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PERIODIC SOLUTIONS FOR NONLINEAR NEUTRAL DELAY INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In this article, we consider a model for the spread of certain infectious disease governed by a delay integro-differential equation. We obtain the existence and the uniqueness of a positive periodic solution, by using Perov's fixed point theorem in generalized metric spaces.

1. Introduction

The existence of positive solutions to the integral equation with non constant delay

$$x(t) = \int_{t-\sigma(t)}^{t} f(s, x(s))ds, \tag{1.1}$$

was considered in [4, 13, 18, 23]. This equation is a mathematical model for the spread of certain infectious diseases with a contact rate that varies seasonally. Here $x(t)$ is the proportion of infectious in population at time t, $\sigma(t)$ is the length of time in which an individual remains infectious, $f(t, x(t))$ is the proportion of new infectious per unit of time (see, for example, [5, 9, 21]).

Ait Dads and Ezzinbi [4] and Ding et al [10] studied the existence of a positive pseudo almost periodic solution. Ezzinbi and Hachimi [13], Torrejón [23], Xu and Yuan [24] showed the existence of a positive almost periodic solution. The existence of a positive almost automorphic solution was studied in [12, 15, 18]. Bica and Mureşan [7, 8] studied the existence and uniqueness of a positive periodic solution of (1.1) by using Perov's fixed point theorem in the case of f depends also on $x'(t)$ and $\sigma(t) = \sigma$ is constant.

Ait Dads and Ezzinbi [3] considered the existence of a positive almost periodic solution, Ding et al [11] studied the existence of positive almost automorphic solutions for the neutral nonlinear delay integral equation

$$x(t) = \gamma x(t-\tau) + (1-\gamma)\int_{t-\tau}^{t} f(s, x(s))ds, \tag{1.2}$$

where $0 \leq \gamma < 1$. We refer to [2, 17] for the meaning of (1.2) in the context of epidemics.

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In this paper, we consider the more general equation

$$x(t) = \gamma x(t-\sigma(t)) + (1-\gamma)\int_{t-\sigma(t)}^{t} f(s, x(s), x'(s))ds, \tag{1.3}$$

Equation (1.3) includes many important integral and functional equations that arise in biomathematics (see for example [1, 6, 8, 9, 14, 16, 18, 19, 21]).

We would to use Perov's fixed point theorem to obtain conditions for the existence and uniqueness of a positive periodic solution to (1.3). This work is motivated by the work of Wei Long and Hui-Sheng Ding [18]. Moreover, the results obtained in this paper generalize several ones obtained in [6, 8, 19, 21], and the main goal in this work is to study the existence and uniqueness of solutions when $\sigma(t)$ is not constant in (1.3).

2. Preliminaries and generalized metric spaces

In this section, we recall the following notation and results in generalized metric spaces.

Definition 2.1 ([20]). Let X be a nonempty set and $d : X \times X \to \mathbb{R}^n$ be a mapping such that for all $x, y, z \in X$, one has:

- (i) $d(x, y) \geq 0_{\mathbb{R}^n}$ and $d(x, y) = 0_{\mathbb{R}^n} \Leftrightarrow x = y$,
- (ii) $d(x, y) = d(y, x)$,
- (iii) $d(x, y) \leq d(x, z) + d(z, y)$,
  where for $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ from $\mathbb{R}^n$, we have $x \leq y \Leftrightarrow x_i \leq y_i$, for $i = \overline{1, n}$.

Then d is called a generalized metric and (X, d) is a generalized metric space.

Definition 2.2 ([22]). Let $(E, \|\cdot\|)$ be a generalized Banach space, the norm $\|\cdot\| : E \to \mathbb{R}^n$ has the following properties:

- (i) $\|x\| \geq 0$ for all $x \in E$ and $\|x\| = 0$ if and only if $x = 0$.
- (ii) $\|\lambda x\| = |\lambda|\, \|x\|$ for all $\lambda \in K = \mathbb{R}$ or $\mathbb{C}$ and for all $x \in E$.
- (iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in E$ (the inequalities are defined by components in $\mathbb{R}^n$).

Remark 2.3. A generalized Banach space is a generalized complete metric space.

