big)ds.
\end{aligned}$$

Since the function $K\mathbf{1}$ is nondecreasing, it follows that

$$\begin{aligned}
K^{m+1}\mathbf{1}(t) &\leq \frac{1}{m!}\int_0^t (K\mathbf{1})^m(s)\Big(\frac{1}{A(s)\rho^2(s)}\int_0^s A(r)\rho^2(r)q(r)\,dr\Big)\,ds\\
&= \frac{1}{m!}\int_0^t (K\mathbf{1})^m(s)(K\mathbf{1})'(s)\,ds\\
&= \frac{1}{(m+1)!}(K\mathbf{1})^{m+1}(t).
\end{aligned}$$

Therefore, the series $\sum_{m=0}^{\infty}(K^m\mathbf{1})(t)$ converges locally uniformly to a function $\varphi\in C([0,\infty))$ satisfying for each $t\geq 0$,

$$\varphi(t) = 1 + \int_0^t \frac{1}{A(s)\rho^2(s)}\Big(\int_0^s A(r)\rho^2(r)q(r)\varphi(r)dr\Big)ds.$$

Hence $\varphi\in C([0,\infty))\cap C^2((0,\infty))$ and φ is a positive solution of (3.3).

Now, we show the uniqueness. Let $u,v\in C([0,\infty))\cap C^2((0,\infty))$ be two positive solutions of (3.3). Then for each $R\in(0,\infty)$ and $t\in[0,R]$ we have

$$|u(t)-v(t)| \leq K(|u-v|)(t).$$

Since K is a nondecreasing operator, we deduce by induction that for each $m\geq 0$,

$$\begin{aligned}
|u(t)-v(t)| &\leq K^m(|u-v|)(t)\\
&\leq \sup_{r\in[0,R]}|u(r)-v(r)|K^m\mathbf{1}(R)\\
&\leq \sup_{r\in[0,R]}|u(r)-v(r)|\frac{(K\mathbf{1})^m(R)}{m!}.
\end{aligned}$$

Letting m tend to infinity, we obtain $|u(t)-v(t)|=0$ for all $t\in[0,R]$. So $u=v$ on $[0,\infty)$. Finally (3.4) follows from the fact that

$$1\leq\varphi(t) = \sum_{m=0}^{\infty}(K^m\mathbf{1})(t) \leq \sum_{m=0}^{\infty}\frac{(K\mathbf{1})^m(t)}{m!} = \exp(K\mathbf{1}(t)) \quad \forall t\geq 0,$$

$$K\mathbf{1}(t) \leq \int_0^{\infty}\frac{1}{A(s)\rho^2(s)}\Big(\int_0^s A(r)\rho^2(r)q(rdr)\Big)ds = \|q\|.$$

□

Remark 3.2. Let q be a nonnegative function in $\mathcal{K}\cap C((0,\infty))$ and φ be the solution of (3.3). It follows that the function ψ defined on $(0,\infty)$ by

$$\psi(t) := \varphi(t)\int_t^\infty \frac{ds}{A(s)\rho^2(s)\varphi^2(s)},$$

is a second solution of the equation

$$\frac{1}{A(t)\rho^2(t)}(A(t)\rho^2(t)u'(t))' - q(t)u(t) = 0, \quad \text{on } (0,\infty),$$

such that φ and ψ are linearly independent.

Furthermore, since for $t>0$,

$$\frac{1}{\varphi^2(\infty)\rho(t)} \le \psi(t) \le \frac{1}{\varphi(t)}\int_t^\infty \frac{ds}{A(s)\rho^2(s)} = \frac{1}{\varphi(t)\rho(t)}, \tag{3.6}$$

it follows that $\lim_{t\to\infty}\psi(t)=0$ and also we have

$$\psi(t) \sim \int_t^\infty \frac{ds}{A(s)\rho^2(s)} = \frac{1}{\rho(t)} \quad \text{as } t\to 0.$$

Hence $\lim_{t\to 0}\rho(t)\psi(t)=1$.

