Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 105, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

WEAK SOLUTIONS TO DISCRETE NONLINEAR TWO-POINT BOUNDARY-VALUE PROBLEMS OF KIRCHHOFF TYPE

BLAISE KONE, ISMAËL NYANQUINI, STANISLAS OUARO

Abstract. In this article, we prove the existence of weak solutions to a family of discrete boundary-value problems whose right-hand side belongs to a discrete Hilbert space. As an extension, we prove the existence of weak solutions for problems whose right-hand side depends on the solution.

1. Introduction

In this article, we study the nonlinear discrete boundary-value problem

$$\begin{gathered} -M(A(k-1,\Delta u(k-1)))\Delta(a(k-1,\Delta u(k-1))) = f(k), \quad k\in\mathbb{Z}[1,T] \\ u(0)=\Delta u(T)=0, \end{gathered} \tag{1.1}$$

where $T\geq 2$ is a positive integer and $\Delta u(k)=u(k+1)-u(k)$ is the forward difference operator. Throughout this paper, we denote by $\mathbb{Z}[a,b]$ the discrete interval $\{a,a+1,\dots,b\}$ where a and b are integers and $a<b$.

We consider in (1.1) two different boundary conditions: a Dirichlet boundary condition ($u(0)=0$) and a Neumann boundary condition ($\Delta u(T)=0$). In the literature, the boundary condition considered in this paper is called a mixed boundary condition. We also consider the function space

$$W=\{v:\mathbb{Z}[0,T+1]\to\mathbb{R} \text{ such that } v(0)=\Delta v(T)=0\}.$$

W is a T-dimensional Hilbert space (see [1]) with the inner product

$$(u,v)=\sum_{k=1}^{T}u(k)v(k),\quad \forall u,v\in W.$$

The associated norm is defined by

$$\|u\|=\Big(\sum_{k=1}^{T}|u(k)|^2\Big)^{1/2}.$$

For the data f and a, we assume the following.

$$f:\mathbb{Z}[1,T]\to\mathbb{R}, \tag{1.2}$$

2010 *Mathematics Subject Classification.* 47A75, 35B38, 35P30, 34L05, 34L30.
Key words and phrases. Kirchhoff type problems; discrete boundary-value problem; critical point; weak solution; electrorheological fluids.

Submitted June 29, 2014. Published April 21, 2015.

$$a(k,.) : \mathbb{R} \to \mathbb{R} \text{ for } k \in \mathbb{Z}[0,T] \text{ and there exists a mapping } A : \mathbb{Z}[0,T] \times \mathbb{R} \to \mathbb{R} \text{ satisfying } a(k,\xi) = \frac{\partial}{\partial \xi} A(k,\xi) \text{ and } A(k,0) = 0 \text{ for all } k \in \mathbb{Z}[0,T], \tag{1.3}$$

$$(a(k,\xi) - a(k,\eta))(\xi - \eta) > 0 \ \forall k \in \mathbb{Z}[0,T] \text{ and } \xi, \eta \in \mathbb{R} \text{ such that } \xi \neq \eta, \tag{1.4}$$

$$|\xi|^{p(k)} \leq a(k,\xi)\xi \leq p(k)A(k,\xi) \quad \forall k \in \mathbb{Z}[0,T] \text{ and } \xi \in \mathbb{R}. \tag{1.5}$$

Moreover, in this paper, we assume that

$$p : \mathbb{Z}[0,T] \to (1, +\infty). \tag{1.6}$$

We also assume that the function $M : (0, +\infty) \to (0, +\infty)$ is continuous and nondecreasing and there exist positive reals number B_1, B_2 with $B_1 \leq B_2$ and $\alpha \geq 1$ such that

$$B_1 t^{\alpha - 1} \leq M(t) \leq B_2 t^{\alpha - 1} \quad \text{for } t \geq t^* > 0. \tag{1.7}$$

The function $M(A(k-1, \Delta u(k-1)))$ in the left-hand side of (1.1) is more general than the one in [7]. Indeed, if we take $M(t) = 1$, (1.1) is the problem studied by Koné et al [7].