Definition 2.4 ([20]). If (E, d) is a generalized complete metric space and $T : E \to E$ which satisfies the inequality

$$d(Tx, Ty) \leq Ad(x, y) \quad \text{for all } x, y \in E,$$

where A is a matrix convergent to zero (the norms of it its eigenvalues are in the interval $[0, 1)$). We say that T is a Picard operator or generalized contraction.

We recall the following Perov's fixed point theorem.

Theorem 2.5 ([20]). *Let (E, d) be a complete generalized metric space. If $T : E \to E$ is a map for which there exists a matrix $A \in M_n(\mathbb{R})$ such that*

$$d(Tx, Ty) \leq Ad(x, y), \quad \forall x, y \in E$$

and the norms of the eigenvalues of A are in the interval $[0, 1)$, then T has a unique fixed point $x^* \in E$ and the sequence of successive approximations $x_m = T^m(x_0)$ converges to x^* for any $x_0 \in E$. Moreover, the following estimation holds

$$d(x_m, x^*) \leq A^m (I_n - A)^{-1} d(x_0, x_1), \quad \forall m \in \mathbb{N}^*.$$

3. Main result

In this section, we study the existence and uniqueness of a positive and periodic solution for the equation (1.3). We consider the following functional spaces

$$\begin{gathered}P(\omega)=\{x\in C(\mathbb{R}):x(t+\omega)=x(t),\ \forall t\in\mathbb{R}\}\\ P^1(\omega)=\{x\in C^1(\mathbb{R}):x(t+\omega)=x(t),\ \forall t\in\mathbb{R}\}\\ K^+(\omega)=\{x\in P^1(\omega):x(t)\geq 0,\ \forall t\in\mathbb{R}\}\end{gathered}$$

and denote by E the product space $E=K^+(\omega)\times P(\omega)$ which is a generalized metric space with the generalized metric $d_C:E\times E\to\mathbb{R}^2$, defined by

$$d_C((x_1,y_1),(x_2,y_2))=(\|x_1-x_2\|+\|x_1'-x_2'\|,\|y_1-y_2\|)$$

where $\|u\|=\max\{|u(t)|:t\in[0,\omega]\}$ for any $u\in P(\omega)$. Before stating the main result, we need the following lemma.

Lemma 3.1. *(E,d_C) is a complete generalized metric space.*

Proof. Let $(x^n)=(x_n,y_n)$ be a Cauchy sequence, then for any $\epsilon=(\epsilon_1,\epsilon_2)>0$, there exists $n_0\in\mathbb{N}$ such that for all $n,m\geq n_0$, we have $d_C((x_m,y_m),(x_n,y_n))\leq\epsilon$. Hence, for all $n,m\geq n_0$, $\|x_m-x_n\|+\|x_m'-x_n'\|\leq\epsilon_1$ and $\|y_m-y_n\|\leq\epsilon_2$. Then, (x_n), (x_n') and (y_n) are Cauchy sequences in $P(\omega)$. It is clear that $(P(\omega),\|\cdot\|)$ is a Banach space, hence there exists $y\in P(\omega)$ such that

$$\lim_{n\to+\infty}\|y_n-y\|=0 \tag{3.1}$$

and there exist $x,w\in P(\omega)$ such that $\lim_{n\to+\infty}\|x_n-x\|=\lim_{n\to+\infty}\|x_n'-w\|=0$. Now, since for all $n\geq n_0$, and all $t\in\mathbb{R}^+$,

$$x_n(t)=x_n(0)+\int_0^t x_n'(s)ds.$$

Then, by Lebesgue's Dominated Convergence Theorem, $x'(t)=w(t)$ for all $t\in\mathbb{R}^+$. Therefore, for all $n\in\mathbb{N}$ and all $t\in\mathbb{R}$, $x_n(t)\geq 0$. Then for all $t\in\mathbb{R}$, $x(t)\geq 0$. As a consequence, $x\in K^+(\omega)$ and

$$\lim_{n\to+\infty}(\|x_n-x\|+\|x_n'-x'\|)=0\,. \tag{3.2}$$

Finally, we deduce, by (3.1) and (3.2), that (x^n) converges to $(x,y)\in E$ and (E,d_C) is a complete generalized metric space. □