Now, following [13, Section 2, p.294], we deduce that $H_q(t,s)$ is given by

$$H_q(t,s) = \begin{cases} A(s)\rho^2(s)\varphi(s)\psi(t), & \text{if } 0<s\le t<\infty,\\ A(s)\rho^2(s)\varphi(t)\psi(s), & \text{if } 0<t\le s<\infty. \end{cases}$$

On the other hand, we deduce that $\{\rho\varphi,\rho\psi\}$ is a fundamental system of solutions of the equation $\frac{1}{A}(Au')' - qu = 0$ on $(0,\infty)$ satisfying

$$A(t)[(\rho\psi)(t)(\rho\varphi)'(t) - (\rho\varphi)(t)(\rho\psi)'(t)] = 1 \quad \text{for } t\in(0,\infty). \tag{3.7}$$

Furthermore, the Green's function $G_q(t,s)$ of problem (3.1) is given by

$$G_q(t,s) = \frac{\rho(t)}{\rho(s)}H_q(t,s) = \begin{cases} A(s)\rho(t)\rho(s)\varphi(s)\psi(t), & \text{if } 0<s\le t<\infty,\\ A(s)\rho(t)\rho(s)\varphi(t)\psi(s), & \text{if } 0<t\le s<\infty. \end{cases}$$

That is,

$$G_q(t,s) = A(s)\rho(t)\rho(s)\varphi(t)\varphi(s)\int_{t\vee s}^\infty \frac{dr}{A(r)\rho^2(r)\varphi^2(r)} \tag{3.8}$$

$$= A(s)\rho(t\wedge s)\rho(t\vee s)\varphi(t\wedge s)\psi(t\vee s), \tag{3.9}$$

where $t\wedge s = \min(t,s)$ and $t\vee s = \max(t,s)$.

Next, we recall that the kernels V and V_q are defined on $\mathcal{B}^+((0,\infty))$ by

$$Vf(t) := \int_0^\infty G(t,s)f(s)ds, \quad V_qf(t) := \int_0^\infty G_q(t,s)f(s)ds, \quad t\ge 0.$$

Proposition 3.3. *Let q be a nonnegative function in $\mathcal{K}\cap C((0,\infty))$, then we have*

$$e^{-2\|q\|}G(t,s) \le G_q(t,s) \le G(t,s). \tag{3.10}$$

In particular for $f\in\mathcal{B}^+((0,\infty))$, we obtain

$$e^{-2\|q\|}Vf \le V_qf \le Vf. \tag{3.11}$$

Proof. Using (3.9), (3.6) and that the function φ is nondecreasing, we obtain inequalities (3.10). Integrating inequalities (3.10), we obtain (3.11). □

Corollary 3.4. *Let q be a nonnegative function in $\mathcal{K} \cap C((0,\infty))$ and let f in $\mathcal{B}^+((0,\infty))$, then the following two statements are equivalent.*

(i) *The function $t \to V_q f(t)$ is continuous on $[0,\infty)$.*
(ii) *The integral $\int_0^\infty A(s)\min(1,\rho(s))f(s)ds$ converges.*

Proposition 3.5. *Let q be a nonnegative function in $\mathcal{K} \cap C((0,\infty))$ and let $f \in \mathcal{B}^+((0,\infty))$ such that $s \to A(s)\min(1,\rho(s))f(s)$ is continuous and integrable on $(0,\infty)$. Then $V_q f$ is the unique nonnegative continuous solution of problem* (3.1).

Proof. Let q be a nonnegative function in $\mathcal{K} \cap C((0,\infty))$ and $f \in \mathcal{B}^+((0,\infty))$. By Corollary 3.4, the function $t \to V_q f(t)$ is continuous on $[0,\infty)$. On the other hand, for $t > 0$, we have

$$\begin{aligned} V_q f(t) &= \int_0^\infty G_q(t,s)f(s)ds \\ &= (\rho\psi)(t)\int_0^t A(s)\rho(s)\varphi(s)f(s)ds + (\rho\varphi)(t)\int_t^\infty A(s)\rho(s)\psi(s)f(s)ds. \end{aligned}$$

So $V_q f$ is differentiable on $(0,\infty)$ and we have for $t > 0$,

$$(V_q f)'(t) = (\rho\psi)'(t)\int_0^t A(s)\rho(s)\varphi(s)f(s)ds + (\rho\varphi)'(t)\int_t^\infty A(s)\rho(s)\psi(s)f(s)ds.$$