Problem (1.1) has its origin in the theory of nonlinear vibration. For instance, the following equation describes the free vibration of a stretched string (see [11])

$$\rho \frac{\partial^2 u}{\partial t^2} = \Big(T_0 + \frac{Ea}{2L} \int_0^L |\frac{\partial u}{\partial x}|^2 dx\Big) \frac{\partial^2 u}{\partial x^2} \tag{1.8}$$

where $\rho > 0$ is the mass per unit length, T_0 is the base tension, E is the Young modulus, a is the area of cross section and L is the initial length of the string. (1.8) takes into account the change of the tension on the string which is caused by the change of its length during the vibration. The nonlocal equation of this type was first proposed by Kirchhoff in 1876 (see [6]). After that, several physicists also consider such equations for their researches in the theory of nonlinear vibrations theoretically or experimentally [3,4,9,11]. Moreover mathematically, the solvability of several Kirchhoff type quasilinear hyperbolic equations have been extensively discussed. As far as we know, the first study which deals with anisotropic discrete boundary-value problems of $p(.)$-Kirchhoff type difference equation was done by Yucedag (see [12]). In this paper, we improve the work by Yucedag [12] since our main operator is more general than the one in [12]. As examples of functions satisfying assumptions (1.3)–(1.7), we can give the following.

- $M(A(k,\xi)) = M(\frac{1}{p(k)}|\xi|^{p(k)}) = 1$, where $M(t) = 1$ and $a(k,\xi) = |\xi|^{p(k)-2}\xi$, for $k \in \mathbb{Z}[0,T]$ and $\xi \in \mathbb{R}$.
- $M(A(k,\xi)) = a + \frac{b}{p(k)}\big[\big(1 + |\xi|^2\big)^{p(k)/2} - 1\big]$, where $M(t) = a + bt$ and $a(k,\xi) = \big(1 + |\xi|^2\big)^{(p(k)-2)/2}\xi$, for all $k \in \mathbb{Z}[0,T]$ and $\xi \in \mathbb{R}$.

The remaining part of this article is organized as follows: section 2 is devoted to mathematical preliminaries. The main existence result is stated and proved in section 3. Finally, in section 4, we discuss some extensions.

2. Preliminaries

We will use the following symbols.

$$p^- = \min_{k \in \mathbb{Z}[0,T]} p(k), \quad p^+ = \max_{k \in \mathbb{Z}[0,T]} p(k).$$

It is useful to introduce other norms on W, namely

$$|u|_m = \Big(\sum_{k=1}^{T} |u(k)|^m\Big)^{1/m} \quad \forall u \in W \text{ and } m \geq 2.$$

We have the following inequalities (see [2, 8]):

$$T^{(2-m)/(2m)}|u|_2 \leq |u|_m \leq T^{1/m}|u|_2 \quad \forall u \in W \text{ and } m \geq 2. \tag{2.1}$$

In the sequel, we will use the following auxiliary result.

Lemma 2.1 ([5]). *There exist two positive constants C_1 and C_2 such that*

$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} \geq C_1\Big(\sum_{k=1}^{T+1} |\Delta u(k-1)|^2\Big)^{p^-/2} - C_2, \tag{2.2}$$

for all $u \in W$ with $\|u\| > 1$.

Next we have the discrete Wirtinger's inequality, see [1, Theorem 12.6.2, page 860].

Lemma 2.2. *For any function $u(k)$, $k \in \mathbb{Z}[0, T]$ satisfying $u(0) = 0$, the following inequality holds*

$$4\sin^2\Big(\frac{\pi}{2(2T+1)}\Big)\sum_{k=1}^{T} |u(k)|^2 \leq \sum_{k=1}^{T} |\Delta u(k-1)|^2. \tag{2.3}$$

3. Existence of weak solutions

In this section, we study the existence of weak solution of (1.1).

Definition 3.1. *A weak solution of* (1.1) *is a function $u \in W$ such that*

$$M(\sum_{k=1}^{T+1} A(k-1, \Delta u(k-1)))\sum_{k=1}^{T+1} a(k-1, \Delta u(k-1))\Delta v(k-1) = \sum_{k=1}^{T} f(k)v(k) \tag{3.1}$$

for all $v \in W$.

Note that, since W is a finite dimensional space, the weak solutions coincide with the classical solutions of the problem (1.1).

Theorem 3.2. *Assume that* (1.2)-(1.7) *hold. Then, there exists at least one weak solution of* (1.1).