Equation (1.3) will be studied under the following assumptions:

- (H1) $f\in C(\mathbb{R}\times\mathbb{R}^+\times\mathbb{R},(0,+\infty))$ and there exists $\omega>0$ such that
$$f(t+\omega,x,y)=f(t,x,y)\ ,\quad\forall(t,x,y)\in\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}.$$
- (H2) There exist α, $\beta>0$ such that
$$|f(t,u,v)-f(t,u',v')|\leq\alpha|u-u'|+\beta|v-v'|,$$
for all $t\in\mathbb{R}$ and all $u,u'\in\mathbb{R}^+$, for all $v,v'\in\mathbb{R}$.
- (H3) $\sigma\in P^1(\omega)$ and $\inf_{t\in[0,\omega]}\sigma(t)=\sigma_0>0$. Let $\sigma_1=\sup_{t\in[0,\omega]}\sigma(t)$, $\sigma_2=\sup_{t\in[0,\omega]}|\sigma'(t)|$ and assume that $\gamma(1+\sigma_2)<1$.

Under the hypothesis (H1)–(H3), we will use Perov's fixed point theorem to prove the following main result.

Theorem 3.2. *If the hypotheses* (H1)–(H3) *hold, and if*

$$\gamma(1+\sigma_2)+(1-\gamma)\beta(2+\sigma_2)+L+\sqrt{\zeta}<2,$$

where, $L=\max(\gamma(1+\sigma_2),\gamma+(1-\gamma)\alpha(2+\sigma_1+\sigma_2))$ *and*

$$\begin{aligned}\zeta&=\gamma^2\sigma_2(2+\sigma_2)+(1-\gamma)^2\beta(2+\sigma_2)(\beta(2+\sigma_2)+4\alpha(2+\sigma_1+\sigma_2))+(L-\gamma)^2\\&\quad+2\gamma\beta(1-\gamma)(1+\sigma_2)(2+\sigma_2)-2L(\beta(1-\gamma)(2+\sigma_2)+\gamma\sigma_2).\end{aligned}$$

Then, (1.3) *has a unique solution in* $K^+(\omega)$.

Proof. If we differentiate (1.3) with respect to t and denoting $x'(t)=y(t)$, for all $t\in\mathbb{R}$, we obtain

$$\begin{aligned}y(t)&=\gamma(1-\sigma'(t))y(t-\sigma(t))+(1-\gamma)\Big[f(t,x(t),y(t))\\&\quad-(1-\sigma'(t))f(t-\sigma(t),x(t-\sigma(t)),y(t-\sigma(t)))\Big],\end{aligned}$$

which leads to

$$\begin{aligned}x(t)&=\gamma x(t-\sigma(t))+(1-\gamma)\int_{t-\sigma(t)}^{t}f(s,x(s),y(s))ds,\\y(t)&=\gamma(1-\sigma'(t))y(t-\sigma(t))+(1-\gamma)\Big[f(t,x(t),y(t))\\&\quad-(1-\sigma'(t))f(t-\sigma(t),x(t-\sigma(t)),y(t-\sigma(t)))\Big].\end{aligned}$$

Let $T:E\to C(\mathbb{R})\times C(\mathbb{R})$ be the map defined by

$$T(x,y)(t)=\begin{pmatrix}T_1(x,y)(t)\\T_2(x,y)(t)\end{pmatrix},$$

where

$$T_1(x,y)(t)=\gamma x(t-\sigma(t))+(1-\gamma)\int_{t-\sigma(t)}^{t}f(s,x(s),y(s))ds,$$

and

$$\begin{aligned}T_2(x,y)(t)&=\gamma(1-\sigma'(t))y(t-\sigma(t))+(1-\gamma)\Big[f(t,x(t),y(t))\\&\quad-(1-\sigma'(t))f(t-\sigma(t),x(t-\sigma(t)),y(t-\sigma(t)))\Big].\end{aligned}\tag{3.3}$$