Therefore by using the fact that $\rho\varphi$ and $\rho\psi$ are solutions of the equation $\frac{1}{A}(Au')' - qu = 0$ on $(0,\infty)$ and (3.7), we obtain

$$\begin{aligned} (A(V_q f)')'(t) &= (A(\rho\psi)')'(t)\int_0^t A(s)\rho(s)\varphi(s)f(s)ds \\ &\quad + (A(\rho\varphi)')'(t)\int_t^\infty A(s)\rho(s)\psi(s)f(s)ds \\ &\quad + A(t)f(t)[A(\rho\varphi)(\rho\psi)' - A(\rho\psi)(\rho\varphi)'](t) \\ &= A(t)q(t)V_q f(t) - A(t)f(t). \end{aligned}$$

So $V_q f$ is a solution of the equation $\frac{1}{A(t)}(A(t)u'(t))' - q(t)u(t) = -f(t)$. Now since $0 \le V_q f \le Vf$, it follows by Corollary 2.2, that $V_q f(0) = 0$ and $\lim_{t\to\infty} \frac{V_q f(t)}{\rho(t)} = 0$.

It remains to prove the uniqueness. Assume that there exist two positive solutions $u, v \in C([0,\infty)) \cap C^2((0,\infty))$ to problem (3.1). Let $\theta := u - v$, then $\theta \in C([0,\infty)) \cap C^2((0,\infty))$ and satisfies

$$\frac{1}{A(t)}(A(t)\theta'(t))' - q(t)\theta(t) = 0 \quad \text{on } (0,\infty),$$
$$\theta(0) = 0, \quad \lim_{t\to\infty} \frac{\theta(t)}{\rho(t)} = 0.$$

Hence, there exists $\lambda, \mu \in \mathbb{R}$, such that

$$\theta(t) = \lambda\rho(t)\varphi(t) + \mu\rho(t)\psi(t), \text{ for } t \ge 0.$$

So using this fact, Proposition 3.1, Remark 3.2 and that

$$\theta(0) = \lim_{t\to\infty} \frac{\theta(t)}{\rho(t)} = 0,$$

we deduce that $\lambda = \mu = 0$. That is, $u = v$. This completes the proof. ◻

Corollary 3.6. *Let q be a nonnegative function in $\mathcal{K} \cap C((0,\infty))$ and let $f \in \mathcal{B}^+((0,\infty))$ such that $s \to A(s)\min(1,\rho(s))f(s)$ is continuous and integrable on $(0,\infty)$. Then $V_q f$ satisfies the resolvent equation*

$$Vf = V_q f + V_q(qVf) = V_q f + V(qV_q f). \tag{3.12}$$

In particular, if $V(qf) < \infty$, we have

$$(I - V_q(q\cdot))(I + V(q\cdot))f = (I + V(q\cdot))(I - V_q(q\cdot))f = f. \tag{3.13}$$

Proof. Let q be a nonnegative function in $\mathcal{K} \cap C((0,\infty))$ and let $f \in \mathcal{B}^+((0,\infty))$ such that $s \to A(s)\min(1,\rho(s))f(s)$ is continuous and integrable on $(0,\infty)$.

By Proposition 2.1 it is clear that the function $t \mapsto q(t)Vf(t)$ is continuous on $(0,\infty)$ and there exists a nonnegative constant c such that

$$Vf(t) \le (1+\rho(t)) \int_0^\infty A(s)\min(1,\rho(s))f(s)ds \le c(1+\rho(t)). \tag{3.14}$$

So we deduce by Proposition 2.4 that

$$\begin{aligned} \int_0^\infty A(s)\min(1,\rho(s))q(s)Vf(s)ds &\le c\int_0^\infty G(1,s)(1+\rho(s))q(s)ds \\ &\le 2c\alpha_q < \infty. \end{aligned}$$

Let $\theta := Vf - V_q f - V_q(qVf)$. By using Corollary 2.2 and Proposition 3.5, the function θ is a solution of the problem

$$\begin{gathered} \frac{1}{A(t)}(A(t)\theta'(t))' - q(t)\theta(t) = 0, \quad t \in (0,\infty), \\ \theta(0) = 0, \quad \lim_{t\to\infty} \frac{\theta(t)}{\rho(t)} = 0. \end{gathered} \tag{3.15}$$

From the uniqueness in Proposition 3.5, we deduce that $\theta = 0$.

Now, by using Corollary 3.4 and (3.11), we deduce that the function $t \mapsto q(t)V_q f(t)$ is continuous on $(0,\infty)$ and that

$$\int_0^\infty A(s)\min(1,\rho(s))q(s)V_q f(s)ds \le \int_0^\infty A(s)\min(1,\rho(s))q(s)Vf(s)ds < \infty.$$

So by similar arguments as above, we obtain $Vf - V_q f - V(qV_q f) = 0$. This completes the proof. □

We recall that for $a, b \ge 0$ such that $a + b > 0$, we have

$$\omega(t) = a + b\rho(t), \; t \in [0,\infty).$$

The next Lemma will be useful for the proof of Theorem 1.2.