For the proof of the above theorem, we define the energy functional corresponding to problem (1.1), $J : W \to \mathbb{R}$ as follows:

$$J(u) = \widehat{M}(\sum_{k=1}^{T+1} A(k-1, \Delta u(k-1))) - \sum_{k=1}^{T} f(k)u(k), \tag{3.2}$$

where $\widehat{M}(t) = \int_0^t M(s)\,ds$. We first establish some basic properties of J.

Proposition 3.3. *The functional J is well-defined on E and is of class $C^1(W,\mathbb{R})$ with derivative given by*

$$\begin{aligned}\langle J'(u),v\rangle = M\Big(\sum_{k=1}^{T+1} A(k-1,\Delta u(k-1))\Big)\sum_{k=1}^{T+1} a(k-1,\Delta u(k-1))\Delta v(k-1)\\ -\sum_{k=1}^{T} f(k)v(k),\end{aligned}\tag{3.3}$$

for all $u,v\in W$.

The proof of the above proposition can be found in [10].

We now define the functional $I:H\to\mathbb{R}$ by

$$I(u)=\widehat{M}\Big(\sum_{k=1}^{T+1} A(k-1,\Delta u(k-1))\Big).$$

We need the following lemma for the proof of Theorem 3.2.

Lemma 3.4. *The functional I is weakly lower semi-continuous.*

Proof. By (1.3) and (1.4), we have that A is convex with respect to the second variable. Thus, it is enough to show that I is lower semi-continuous. For this, we fix $u\in H$ and $\epsilon>0$. Since I is convex, we deduce that for any $v\in H$,

$$\begin{aligned}I(v) &\ge I(u)+\langle I'(u),v-u\rangle\\ &\ge I(u)+M\Big(\sum_{k=1}^{T+1} A(k-1,\Delta u(k-1))\Big)\sum_{k=1}^{T+1} a(k-1,\Delta u(k-1))\\ &\quad\times(\Delta v(k-1)-\Delta u(k-1))\\ &\ge I(u)-\Big(M\Big(\sum_{k=1}^{T+1} A(k-1,\Delta u(k-1))\Big)\Big)\sum_{k=1}^{T+1} |a(k-1,\Delta u(k-1))|\\ &\quad\times|\Delta v(k-1)-\Delta u(k-1)|\\ &\ge I(u)-C_0\sum_{k=1}^{T} |a(k-1,\Delta u(k-1))|(|v(k)-u(k)+u(k-1)-v(k-1)|)\\ &\ge I(u)-C_0\sum_{k=1}^{T} |a(k-1,\Delta u(k-1))|(|v(k)-u(k)|+|u(k-1)-v(k-1)|),\end{aligned}$$

where $C_0=\Big(1+M(\sum_{k=1}^{T+1} A(k-1,\Delta u(k-1)))\Big)$. We denote

$$X_1=\sum_{k=1}^{T} |a(k-1,\Delta u(k-1))||v(k)-u(k)|,$$

$$X_2=\sum_{k=1}^{T} |a(k-1,\Delta u(k-1))||u(k-1)-v(k-1)|.$$

By using Schwartz inequality, we obtain

$$X_1\le\Big(\sum_{k=1}^{T} |a(k-1,\Delta u(k-1))|^2\Big)^{1/2}\Big(\sum_{k=1}^{T} |v(k)-u(k)|^2\Big)^{1/2}$$

$$\le \Big(\sum_{k=1}^{T+1}|a(k-1,\Delta u(k-1))|^2\Big)^{1/2}\|v-u\|$$

and

$$\begin{aligned}X_2 &\le \Big(\sum_{k=1}^{T}|a(k-1,\Delta u(k-1))|^2\Big)^{1/2}\Big(\sum_{k=1}^{T}|u(k-1)-v(k-1)|^2\Big)^{1/2}\\&\le \Big(\sum_{k=1}^{T+1}|a(k-1,\Delta u(k-1))|^2\Big)^{1/2}\|v-u\|.\end{aligned}$$