From Conditions (H1) and (H3), one has that $T_1(E)\subset C^1(\mathbb{R})$. Hence, from the condition that f is ω-periodic with respect to t, it follows that $T_1(E)\subset K^+(\omega)$. indeed

$$\begin{aligned}T_1(x,y)(t+\omega)&=\gamma x(t+\omega-\sigma(t+\omega))+(1-\gamma)\int_{t+\omega-\sigma(t+\omega)}^{t+\omega}f(s,x(s),y(s))ds\\&=\gamma x(t-\sigma(t))+(1-\gamma)\int_{t-\sigma(t)}^{t}f(s-\omega,x(s-\omega),y(s-\omega))ds\\&=T_1(x,y)(t),\quad\forall t\in\mathbb{R},\ \forall(x,y)\in E.\end{aligned}$$

In addition in the same way, one has $T_2(x,y)(t+\omega)=T_2(x,y)(t)$. Consequently, $T(E)\subset E$. Moreover, from Conditions (H2),

$$\begin{aligned}&|T_1(x_1,y_1)(t)-T_1(x_2,y_2)(t)|+|T_1'(x_1,y_1)(t)-T_1'(x_2,y_2)(t)|\\&\le\gamma|x_1(t-\sigma(t))-x_2(t-\sigma(t))|\end{aligned}$$

$$
\begin{aligned}
&+(1-\gamma)\int_{t-\sigma(t)}^{t}[\alpha|x_1(s)-x_2(s)|+\beta|y_1(s)-y_2(s)|]ds\\
&+\gamma(1-\sigma'(t))|x_1'(t-\sigma(t))-x_2'(t-\sigma(t))|\\
&+(1-\gamma)(\alpha|x_1(t)-x_2(t)|+\beta|y_1(t)-y_2(t)|)\\
&+(1-\gamma)|1-\sigma'(t)|\alpha|x_1(t-\sigma(t))-x_2(t-\sigma(t))|\\
&+(1-\gamma)|1-\sigma'(t)|\beta|y_1(t-\sigma(t))-y_2(t-\sigma(t))|\\
\le{}& L(\|x_1-x_2\|+\|x_1'-x_2'\|)+(1-\gamma)\beta(1+\sigma_1+\sigma_2)\|y_1-y_2\|
\end{aligned}
$$

where

$$
L=\max(\underbrace{\gamma(1+\sigma_2)}_{r_1},\underbrace{\gamma+(1-\gamma)\alpha(2+\sigma_1+\sigma_2)}_{r_2}).
$$

Similarly, one has

$$
\begin{aligned}
&|T_2(x_1,y_1)(t)-T_2(x_2,y_2)(t)|\\
&\le(2+\sigma_2)(1-\gamma)\alpha\|x_1-x_2\|+[\gamma(1+\sigma_2)+(1-\gamma)\beta(2+\sigma_2)]\|y_1-y_2\|\\
&\le(2+\sigma_2)(1-\gamma)\alpha(\|x_1-x_2\|+\|x_1'-x_2'\|)\\
&\quad+[\gamma(1+\sigma_2)+(1-\gamma)\beta(2+\sigma_2)]\|y_1-y_2\|.
\end{aligned}
$$

So

$$
d_C(T(x_1,y_1),T(x_2,y_2))\le A\begin{pmatrix}\|x_1-x_2\|+\|x_1'-x_2'\|\\ \|y_1-y_2\|\end{pmatrix},
$$

where

$$
A=\begin{pmatrix} L & (1-\gamma)\beta(2+\sigma_1+\sigma_2)\\ (2+\sigma_2)(1-\gamma)\alpha & \gamma(1+\sigma_2)+(1-\gamma)\beta(2+\sigma_2)\end{pmatrix}.
$$

The eigenvalues of this matrix are

$$
\begin{aligned}
\lambda_1&=\frac{1}{2}[\gamma(1+\sigma_2)+(1-\gamma)\beta(2+\sigma_2)+L+\sqrt{\zeta}],\\
\lambda_2&=\frac{1}{2}[\gamma(1+\sigma_2)+(1-\gamma)\beta(2+\sigma_2)+L-\sqrt{\zeta}]
\end{aligned}
$$

where

$$
\begin{aligned}
\zeta&=\gamma^2\sigma_2(2+\sigma_2)+(1-\gamma)^2\beta(2+\sigma_2)(\beta(2+\sigma_2)+4\alpha(2+\sigma_1+\sigma_2))+(L-\gamma)^2\\
&\quad+2\gamma\beta(1-\gamma)(1+\sigma_2)(2+\sigma_2)-2L(\beta(1-\gamma)(2+\sigma_2)+\gamma\sigma_2).
\end{aligned}
$$

In what follows, we show that the eigenvalues of the matrix A are nonnegative real numbers ($\lambda_1,\lambda_2\in\mathbb{R}^+$).