Lemma 3.7. *Let q be a nonnegative function in $\mathcal{K} \cap C((0,\infty))$, then we have*

$$e^{-2\|q\|}\omega \le \omega - V_q(q\omega) \le \omega.$$

Proof. Let $\theta := \omega - V_q(q\omega)$. It is clear that $\theta \le \omega$. Now since $\{\rho\varphi, \rho\psi\}$ is a fundamental system of solutions of the equation

$$\frac{1}{A(t)}(A(t)u'(t))' - q(t)u(t) = 0, \tag{3.16}$$

and the function θ is also a solution of this equation with $\theta(0) = a$ and $\lim_{t\to\infty} \frac{\theta(t)}{\rho(t)} = b$, we deduce by using Proposition 3.1 and Remark 3.2 that

$$\theta(t) = \frac{b}{\varphi(\infty)}\rho(t)\varphi(t) + a\rho(t)\psi(t), \quad t > 0.$$

Using Proposition 3.1 and (3.6), this implies that

$$\theta = \omega - V_q(q\omega) \geq \frac{b}{\varphi(\infty)}\rho + \frac{a}{\varphi^2(\infty)} \geq \frac{1}{\varphi^2(\infty)}\omega \geq e^{-2\|q\|}\omega.$$

The proof is complete. □

Proof of Theorem 1.2. Since g satisfies (H2), there exists a nonnegative continuous function q in $\mathcal{K}$ such for each $t \in (0,\infty)$, the map $s \to s(q(t) - g(t, s\omega(t)))$ is nondecreasing on $[0,1]$. Let

$$\Lambda := \big\{u \in \mathcal{B}^+((0,\infty)) : e^{-2\|q\|}\omega \leq u \leq \omega\big\},$$

and define the operator T on Λ by

$$Tu = \omega - V_q(q\omega) + V_q((q - g(\cdot, u))u).$$

By (H2), we have

$$0 \leq g(.,u) \leq q, \quad \text{for all } u \in \Lambda. \tag{3.17}$$

We claim that Λ is invariant under T. Indeed, since g is nonnegative, we have for $u \in \Lambda$

$$Tu \leq \omega - V_q(q\omega) + V_q(qu) \leq \omega$$

and by (3.17) and Lemma 3.7,

$$Tu \geq \omega - V_q(q\omega) \geq e^{-2\|q\|}\omega.$$

Next, we will prove that the operator T is nondecreasing on Λ. Indeed, let $u, v \in \Lambda$ be such that $u \leq v$. Since for $t \in (0,\infty)$, the map $s \to s(q(t) - g(t, s\omega(t)))$ is nondecreasing on $[0,1]$, then we obtain

$$Tv - Tu = V_q([v(q - g(\cdot, v)) - u(q - g(\cdot, u))]) \geq 0.$$

Now, we consider the sequence $\{u_n\}$ defined by $u_0 = e^{-2\|q\|}\omega$ and $u_{n+1} = Tu_n$, for $n \in \mathbb{N}$. Since Λ is invariant under T, we have $u_1 = Tu_0 \geq u_0$ and by the monotonicity of T, we deduce that

$$e^{-2\|q\|}\omega = u_0 \leq u_1 \leq \dots \leq u_n \leq u_{n+1} \leq \omega.$$

So the sequence $\{u_n\}$ converges to a function $u \in \Lambda$. Using hypotheses (H1)–(H2) and the monotone convergence theorem, we deduce that

$$u = (I - V_q(q\cdot))\omega + V_q((q - g(.,u))u).$$

That is,

$$(I - V_q(q\cdot))u = (I - V_q(q\cdot))\omega - V_q(ug(\cdot, u)).$$

On the other hand, since $u \leq \omega$, then by (2.7), we obtain $V(qu) \leq V(q\omega) \leq \alpha_q\omega < \infty$. So by applying the operator $(I + V(q\cdot))$ on both sides of the above equality and using (3.12) and (3.13), we conclude that u satisfies

$$u = \omega - V(ug(\cdot, u)). \tag{3.18}$$

Next we aim at proving that u is a solution of problem (1.1). To this end, we remark by (3.17) that

$$ug(\cdot, u) \leq q\omega. \tag{3.19}$$

By Proposition 2.1 (ii) and (2.7), this implies that the function $t \to V(ug(\cdot,u))(t)$ is continuous on $[0,\infty)$ and so by (3.18), u is continuous on $[0,\infty)$. Now, since by (H_1) and (3.19), the function $s \to A(s)\min(1,\rho(s))u(s)g(s,u(s))$ is continuous and integrable on $(0,\infty)$, we conclude by Corollary 2.2 that u is the required solution.