Finally, we have

$$\begin{aligned}I(v) &\ge I(u) - C_0\Big(1+2\sum_{k=1}^{T+1}|a(k-1,\Delta u(k-1))|^2\Big)^{1/2}\|v-u\|\\&\ge I(u)-\epsilon,\end{aligned}$$

for all $v\in W$ with $\|v-u\|<\delta=\epsilon/K(T,u)$, where

$$K(T,u)=C_0\Big(1+2\sum_{k=1}^{T+1}|a(k-1,\Delta u(k-1))|^2\Big)^{1/2}.$$

We conclude that I is weakly lower semi-continuous. The proof of Lemma 3.4 is then complete. □

Proposition 3.5. *The functional J is bounded from below, coercive and weakly lower semi-continuous.*

Proof. By Lemma 3.4, J is weakly lower semicontinuous. We will only prove the coerciveness of the energy functional since the boundeness from below of J is a consequence of coerciveness. By (1.5) and (1.7), we deduce that

$$\begin{aligned}J(u)&=\widehat{M}\Big(\sum_{k=1}^{T+1}A(k-1,\Delta u(k-1))\Big)-\sum_{k=1}^{T}f(k)u(k)\\&\ge \frac{B_1}{\alpha(p^+)^\alpha}\Big(\sum_{k=0}^{T+1}|\Delta u(k-1)|^{p(k-1)}\Big)^\alpha-\sum_{k=1}^{T}f(k)u(k).\end{aligned}$$

To prove the coercivity of J, we may assume that $\|u\|>1$ and we get from the above inequality and Lemma 2.1, that

$$\begin{aligned}J(u)&\ge \frac{B_1}{\alpha(p^+)^\alpha}\Big[C_1\Big(\sum_{k=1}^{T+1}|\Delta u(k-1)|^2\Big)^{p^-/2}-C_2\Big]^\alpha-\sum_{k=1}^{T}f(k)u(k)\\&\ge \frac{B_1C_1^\alpha}{\alpha(p^+)^\alpha}\Big(\sum_{k=1}^{T+1}|\Delta u(k-1)|^2\Big)^{\alpha p^-/2}-K(\alpha,C_2)C_2^\alpha\\&\quad-\Big(\sum_{k=1}^{T}|f(k)|^2\Big)^{1/2}\Big(\sum_{k=1}^{T}|u(k)|^2\Big)^{1/2}.\end{aligned}$$

Using Wirtinger's discrete inequality, we obtain

$$J(u)\ge \frac{B_1C_1^\alpha}{\alpha(p^+)^\alpha}\Big(4\sin^2\Big(\frac{\pi}{2(2T+1)}\Big)\sum_{k=1}^{T+1}|u(k)|^2\Big)^{\alpha p^-/2}$$

$$
\begin{aligned}
&- K' - \Big(\sum_{k=1}^{T}|f(k)|^2\Big)^{1/2}\Big(\sum_{k=1}^{T}|u(k)|^2\Big)^{1/2}\\
&\geq \frac{B_1C_1^{\alpha}2^{\alpha p^-}}{\alpha(p^+)^{\alpha}}\Big(\sin^{\alpha p^-}\Big(\frac{\pi}{2(2T+1)}\Big)\Big)\Big(\sum_{k=1}^{T+1}|u(k)|^2\Big)^{\alpha p^-/2} - K' - K_1\|u\|\\
&\geq \frac{B_1C_1^{\alpha}2^{\alpha p^-}}{\alpha(p^+)^{\alpha}}\Big(\sin^{\alpha p^-}\Big(\frac{\pi}{2(2T+1)}\Big)\Big)\|u\|^{\alpha p^-} - K' - K_1\|u\|,
\end{aligned}
$$

where K_1 and K' are two positive constants. Hence, as $\alpha p^- > 1$, then J is coercive. □

By Proposition 3.5, J has a minimizer which is a weak solution of (1.1).

4. Extensions

4.1. **Extension 1.** In this section, we show that the existence result obtained for (1.1) can be extended to more general discrete boundary-value problems of the form

$$
\begin{gathered}
\begin{aligned}
&- M(A(k-1,\Delta u(k-1)))\Delta(a(k-1,\Delta u(k-1))) + |u(k)|^{q(k)-2}u(k)\\
&= f(k), \quad k\in\mathbb{Z}[1,T]
\end{aligned}\\
u(0) = \Delta u(T) = 0,
\end{gathered} \tag{4.1}
$$

where $T \geq 2$ is a positive integer and where we assume that $q:\mathbb{Z}[1,T]\to(1,+\infty)$. By a weak solution of problem (4.1), we understand a function $u\in W$ such that

$$
\begin{aligned}
&M\Big(\sum_{k=1}^{T+1}A(k-1,\Delta u(k-1))\Big)\sum_{k=1}^{T+1}a(k-1,\Delta u(k-1))\Delta v(k-1)\\
&+\sum_{k=1}^{T}|u(k)|^{q(k)-2}u(k)v(k)\\
&=\sum_{k=1}^{T}f(k)v(k), \quad \text{for any } v\in W.
\end{aligned} \tag{4.2}
$$