Step 1: We show that λ_1,λ_2 are real numbers. It suffices to prove that $\zeta\ge 0$. We have two cases:

Case 1: If $L=r_1$, then

$$
\begin{aligned}
\zeta&=\gamma^2\sigma_2(2+\sigma_2)+(1-\gamma)^2\beta(2+\sigma_2)(\beta(2+\sigma_2)+4\alpha(2+\sigma_1+\sigma_2))+\gamma^2\sigma_2^2\\
&\quad+2\gamma\beta(1-\gamma)(1+\sigma_2)(2+\sigma_2)-2\gamma(1+\sigma_2)(\beta(1-\gamma)(2+\sigma_2)+\gamma\sigma_2)\\
&=\gamma^2\sigma_2(2+\sigma_2)+(1-\gamma)^2\beta(2+\sigma_2)(\beta(2+\sigma_2)+4\alpha(2+\sigma_1+\sigma_2))+\gamma^2\sigma_2^2\\
&\quad-2\gamma^2\sigma_2(1+\sigma_2)\\
&=(1-\gamma)^2\beta(2+\sigma_2)(\beta(2+\sigma_2)+4\alpha(2+\sigma_1+\sigma_2))\ge 0.
\end{aligned}
$$

Case 2: If $L = r_2$, then

$$\begin{aligned}
\zeta &= \gamma^2\sigma_2(2+\sigma_2) + (1-\gamma)^2\beta(2+\sigma_2)(\beta(2+\sigma_2) + 4\alpha(2+\sigma_1+\sigma_2)) \\
&\quad + (1-\gamma)^2\alpha^2(2+\sigma_1+\sigma_2)^2 + 2\gamma\beta(1-\gamma)(1+\sigma_2)(2+\sigma_2) \\
&\quad - 2(\gamma + (1-\gamma)\alpha(2+\sigma_1+\sigma_2))(\beta(1-\gamma)(2+\sigma_2) + \gamma\sigma_2) \\
&= \gamma^2\sigma_2(2+\sigma_2) + (1-\gamma)^2\beta(2+\sigma_2)(\beta(2+\sigma_2) + 2\alpha(2+\sigma_1+\sigma_2)) \\
&\quad + (1-\gamma)^2\alpha^2(2+\sigma_1+\sigma_2)^2 + 2\gamma\beta(1-\gamma)\sigma_2(2+\sigma_2) \\
&\quad - 2\gamma\sigma_2(\gamma + (1-\gamma)\alpha(2+\sigma_1+\sigma_2)) \\
&= \gamma^2\sigma_2^2 + (1-\gamma)^2\beta(2+\sigma_2)(\beta(2+\sigma_2) + 2\alpha(2+\sigma_1+\sigma_2)) \\
&\quad + (1-\gamma)^2\alpha^2(2+\sigma_1+\sigma_2)^2 + 2\gamma\beta(1-\gamma)\sigma_2(2+\sigma_2) \\
&\quad - 2\gamma\sigma_2(1-\gamma)\alpha(2+\sigma_1+\sigma_2) \\
&= (1-\gamma)^2\beta(2+\sigma_2)(\beta(2+\sigma_2) + 2\alpha(2+\sigma_1+\sigma_2)) + 2\gamma\beta(1-\gamma)\sigma_2(2+\sigma_2) \\
&\quad + ((1-\gamma)\alpha(2+\sigma_1+\sigma_2) - \gamma\sigma_2)^2 \geq 0.
\end{aligned}$$