It remains to prove that u is the unique solution to (1.1). Assume that $v \in C([0,\infty)) \cap C^2((0,\infty))$ is another nonnegative solution to problem (1.1). Then we have

$$v = \omega - V(vg(\cdot,v)). \tag{3.20}$$

Now let h be the function defined on $(0,\infty)$ by

$$h(t) = \begin{cases} \frac{v(t)g(t,v(t))-u(t)g(t,u(t))}{v(t)-u(t)} & \text{if } v(t) \neq u(t), \\ 0 & \text{if } v(t) = u(t). \end{cases}$$

From (H3), we have $h \in \mathcal{B}^+((0,\infty))$ and by using (3.18) and (3.20), we obtain

$$(I + V(h.))(v-u) = 0.$$

On the other hand, since by (H2), we have $h \leq q$, then by using (2.7) we deduce that

$$V(h|v-u|) \leq 2V(q\omega) \leq 2\alpha_q \omega < \infty.$$

Hence by (3.13), we conclude that $u = v$. This completes the proof. □

Proof of Corollary 1.3. Let $g(t,s) = p(t)f(s)$ and $\theta(s) = sf(s)$, and let $q(t) = p(t)\max_{0\leq\xi\leq\omega(t)}\theta'(\xi) \in \mathcal{K}$. It is clear that hypotheses (H1) and (H3) are satisfied. Moreover, by a simple computation, we obtain

$$\frac{d}{ds}[s(q(t) - g(t,s\omega(t)))] = q(t) - p(t)\theta'(s\omega(t)) \geq 0 \quad \text{for } s \in [0,1] \text{ and } t > 0.$$

This implies that the function g satisfies hypothesis (H2). So the result follows by Theorem 1.2. □

Example 3.8. Let $a \geq 0$ and $b \geq 0$ with $a + b > 0$. Let $\sigma \geq 0$, and p be a positive continuous function on $(0,\infty)$ such that

$$\int_0^\infty A(r)\rho(r)(\omega(t))^\sigma p(r)dr < \infty.$$

Since the function $q(t) := (\sigma+1)p(t)(\omega(t))^\sigma$ belongs to the class $\mathcal{K}$, the problem

$$\frac{1}{A(t)}(A(t)u'(t))' = p(t)u^{\sigma+1}(t), \quad t \in (0,\infty),$$

$$u(0) = a, \quad \lim_{t\to\infty}\frac{u(t)}{\rho(t)} = b,$$

has a unique positive solution $u \in C([0,\infty)) \cap C^2((0,\infty))$ satisfying

$$e^{-2\|q\|}\omega(t) \leq u(t) \leq \omega(t), \quad t \geq 0.$$

Example 3.9. Let $a \geq 0$ and $b \geq 0$ with $a + b > 0$. Let $\sigma \geq 0$, $\gamma > 0$ and p be a positive continuous function on $(0,\infty)$ such that

$$\int_0^\infty A(r)\rho(r)(\omega(t))^{\sigma+\gamma}p(r)dr < \infty.$$

Let $\theta(s) = s^{\sigma+1}\log(1+s^\gamma)$. Since the function $q(t) := p(t)\max_{0\le\xi\le\omega(t)}\theta'(\xi)$ belongs to the class $\mathcal{K}$, then the problem

$$\frac{1}{A(t)}(A(t)u'(t))' = p(t)u^{\sigma+1}(t)\log(1+u^\gamma(t)), \quad t \in (0,\infty),$$

$$u(0) = a, \quad \lim_{t\to\infty}\frac{u(t)}{\rho(t)} = b,$$

has a unique positive solution $u \in C([0,\infty)) \cap C^2((0,\infty))$ satisfying

$$e^{-2\|q\|}\omega(t) \le u(t) \le \omega(t), \quad t \ge 0.$$

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