Theorem 4.1. *Under assumptions* (1.2)-(1.7)*, problem* (4.1) *has at least one weak solution.*

Proof. For $u\in W$,

$$
J(u) = \widehat{M}\Big(\sum_{k=1}^{T+1}A(k-1,\Delta u(k-1))\Big) + \sum_{k=1}^{T}\frac{1}{q(k)}|u(k)|^{q(k)} - \sum_{k=1}^{T}f(k)u(k) \tag{4.3}
$$

is such that $J\in C^1(W;\mathbb{R})$ is weakly lower semi-continuous and we have

$$
\begin{aligned}
\langle J'(u),v\rangle &= M\Big(\sum_{k=1}^{T+1}A(k-1,\Delta u(k-1))\Big)\sum_{k=1}^{T+1}a(k-1,\Delta u(k-1))\Delta v(k-1)\\
&\quad+\sum_{k=1}^{T}|u(k)|^{q(k)-2}u(k)v(k) - \sum_{k=1}^{T}f(k)v(k),
\end{aligned}
$$

for all $u,v\in W$. This implies that the weak solutions of (4.1) coincide with the critical points of J. Next, we prove that J is bounded from below and coercive in order to complete the proof. □

Since
$$\sum_{k=1}^{T}\frac{1}{q(k)}|u(k)|^{q(k)}\geq 0,$$
it follows that
$$J(u)\geq \widehat{M}\Big(\sum_{k=1}^{T+1}A(k-1,\Delta u(k-1))\Big)-\sum_{k=1}^{T}f(k)u(k). \tag{4.4}$$
Using Proposition 3.5, we deduce that J is bounded from below and coercive.

4.2. **Extension 2.** In this section, we show that the existence result obtained for (1.1) can be extended to more general discrete boundary-value problems of the form
$$\begin{gathered}
-M(A(k-1,\Delta u(k-1)))\Delta(a(k-1,\Delta u(k-1)))+\lambda|u(k)|^{\beta^+-2}u(k)\\
=f(k,u(k)),\quad k\in\mathbb{Z}[1,T]\\
u(0)=\Delta u(T)=0,
\end{gathered} \tag{4.5}$$
where $T\geq 2$ is a positive integer, $\lambda\in\mathbb{R}^+$ and $f:\mathbb{Z}[1,T]\times\mathbb{R}\to\mathbb{R}$ is a continuous function with respect to the second variable for all $(k,z)\in\mathbb{Z}[1,T]\times\mathbb{R}$.

For $k\in\mathbb{Z}[1,T]$ and every $t\in\mathbb{R}$, we put $F(k,t)=\int_0^t f(k,\tau)d\tau$. By a weak solution of problem (4.5), we understand a function $u\in W$ such that
$$\begin{aligned}
&M\Big(\sum_{k=1}^{T+1}A(k-1,\Delta u(k-1))\Big)\sum_{k=1}^{T+1}a(k-1,\Delta u(k-1))\Delta v(k-1)\\
&+\lambda\sum_{k=1}^{T}|u(k)|^{\beta^+-2}u(k)v(k)\\
&=\sum_{k=1}^{T}f(k,u(k))v(k),\quad\text{for all } v\in W.
\end{aligned} \tag{4.6}$$
We assume that there exists a positive constant C_3 such that
$$|f(k,t)|\leq C_3(1+|t|^{\beta(k)-1}),\quad\text{for all } (k,t)\in\mathbb{Z}[1,T]\times\mathbb{R}, \tag{4.7}$$
where $1<\beta^-<\alpha p^-$.