Step 2: We show that λ_2 is nonnegative. We have two cases:
Case 1: If $L = r_1$, then

$$\begin{aligned}
&[\gamma(1+\sigma_2) + (1-\gamma)\beta(2+\sigma_2) + L]^2 - \zeta \\
&= 4\gamma^2(1+\sigma_2)^2 + (1-\gamma)^2\beta^2(2+\sigma_2)^2 + 4\gamma(1+\sigma_2)(1-\gamma)\beta(1+\sigma_2) - \zeta \\
&= 4\gamma^2(1+\sigma_2)^2 + 4(1-\gamma)\beta(2+\sigma_2)(r_1 - r_2 + \gamma) \geq 0.
\end{aligned}$$

Case 2: If $L = r_2$, then

$$\begin{aligned}
&[\gamma(1+\sigma_2) + (1-\gamma)\beta(2+\sigma_2) + L]^2 - \zeta \\
&= [\gamma(2+\sigma_2) + (1-\gamma)\beta(2+\sigma_2) + (1-\gamma)\alpha(2+\sigma_1+\sigma_2)]^2 - \zeta \\
&= \gamma^2(2+\sigma_2)^2 + (1-\gamma)^2\beta(2+\sigma_2)(\beta(2+\sigma_2) + 2\alpha(2+\sigma_1+\sigma_2)) \\
&\quad + 2\gamma\beta(1-\gamma)(2+\sigma_2)^2 + (1-\gamma)^2\alpha^2(2+\sigma_1+\sigma_2)^2 \\
&\quad + 2\gamma(1-\gamma)(2+\sigma_2)\alpha(2+\sigma_1+\sigma_2) - \zeta \\
&= 4\gamma^2(1+\sigma_2) + 4\gamma\beta(1-\gamma)(2+\sigma_2) + 4\gamma(1-\gamma)\alpha(2+\sigma_1+\sigma_2)(1+\sigma_2) \geq 0.
\end{aligned}$$

Which implies that $\lambda_2 \geq 0$.

We remark that $\lambda_1 > \lambda_2$, this implies that λ_1 and λ_2 belong to the open unit disc of $\mathbb{R}^2$ if and only if $\lambda_1 < 1$, which is equivalent to

$$\gamma(1+\sigma_2) + (1-\gamma)\beta(2+\sigma_2) + L + \sqrt{\zeta} < 2.$$

Then, by Perov's fixed point theorem, the operator T has a unique solution $x^* = (x_*, y_*) \in K^+(\omega) \times P(\omega)$, which implies that $x_* \in C^1(\mathbb{R})$, and for all $t \in \mathbb{R}$,

$$\begin{aligned}
(x_*)'(t) = \gamma(1-\sigma'(t))(x_*)'(t-\sigma(t)) + (1-\gamma)\Big[& f(t, x_*(t), y_*(t)) \\
& - (1-\sigma'(t)) f(t-\sigma(t), x_*(t-\sigma(t)), y_*(t-\sigma(t)))\Big].
\end{aligned}$$

Hence, by using (3.3), for all $t \in \mathbb{R}$,

$$((x_*)' - y_*)(t) = \gamma(1-\sigma'(t))((x_*)' - y_*)(t-\sigma(t)).$$

Then, $\|(x_*)' - y_*\| \leq \gamma(1+\sigma_2)\|(x_*)' - y_*\|$. We deduce, by Condition (H3), that $(x_*)' = y_*$ and x_* is the unique solution of (1.3). $\square$

To illustrate this result, we have the following example.

Example 3.3. Consider (1.3) where f is ω-periodic with respect to t and σ is ω-periodic, $\gamma = \sigma_1 = \sigma_2 = \frac{1}{4}$, $\alpha = \frac{1}{6}$, $\beta = \frac{1}{5}$, then $\gamma(1+\sigma_2) = \frac{5}{16} < 1$, $L = \max(\frac{5}{16}, \frac{9}{16}) = \frac{9}{16}$ and

$$\gamma(1+\sigma_2) + (1-\gamma)\beta(2+\sigma_2) + L + \sqrt{\zeta} = \frac{67+\sqrt{2749}}{80} \cong 1.86 < 2.$$

Thus, by Theorem 3.2, Equation (1.3) has a unique positive ω-periodic solution.