Theorem 4.2. *Under assumptions* (1.3)–(1.7) *and* (4.7), *there exists* $\lambda^*>0$ *such that for* $\lambda\in[\lambda^*,+\infty)$, *problem* (4.5) *has at least one weak solution.*

Proof. Let $g(u)=\sum_{k=1}^{T}F(k,u(k))$, then $g':W\to W$ is completely continuous and thus, g is weakly lower semi-continuous. Therefore, for $u\in W$,
$$J(u)=\widehat{M}\Big(\sum_{k=1}^{T+1}A(k-1,\Delta u(k-1))\Big)+\frac{\lambda}{\beta^+}\sum_{k=1}^{T}|u(k)|^{\beta^+}-\sum_{k=1}^{T}F(k,u(k)) \tag{4.8}$$
is such that $J\in C^1(W;\mathbb{R})$, is weakly lower semi-continuous and we have
$$\begin{aligned}
\langle J'(u),v\rangle&=M\Big(\sum_{k=1}^{T+1}A(k-1,\Delta u(k-1))\Big)\sum_{k=1}^{T+1}a(k-1,\Delta u(k-1))\Delta v(k-1)\\
&\quad+\lambda\sum_{k=1}^{T}|u(k)|^{\beta^+-2}u(k)v(k)-\sum_{k=1}^{T}f(k,u(k))v(k),
\end{aligned}$$

for all $u, v \in W$. This implies that the weak solutions of problem (4.5) coincide with the critical points of J. We then have to prove that J is bounded below and coercive to complete the proof.

For $u \in W$ such that $\|u\| > 1$,

$$
\begin{aligned}
J(u) &\geq \frac{B_1 C_1^{\alpha} 2^{\alpha p^-}}{\alpha (p^+)^{\alpha}} \Big(\sin^{\alpha p^-} \Big(\frac{\pi}{2(2T+1)} \Big) \Big) \|u\|^{\alpha p^-} \\
&\quad + \frac{\lambda}{\beta^+} \sum_{k=1}^{T} |u(k)|^{\beta^+} - K' - \sum_{k=1}^{T} F(k, u(k)) \\
&\geq \frac{B_1 C_1^{\alpha} 2^{\alpha p^-}}{\alpha (p^+)^{\alpha}} \Big(\sin^{\alpha p^-} \Big(\frac{\pi}{2(2T+1)} \Big) \Big) \|u\|^{\alpha p^-} + \frac{\lambda}{\beta^+} \sum_{k=1}^{T} |u(k)|^{\beta^+} - K' \\
&\quad - C' \sum_{k=1}^{T} \Big(1 + |u(k)|^{\beta(k)} \Big) \\
&\geq \frac{B_1 C_1^{\alpha} 2^{\alpha p^-}}{\alpha (p^+)^{\alpha}} \Big(\sin^{\alpha p^-} \Big(\frac{\pi}{2(2T+1)} \Big) \Big) \|u\|^{\alpha p^-} + \frac{\lambda}{\beta^+} \sum_{k=1}^{T} |u(k)|^{\beta^+} - K' - C'T \\
&\quad - C' \Big(\sum_{k=1}^{T} |u(k)|^{\beta(k)} \Big) \\
&\geq \frac{B_1 C_1^{\alpha} 2^{\alpha p^-}}{\alpha (p^+)^{\alpha}} \Big(\sin^{\alpha p^-} \Big(\frac{\pi}{2(2T+1)} \Big) \Big) \|u\|^{\alpha p^-} + \frac{\lambda}{\beta^+} \sum_{k=1}^{T} |u(k)|^{\beta^+} - K' - C'T \\
&\quad - C' \Big(\sum_{k=1}^{T} |u(k)|^{\beta^-} + \sum_{k=1}^{T} |u(k)|^{\beta^+} \Big) \\
&\geq \frac{B_1 C_1^{\alpha} 2^{\alpha p^-}}{\alpha (p^+)^{\alpha}} \Big(\sin^{\alpha p^-} \Big(\frac{\pi}{2(2T+1)} \Big) \Big) \|u\|^{\alpha p^-} \\
&\quad + (\frac{\lambda}{\beta^+} - C') \sum_{k=1}^{T} |u(k)|^{\beta^+} - K' - C'T - K\|u\|^{\beta^-} \\
&\geq \frac{B_1 C_1^{\alpha} 2^{\alpha p^-}}{\alpha (p^+)^{\alpha}} \Big(\sin^{\alpha p^-} \Big(\frac{\pi}{2(2T+1)} \Big) \Big) \|u\|^{\alpha p^-} - K' - C'T - K\|u\|^{\beta^-},
\end{aligned}
$$

where we put $\lambda^* = C'\beta^+$ and where K, K' and C' are positive constants. Furthermore, by the fact that $1 < \beta^- < \alpha p^-$, it turns out that

$$
J(u) \geq \frac{B_1 C_1^{\alpha} 2^{2^{\alpha p^-}}}{\alpha (p^+)^{\alpha}} \Big(\sin^{\alpha p^-} \Big(\frac{\pi}{2(2T+1)} \Big) \Big) \|u\|^{\alpha p^-} - K' - C'T - K\|u\|^{\beta^-} \to +\infty
$$

as $\|u\| \to +\infty$, where K is a positive constant. Therefore, J is coercive. □

4.3. **Extension 3.** In this section, we show that the existence result obtained for (1.1) can be extended to more general discrete boundary-value problems of the form

$$
\begin{gathered}
-M(A(k-1, \Delta u(k-1)))\Delta(a(k-1, \Delta u(k-1))) = f(k, u(k)), \quad k \in \mathbb{Z}[1, T] \\
u(0) = \Delta u(T) = 0,
\end{gathered}
\tag{4.9}
$$

where $T \geq 2$. We suppose that $F^+(k,t) = \int_0^t f^+(k,\tau)d\tau$ is such that there exist two positive constants C_4 and C_5 such that

$$f^+(k,t) \leq C_4 + C_5|t|^{\beta-1}, \quad \text{for all } (k,t) \in \mathbb{Z}[1,T] \times \mathbb{R}, \tag{4.10}$$

where $1 < \beta < \alpha p^-$. By a weak solution of problem (4.9), we understand a function $u \in W$ such that

$$\begin{aligned} &M(\sum_{k=1}^{T+1} A(k-1,\Delta u(k-1))) \sum_{k=1}^{T+1} a(k-1,\Delta u(k-1))\Delta v(k-1) \\ &= \sum_{k=1}^{T} f(k,u(k))v(k), \quad \text{for all } v \in W. \end{aligned} \tag{4.11}$$

Theorem 4.3. *Under assumptions* (1.3)-(1.7) *and* (4.10)*, problem* (4.9) *has at least one weak solution.*

Proof. As $f = f^+ - f^-$, letting

$$F^+(k,t) = \int_0^t f^+(k,\tau)d\tau, \quad F^-(k,t) = \int_0^t f^-(k,\tau)d\tau,$$

we have

$$\begin{aligned} J(u) &= \widehat{M}(\sum_{k=1}^{T+1} A(k-1,\Delta u(k-1))) - \sum_{k=1}^{T} F(k,u(k)) \\ &= \widehat{M}(\sum_{k=1}^{T+1} A(k-1,\Delta u(k-1))) - \sum_{k=1}^{T} F^+(k,u(k)) + \sum_{k=1}^{T} F^-(k,u(k)) \\ &\geq \widehat{M}(\sum_{k=1}^{T+1} A(k-1,\Delta u(k-1))) - \sum_{k=1}^{T} F^+(k,u(k)). \end{aligned}$$

Therefore, similarly to the proof of Theorem 4.2, the statement of Theorem 4.3 follows immediately. □

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Blaise Kone
Laboratoire de Mathématiques et Informatique (LAMI), Institut Burkinabé des Arts et Métiers, Université de Ouagadougou, 03 BP 7021 Ouaga 03 Ouagadougou, Burkina Faso
E-mail address: leizon71@yahoo.fr

Ismaël Nyanquini
Laboratoire de Mathématiques et Informatique (LAMI), Institut des Sciences Exactes et Appliquées, Université Polytechnique de Bobo-Dioulasso, 01 BP 1091 Bobo-Dioulasso 01, Bobo Dioulasso, Burkina Faso
E-mail address: nyanquis@yahoo.fr

Stanislas Ouaro
Laboratoire de Mathématiques et Informatique (LAMI), UFR Sciences Exactes et Appliquées, Université de Ouagadougou, 03 BP 7021 Ouaga 03 Ouagadougou, Burkina Faso
E-mail address: souaro@univ-ouaga.bf, ouaro@yahoo.fr