The following proposition gives an estimation of the error between the exact solution and the approximate solution of (1.3).

Proposition 3.4. *Under the assumptions of Theorem 3.2, the solution of* (1.3)*, which is obtained by the successive approximations method starting from any $x^0 = (x_0, y_0) \in E$, satisfies the estimate*

$$d_C(x^m, x^*) \leq \frac{1}{\mu(\lambda_1 - \lambda_2)} \begin{pmatrix} e_1\lambda_1^m + e_2\lambda_2^m & e_3\lambda_1^m + e_4\lambda_2^m \\ e_5\lambda_1^m + e_6\lambda_2^m & e_7\lambda_1^m + e_8\lambda_2^m \end{pmatrix} \times d_C(x^1, x^0),$$

where $\mu = (1-L)(1-\gamma(1+\sigma_2) - \beta(2+\sigma_2)(1-\gamma)) - (1-\gamma)^2\alpha\beta(2+\sigma_2)(2+\sigma_1+\sigma_2)$, $x^m = T(x^{m-1})$, $x^m = (x_m, y_m)$, for all $m \in \mathbb{N}^*$ and

$$\begin{aligned} &e_1 = (a(L-\lambda_2) - c^2), e_2 = (a(\lambda_1 - L) + c^2) \\ &e_3 = (b(L-\lambda_2) - c(1-L)), e_4 = (b(\lambda_1 - L) + c(1-L)) \\ &e_5 = (L-\lambda_1)(\frac{a(L-\lambda_2)}{c} - c), e_6 = (L-\lambda_2)(c - \frac{a(L-\lambda_1)}{c}) \\ &e_7 = (L-\lambda_1)(\frac{b(L-\lambda_2)}{c} + L - 1), e_8 = (L-\lambda_2)(1 - L - \frac{b(L-\lambda_1)}{c}) \end{aligned} \tag{3.4}$$

such that

$$\begin{aligned} a &= 1 - \gamma(1+\sigma_2) - \beta(2+\sigma_2)(1-\gamma) \\ b &= (1-\gamma)\beta(2+\sigma_1+\sigma_2) \\ c &= (2+\sigma_2)(1-\gamma)\alpha. \end{aligned}$$

Proof. From Theorem 2.5, by the conditions of Theorem 3.2, one has

$$d_C(x^m, x^*) \leq A^m(I-A)^{-1} d_C(x^1, x^0), \quad \forall m \in \mathbb{N}^*.$$

We have

$$A^m = \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} (L-\lambda_2)\lambda_1^m + (\lambda_1 - L)\lambda_2^m & (1-\gamma)\alpha(2+\sigma_2)(\lambda_2^m - \lambda_1^m) \\ \frac{(L-\lambda_1)(L-\lambda_2)(\lambda_1^m - \lambda_2^m)}{(1-\gamma)\alpha(2+\sigma_2)} & (\lambda_1 - L)\lambda_1^m + (L-\lambda_2)\lambda_2^m \end{pmatrix},$$

and

$$(I-A)^{-1} = \frac{1}{\mu} \begin{pmatrix} \underbrace{1-\gamma(1+\sigma_2) - \beta(2+\sigma_2)(1-\gamma)}_{a} & \underbrace{(1-\gamma)\beta(2+\sigma_1+\sigma_2)}_{b} \\ \underbrace{(2+\sigma_2)(1-\gamma)\alpha}_{c} & 1-L \end{pmatrix},$$

where $\mu = (1-L)(1-\gamma(1+\sigma_2)-\beta(2+\sigma_2)(1-\gamma)) - (1-\gamma)^2\alpha\beta(2+\sigma_2)(2+\sigma_1+\sigma_2)$. Which implies

$$A^m(I-A)^{-1} = \frac{1}{\mu(\lambda_1-\lambda_2)}\begin{pmatrix} e_1\lambda_1^m + e_2\lambda_2^m & e_3\lambda_1^m + e_4\lambda_2^m \\ e_5\lambda_1^m + e_6\lambda_2^m & e_7\lambda_1^m + e_8\lambda_2^m \end{pmatrix},$$

where e_i, $i = 1, \dots, 8$ are given by (3.4). □

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